

Characterizations based on conditional expectations¹

Jyh-Cherng Su and Wen-Jang Huang

Received: October 15, 1998; revised version: August 10, 1999

For given real functions g and h , first we give necessary and sufficient conditions such that there exists a random variable X satisfying that $E(g(X)|X \geq y) = h(y)r_X(y)$, $\forall y \in C_X$, where C_X and r_X are the support and the failure rate function of X , respectively. These extend the results of Ruiz and Navarro (1994) and Ghitany et al. (1995). Next we investigate necessary and sufficient conditions such that $h(y) = E(g(X)|X \geq y)$, for a given function h .

Key words: Characterization; conditional expectation; failure rate function.

AMS 1991 subject classifications: Primary 60E05; secondary 62E10.

1. Introduction

The classical characterization of the exponential distributions is by the memoryless property. Here a nonnegative random variable X is said to have the memoryless property if it satisfies

$$(1) \quad P(X \geq x + y | X \geq y) = P(X \geq x), \forall x, y \geq 0.$$

This has subsequently been superseded by Cox (1962), Cundy (1966), Reinhardt (1968) and Shanbhag (1970) with a characterization which uses the property of constant expected residual life. Further related results

¹Support for this research was provided in part by the National Science Council of the Republic of China, Grant No. NSC 86-2115-M-110-014 and NSC 88-2118-M-110-001.

can be found in Dallas (1976), Galambos and Hagwood (1992), Rao and Shanbhag (1994) and references therein. For example, if a nonnegative and nondegenerate random variable X satisfies that

$$(2) \quad E(G(X - y)|X \geq y) = c, \quad \forall y \geq 0,$$

where the function G satisfies some mild conditions, then X has an exponential distribution. Furthermore, for a given real, continuous and strictly monotone function g , Zoroa et al. (1990) gave necessary and sufficient conditions for a real function $h(y)$ to be the conditional expectation $E(g(X)|X \geq y)$ of some continuous random variable X . Ruiz et al. (1993) extended this result to continuous multivariate random variables. Ruiz and Navarro (1995) and Marín et al. (1996) gave the related characterizations for discrete cases. Franco and Ruiz (1995), (1996) also gave the corresponding characterizations about order statistics and record values, respectively.

On the other hand, Shanbhag (1970), Osaki and Li (1988), Ahmed (1991), Nair and Sankaran (1991), Ruiz and Navarro (1994) and Ghitany et al. (1995) used relationship between conditional expectation and failure (or hazard) rate function to establish some characterization results. More precisely, under certain conditions, Ghitany et al. (1995) proved that the continuous nonnegative random variable X has a probability density function (*p.d.f.*) $f(x) = \exp\{-q(x)\}$, $x \geq 0$, if and only if for a real-valued differentiable function $k(y) \neq 0$, $\forall y \geq 0$,

$$(3) \quad E \left\{ \left[1 + \frac{q''(X)}{[q'(X)]^2} \right] k(X) - \frac{k'(X)}{q'(X)} \middle| X \geq y \right\} = \frac{k(y)}{q'(y)} r_X(y), \quad \forall y \geq 0,$$

where $r_X(x) = f(x)/(1 - F(x-))$, $x \in R$, is the failure rate function of X , and F is the distribution function of X . They also gave some applications to characterize the gamma, Weibull, and Gompertz distributions. For a given real function h and constant c , Ruiz and Navarro (1994) characterized the distribution of X by the equation $E(X|X \geq y) = c + h(y)r_X(y)$, where X is allowed to be discrete or absolutely continuous. The characterization theorems of Ruiz and Navarro (1994) extended the results of Shanbhag (1970), Osaki and Li (1988), Ahmed (1991) and Nair and Sankaran (1991), which gave the characterizations of some special distributions.

In this paper, for a random variable X with *p.d.f.* f , let $C_X = \{x|f(x) \neq 0\}$ denote the support of X . First we extend the results of Ruiz and Navarro (1994) and Ghitany et al. (1995). For given real continuous functions g and h , we will give necessary and sufficient conditions

such that there exists a continuous random variable X satisfying that $E(g(X)|X > y) = h(y)r_X(y)$, $\forall y \in C_X$. The corresponding characterization for discrete random variable is also given. Next, replacing the strictly monotone function g , which is assumed in Zoroa et al. (1990), Marín et al. (1996) and Franco and Ruiz (1996), by a more general function, some related characterizations are investigated.

2. Characterizations based on a relationship between the conditional expectation and the failure rate function

In this section, first we give the following characterization theorem based on a relationship between the conditional expectation and failure rate function.

Theorem 1. Let $a < b$, be extended real numbers, and g and h be the real functions defined on (a, b) . Assume g is continuous and $h(y) \neq 0$, $\forall y \in (a, b)$. Then there exists an absolutely continuous random variable X with $C_X = (a, b)$, such that $E(g(X)|X \geq y)$ is finite, $\forall y \in C_X$, and

$$(4) \quad E(g(X)|X \geq y) = h(y)r_X(y), \quad \forall y \in C_X,$$

if and only if for any fixed $w \in (a, b)$, the following conditions hold.

- (i) $\int_x^y g(u)/h(u)du$ is finite, $\forall x, y \in (a, b)$.
- (ii) $\int_a^b \exp\{-\int_w^y g(u)/h(u)du\}/|h(y)|dy < \infty$.
- (iii) $\lim_{y \rightarrow b} \int_w^y g(u)/h(u)du = \infty$.

Moreover, the *p.d.f.* of the random variable X which satisfies (4) with $C_X = (a, b)$, is

$$(5) \quad f(y) = \frac{1}{\alpha_w |h(y)|} e^{-\int_w^y g(u)/h(u)du}, \quad \forall a < y < b,$$

where $\alpha_w = \int_a^b \exp\{-\int_w^y g(u)/h(u)du\}/|h(y)|dy$.

Proof. First we prove the necessity. From (4), we have

$$(6) \quad \int_y^b g(u)f(u)du = h(y)f(y), \quad \forall a < y < b.$$

This in turn implies that

$$(7) \quad \int_x^y \frac{g(u)}{h(u)}du = \int_x^y \frac{g(v)f(v)dv}{\int_v^b g(u)f(u)du}$$

$$\begin{aligned}
&= -\ln \left(\left| \int_y^b g(u)f(u)du \right| \right) + \ln \left(\left| \int_x^b g(u)f(u)du \right| \right) \\
&= -\ln(|h(y)f(y)|) + \ln(|h(x)f(x)|), \quad \forall x, y \in (a, b),
\end{aligned}$$

thus $\int_x^y g(u)/h(u)du$ is finite, $\forall x, y \in (a, b)$. For any $w \in (a, b)$, from (7), we have

$$(8) \quad f(y) = \frac{|h(w)|f(w)}{|h(y)|} e^{-\int_w^y g(u)/h(u)du}, \quad \forall a < y < b.$$

Since $\int_a^b f(y)dy = 1$, we have $\int_a^b \exp\{-\int_w^y g(u)/h(u)du\}/|h(y)|dy < \infty$ and the *p.d.f.* of the absolutely continuous random variable X satisfies (4), with $C_X = (a, b)$, is given in (5). Also it is easy to see that (6) implies $\lim_{y \rightarrow b} h(y)f(y) = 0$. Hence $\lim_{y \rightarrow b} \exp\{-\int_w^y g(u)/h(u)du\} = 0$, i.e. $\lim_{y \rightarrow b} \int_w^y g(u)/h(u)du = \infty$.

Next, assume conditions (i)-(iii) hold. For any $w \in (a, b)$, let f be defined as in (5). Conditions (i) and (ii) imply that f is a *p.d.f.* of some random variable X with $C_X = (a, b)$. Also it can be shown that

$$(9) \quad -\int_w^y \frac{g(u)}{h(u)}du = \ln(|h(y)f(y)|) + \ln \alpha_w, \quad \forall a < y < b.$$

As the left side of (9) is differentiable with respect to y , $s(y) \equiv h(y)f(y)$ is also differentiable with respect to y . Taking the derivatives of both sides of (9) with respect to y , after some manipulations, we obtain

$$(10) \quad -g(u)f(u)du = d(h(u)f(u)), \quad \forall u \in (a, b).$$

As condition (iii) is equivalent to $\lim_{u \rightarrow b} h(u)f(u) = 0$, (10) implies

$$(11) \quad \int_y^b g(x)f(x)dx = h(y)f(y), \quad \forall a < y < b.$$

From (11), we obtain that $E(g(X)|X \geq y)$ is finite, $\forall y \in C_X$, and $E(g(X)|X \geq y) = h(y)r_X(y)$, $\forall y \in C_X$. The sufficiency is proved.

Example 1. Assume $\lambda > 0$. By Theorem 1, X has an exponential distribution with parameter λ , if and only if $C_X = (0, \infty)$ and

$$(12) \quad E(X^2|X \geq y) = \frac{\lambda^2 y^2 + 2\lambda y + 2}{\lambda^3} r_X(y), \quad \forall y > 0.$$

Example 2. Let $a < b$. By Theorem 1, X has a uniform distribution on (a, b) , if and only if $C_X = (a, b)$ and

$$(13) \quad E(X(X - \frac{2}{3}b)|X \geq y) = \frac{1}{3}y^2(b - y)r_X(y), \quad \forall y \in (a, b).$$

We now explain that Theorem 1 is indeed a generalization of Ruiz and Navarro (1994) and Ghitany et al. (1995). Assume $a = 0$, $b = \infty$, $g(y) + h'(y) \neq 0, \forall y > 0$, and g and h' are differentiable. By letting k and q in (3) satisfy $k(y) = g(y) + h'(y)$ and $q'(y) = (g(y) + h'(y))/h(y)$, $\forall y \geq 0$, we obtain Theorem 1 of Ghitany et al. (1995). Yet Example 2 can not be obtained by Ghitany et al. (1995).

Next, it can be seen that if $g(x) = x - c$, equations (4) and (5) are equivalent to (3-3b) and (3-3a) of Ruiz and Navarro (1994), respectively, and Theorem 3 of Ruiz and Navarro (1994) can be obtained from Theorem 1. In Example 1, if (12) is replaced by

$$(14) \quad E(X|X \geq y) = \frac{\lambda y + 1}{\lambda^2} r_X(y), \quad \forall y > 0,$$

then the assertions still hold. By using Theorem 3 of Ruiz and Navarro (1994), the distribution of X also can be determined by (14). But Ruiz and Navarro (1994) cannot determine the distribution of X by giving (12) holds. Similar comments can be applied to Example 2.

Corresponding to Theorem 1, we have the following characterization for discrete case. Throughout the rest of this paper, for convenience when $m = -\infty$, $m \leq i$ means $-\infty < i$, and when $n = \infty$, $i \leq n$ means $i < \infty$.

Theorem 2. Let $m < n$, where m and n are extended integers, and $\{a_i, m \leq i \leq n\}$ be a sequence of real numbers with $a_i < a_{i+1}, \forall m \leq i < n$. Also let g and h be the real functions defined on $\{a_i, m \leq i \leq n\}$. Assume that $h(a_i) \neq 0, \forall m + 1 \leq i \leq n$. Then there exists a discrete random variable X with $C_X = \{a_i, m \leq i \leq n\}$ such that $E(g(X)|X \geq y)$ is finite, $\forall y \in C_X$, and

$$(15) \quad E(g(X)|X \geq y) = h(y)r_X(y), \quad \forall y \in C_X,$$

if and only if for any fixed integer p , $m \leq p \leq n$, define

$$(16) \quad \gamma_p \equiv \sum_{k=p+1}^n \frac{\prod_{i=p}^{k-1} (h(a_i) - g(a_i))}{\prod_{i=p}^{k-1} h(a_{i+1})},$$

where $\gamma_p \equiv 0$ if $p = n < \infty$, and

$$(17) \quad \beta_p \equiv \sum_{k=m}^{p-1} \frac{\prod_{i=k}^{p-1} h(a_{i+1})}{\prod_{i=k}^{p-1} (h(a_i) - g(a_i))},$$

where $\beta_p \equiv 0$ if $p = m > -\infty$, the following conditions hold.

(i) $h(a_{k+1})(h(a_k) - g(a_k)) > 0, \forall m \leq k \leq n - 1$.

(ii) $\gamma_p < \infty$.(iii) $\beta_p < \infty$.(iv) If $n < \infty$, then $h(a_n) = g(a_n)$, and if $n = \infty$ then

$$(18) \quad \lim_{k \rightarrow \infty} \frac{\prod_{i=p}^{k-1} (h(a_i) - g(a_i))}{\prod_{i=p}^{k-2} h(a_{i+1})} = 0.$$

Moreover, the *p.d.f.* of the random variable X satisfies (15) with $C_X = \{a_i, m \leq i \leq n\}$ is

$$(19) \quad f(a_k) = \begin{cases} \frac{1}{1+\beta_p+\gamma_p} \cdot \frac{\prod_{i=p}^{k-1} (h(a_i) - g(a_i))}{\prod_{i=p}^{k-1} h(a_{i+1})}, & \text{for } p+1 \leq k \leq n, \\ \frac{1}{1+\beta_p+\gamma_p}, & \text{for } k = p, \\ \frac{1}{1+\beta_p+\gamma_p} \cdot \frac{\prod_{i=k}^{p-1} h(a_{i+1})}{\prod_{i=k}^{p-1} (h(a_i) - g(a_i))}, & \text{for } m \leq k \leq p-1. \end{cases}$$

Proof. First assume that there exists a random variable X with $C_X = \{a_i, m \leq i \leq n\}$, such that (15) holds. Let f be the *p.d.f.* of X , (15) implies that

$$(20) \quad \sum_{i=k}^n g(a_i) f(a_i) = h(a_k) f(a_k), \quad \forall m \leq k \leq n.$$

From (20), we have

$$(21) \quad \sum_{i=k}^n g(a_i) f(a_i) = g(a_k) f(a_k) + h(a_{k+1}) f(a_{k+1}), \quad \forall m \leq k \leq n-1.$$

In view of (20) and (21), we have

$$(22) \quad h(a_{k+1}) f(a_{k+1}) = f(a_k) (h(a_k) - g(a_k)), \quad \forall m \leq k \leq n-1.$$

As $f(a_k)$ and $f(a_{k+1})$ are positive and $h(a_{k+1})$ is nonzero, we have $h(a_{k+1})(h(a_k) - g(a_k)) > 0$, $\forall m \leq k \leq n-1$. Let p be any fixed integer, where $m \leq p \leq n$, from (22), we have

$$(23) \quad f(a_k) = f(a_p) \frac{\prod_{i=p}^{k-1} (h(a_i) - g(a_i))}{\prod_{i=p}^{k-1} h(a_{i+1})}, \quad \forall p+1 \leq k \leq n,$$

and

$$(24) \quad f(a_k) = f(a_p) \frac{\prod_{i=k}^{p-1} h(a_{i+1})}{\prod_{i=k}^{p-1} (h(a_i) - g(a_i))}, \quad \forall m \leq k \leq p-1.$$

By (23), (24) and the fact that $\sum_{i=m}^n f(a_i) = 1$, it is clear that $\gamma_p < \infty$, $\beta_p < \infty$ and $f(a_p) = (1 + \beta_p + \gamma_p)^{-1}$, where β_p and γ_p are defined as in (16) and (17). Hence the *p.d.f.* of X is given as in (19). Also from (20), it is easy to see that if $n < \infty$, then $h(a_n) = g(a_n)$, and if $n = \infty$, then $\lim_{k \rightarrow \infty} h(a_k)f(a_k) = 0$, i.e. $\lim_{k \rightarrow \infty} \prod_{i=p}^{k-1} (h(a_i) - g(a_i)) / \prod_{i=p}^{k-2} h(a_{i+1}) = 0$. The necessity is proved.

On the other hand, assume for some fixed integer p , $m \leq p \leq n$, conditions (i)-(iv) hold. Define the function f as in (19). It is easy to see that conditions (i)-(iii) imply that f is a *p.d.f.* of some random variable, say X with $C_X = \{a_i, m \leq i \leq n\}$. From the definition of the function f , we have

$$(25) \quad h(a_{k+1})f(a_{k+1}) = h(a_k)f(a_k) - g(a_k)f(a_k), \quad \forall m \leq k \leq n-1.$$

Also it can be seen that if $n = \infty$, (18) is equivalent to $\lim_{k \rightarrow \infty} h(a_k)f(a_k) = 0$. Now for any fixed integer j , $m \leq j \leq n-1$, summing both sides of (25) up for $k = j, \dots, n-1$, and using condition (iv), we obtain $h(a_j)f(a_j) = \sum_{k=j}^n g(a_k)f(a_k)$, and this implies that $E(g(X)|X \geq y)$ is finite, $\forall y \in C_X$, and $E(g(X)|X \geq y) = h(y)r_X(y)$, $\forall y \in C_X$.

Example 3. Assume $0 < \theta < 1$. By Theorem 2, the *p.d.f.* of X is given by $f(k) = \theta(1-\theta)^k$, $\forall k = 0, 1, 2, \dots$, if and only if $C_X = \{0, 1, 2, \dots\}$ and

$$(26) \quad E(X|X \geq k) = \frac{1-\theta}{\theta} + \frac{k}{\theta}r_X(k), \quad \forall k = 0, 1, 2, \dots$$

Example 4. Assume $m < n$ are two finite integers. By Theorem 2, the *p.d.f.* of X is given by $f(k) = (n-m+1)^{-1}$, $m \leq k \leq n$, if and only if $C_X = \{m, m+1, \dots, n\}$ and

$$(27) \quad E(X^2|X \geq k) = (k^2 + (k+1)^2 + \dots + n^2)r_X(k), \quad \forall m \leq k \leq n.$$

Note that if $g(x) = x - c$, then (15) is equivalent to (3-5b) of Ruiz and Navarro (1994). Hence Theorem 2 is an extensions of Theorem 4 of Ruiz and Navarro (1994).

3. Characterizations based on the functions of conditional expectations

In this section, we will give some characterizations based on the functions of conditional expectations. First we have Theorem 3 which is

a generalization of Laurent (1974) and can be compared with Zoroa et al. (1990), where the function g in the theorem is assumed to be strictly monotone. The proof is standard, hence is omitted. Note that if $g(x) = x$ and $h(y) = y + c$, then (28) in Theorem 3 is the same as (2) with $G(x) = x$, and this is a much weaker condition than (1).

Theorem 3. Let g be a given real, continuous function. Assume h is a real function with $h(x) - g(x) \neq 0$, a.e. Then there exists an absolutely continuous random variable X such that $E(g(X)|X \geq y)$ is finite, $\forall y \in D$, and

$$(28) \quad h(y) = E(g(X)|X \geq y), \quad \forall y \in D,$$

where $D = \{y|P(X \geq y) > 0\}$, if and only if the following conditions hold.

(i) $D = (-\infty, b)$, where $b \in (-\infty, \infty) \cup \{\infty\}$.

(ii) h is differentiable almost everywhere in D .

(iii) Let $Q(y) = \int_{-\infty}^y \frac{dh(u)}{h(u)-g(u)}, y < b$, then Q is a nonnegative and increasing function.

(iv) $\int_{-\infty}^b \frac{dh(u)}{h(u)-g(u)} = \infty$.

(v) $\lim_{y \rightarrow b} h(y) \exp\{-\int_{-\infty}^y \frac{dh(u)}{h(u)-g(u)}\} = 0$.

Moreover, the distribution function of X satisfies (28) is

$$(29) \quad F(y) = 1 - e^{-\int_{-\infty}^y \frac{dh(u)}{h(u)-g(u)}}, \quad y \in D.$$

If g is strictly increasing and (28) is satisfied, it is easy to see that $g(y) < h(y), \forall y \in D$, and thus $h(y) - g(y) \neq 0$, a.e. In fact, if there does not exist an interval (d_1, d_2) such that $g(y)$ is constant on (d_1, d_2) , it can be proved that (28) implies that $h(y) - g(y) \neq 0$, a.e.

Example 5. Let $a < b$ be two real numbers, it can be shown that X has a uniform distribution on (a, b) , if and only if

$$(30) \quad E(X^2|X \geq y) = \begin{cases} \frac{1}{3}(a^2 + ab + b^2), & \forall y \leq a, \\ \frac{1}{3}(y^2 + by + b^2), & \forall a < y < b. \end{cases}$$

Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the order statistics from a random sample having the common distribution function F . Theorem 3 of Wu and Ouyang (1996) characterized the absolutely continuous distribution

function F by

$$(31) \quad E(g(X_{1:n})|X_{1:n} > t) = g(t) + \frac{c}{n}, \quad \forall a < y < b,$$

where c is a constant and g is a nonconstant differentiable function. In fact, from Theorem 3, we know that (31) characterizes the distribution of $X_{1:n}$, and hence the distribution function F can also be uniquely determined. Moreover, by Theorem 3, more general results than Wu and Ouyang (1996) can also be obtained.

Again, corresponding to Theorem 3, we have the following theorem for discrete case.

Theorem 4. Let $m < n$, where m and n are extended integers, and $\{a_i, m \leq i \leq n\}$ be a sequence of numbers with $a_i < a_{i+1}$, $\forall m \leq i < i+1 \leq n$. Also let g and h be two given functions defined on $\{a_i, m \leq i \leq n\}$ such that $h(a_i) \neq h(a_{i+1})$, $\forall m \leq i < i+1 \leq n$. Then there exists a random variable X with $C_X = \{a_i, m \leq i \leq n\}$, such that $E(g(X)|X \geq y)$ is finite, $\forall y \in C_X$, and

$$(32) \quad h(y) = E(g(X)|X \geq y), \quad \forall y \in C_X,$$

if and only if the following conditions hold.

(i) $0 < \frac{h(a_i) - g(a_i)}{h(a_{i+1}) - g(a_i)} < 1$, $\forall m \leq i \leq n-1$, and $\prod_{m \leq i \leq k-1} \frac{h(a_i) - g(a_i)}{h(a_{i+1}) - g(a_i)} > 0$, $\forall m+1 \leq k \leq n$.

(ii) If $n < \infty$, then $h(a_n) = g(a_n)$ and if $n = \infty$, then $\lim_{k \rightarrow \infty} h(a_{k+1}) \cdot \prod_{m \leq i \leq k} \frac{h(a_i) - g(a_i)}{h(a_{i+1}) - g(a_i)} = 0$.

Moreover, the *p.d.f.* of X satisfies (32) with $C_X = \{a_i, m \leq i \leq n\}$ is

$$(33) \quad f(a_k) = \frac{h(a_{k+1}) - h(a_k)}{h(a_{k+1}) - g(a_k)} \prod_{m \leq i \leq k-1} \frac{h(a_i) - g(a_i)}{h(a_{i+1}) - g(a_i)}, \quad m \leq k \leq n,$$

where $\prod_{m \leq i \leq m-1} \frac{h(a_i) - g(a_i)}{h(a_{i+1}) - g(a_i)} \equiv 1$ and $\frac{h(a_{n+1}) - h(a_n)}{h(a_{n+1}) - g(a_n)} \equiv 1$, if $m, n < \infty$.

Proof. Following Theorem 3.1 of Marin et al. (1996), the necessity can be proved easily, hence the proof of this part is omitted. We now prove the sufficiency. Assume conditions (i) and (ii) hold. Define the function f as in (33). Condition (i) implies $0 < f(a_k) < 1$, $\forall m \leq k \leq n$. Also define the function S as $S(k) = \prod_{m \leq i \leq k-1} \frac{h(a_i) - g(a_i)}{h(a_{i+1}) - g(a_i)}$, $\forall m \leq k \leq n$, where $S(m) \equiv 1$, if $m < \infty$. It can be seen that $f(a_k) = S(k) - S(k+1)$, $\forall m \leq k \leq n-1$, and $f(a_n) = S(n)$ if $n < \infty$. Condition (ii) implies that

$\sum_{k=m}^n f(a_k) = 1$ and $S(k) = \sum_{i=k}^n f(a_i), \forall m \leq k \leq n$. Hence f is a *p.d.f.* of some random variable X with $C_X = \{a_i, m \leq i \leq n\}$ and

$$(34) \quad \frac{\sum_{i=k+1}^n f(a_i)}{\sum_{i=k}^n f(a_i)} = \frac{S(k+1)}{S(k)} = \frac{h(a_k) - g(a_k)}{h(a_{k+1}) - g(a_k)}, \quad \forall m \leq k \leq n - 1.$$

This implies

$$(35) \quad (h(a_k) - g(a_k)) \sum_{i=k}^n f(a_i) = (h(a_{k+1}) - g(a_k)) \sum_{i=k+1}^n f(a_i),$$

$$\forall m \leq k \leq n - 1.$$

Subtracting $(h(a_k) - g(a_k)) \sum_{i=k+1}^n f(a_i)$ from both sides of (35), after some manipulations, we obtain

$$(36) \quad h(a_k) \sum_{i=k}^n f(a_i) = g(a_k)f(a_k) + h(a_{k+1}) \sum_{i=k+1}^n f(a_i),$$

$$\forall m \leq k \leq n - 1.$$

From condition (ii), (36) implies that $h(a_k) \sum_{i=k}^n f(a_i) = \sum_{i=k}^n g(a_i)f(a_i), \forall m \leq k \leq n$. and this implies that $E(g(X)|X \geq y)$ is finite, $\forall y \in C_X$, and $h(y) = E(g(X)|X \geq y), \forall y \in C_X$.

Ruiz and Navarro (1995), (1996) gave the characterizations for both cases of discrete and continuous distributions using the doubly truncated mean function $h(x, y) = E(g(X)|x \leq X \leq y)$, where g is a monotone function. They also investigated the necessary and sufficient conditions as in Theorems 3 and 4. If the function g is not restricted to be monotone, for example only assumed to be continuous, so far we could not give the explicit form of necessary and sufficient conditions.

In the rest of this paper, we give some related results on point processes. Assume $N \equiv \{N(t), t \geq 0\}$ is a renewal process with $N(0) = 0$. Huang et al. (1993) proved that if $E(S_n|N(t) = n) = nt/(n + 1)$, where S_n is the n th arrival time of N , then N is a Poisson process. For non-homogeneous Poisson processes defined on $(-\infty, \infty)$, we have some more interesting results. In the following, let g be a real continuous function. For a positive integer n , denote \mathcal{F}_n as the set of nonhomogeneous Poisson processes, where it is assumed that for each $M \in \mathcal{F}_n$, the mean function $m(t) \equiv E(M((-\infty, t])) < \infty, \forall t \in R$, m is differentiable, and $\int_{-\infty}^t g(u)dm^n(u)$ is finite, $\forall t \in R$. We state the following theorem without proof.

Theorem 5. Let n be a fixed positive integer, $a \in (-\infty, \infty) \cup \{-\infty\}$ and h be any real function with $g(x) - h(x) \neq 0$, for almost all $x > a$. Then there exists a nonhomogeneous Poisson process $M \in \mathcal{F}_n$, such that $a = \inf\{t | E(M((-\infty, t])) \neq 0\}$ and

$$(37) \quad h(t) = E(g(S_n) | M(t) = n), \quad \forall t > a,$$

where S_n is the n th arrival time of the process M , if and only if the following conditions hold.

(i) h is differentiable on (a, ∞) .

(ii) $\int_x^y \frac{dh(u)}{g(u)-h(u)}$ is finite, $\forall x, y \in (a, \infty)$.

(iii) $\int_x^y \frac{dh(u)}{g(u)-h(u)}$ is increasing in y , $\forall x, y \in (a, \infty)$, and $\lim_{x \rightarrow a} \int_x^y \frac{dh(u)}{g(u)-h(u)} = \infty$.

(iv) $\lim_{x \rightarrow a} h(t) \exp\{-\int_x^y \frac{dh(u)}{g(u)-h(u)}\} = 0$, $\forall y \in (a, \infty)$.

Moreover, the mean function m of the nonhomogeneous Poisson process M satisfies (37) with $a = \inf\{t | E(M((-\infty, t])) \neq 0\}$ is $m(y) = m(x) \cdot \exp\{\int_x^y \frac{dh(u)}{g(u)-h(u)} / n\}$, $\forall x, y > a$.

Again it is known that the sequence of upper record values from a population with continuous distribution F forms the sequence of arrival times of a nonhomogeneous Poisson process, say $\{B(t), t \geq 0\}$, with $m(t) = E(B(t)) = -\ln(1 - F(t))$, where $\lim_{t \rightarrow \infty} m(t) = \infty$. Now denote \mathcal{F}'_n as a subset of \mathcal{F}_n , with $\lim_{t \rightarrow \infty} m(t) = \infty$, where $m(t) = E(M((-\infty, t]))$, $\forall M \in \mathcal{F}_n$. We have the following theorem. Once again the proof is standard hence is omitted.

Theorem 6. Let $a < b$, be two extended real numbers, and h be any real function with $g(x) - h(x) \neq 0$, for almost all $x \in (a, b)$. Then there exists a nonhomogeneous Poisson process $M \in \mathcal{F}'_n$, such that $a = \inf\{t | E(M((-\infty, t])) \neq 0\}$, $b = \inf\{t | E(M((-\infty, t])) = \infty\}$ and $h(t) = E(g(S_n) | M(t) = n)$, $\forall a < t < b$, where as usual S_n denote the n th arrival time of the process M , if and only if the conditions (i)-(iv) in Theorem 5 with (a, ∞) being replaced by (a, b) , and the following condition (v) are satisfied.

(v) $\lim_{y \rightarrow b} \int_x^y \frac{dh(u)}{g(u)-h(u)} = \infty$, $\forall x \in (a, b)$.

In Theorems 5 and 6, if g is strictly increasing, then the result still hold without the assumption that m is differentiable. In this case, Theorem 6 is equivalent to Theorem 3.2 of Franco and Ruiz (1996).

Acknowledgements

The authors would like to thank the referees for many helpful comments which greatly improved the presentation of the paper.

References

1. Ahmed, A. N. (1991). Characterization of Beta, Binomial, and Poisson distributions. *IEEE Trans. Reliability* **40**, 290-295.
2. Cox, D. R. (1962). *Renewal Theory*. Methuen, London.
3. Cundy, H. (1966). Birds and atoms. *Math. Gazette*. **50**, 294-295.
4. Dallas, A. C. (1976). Characterizing the Pareto and power distribution. *Ann. Inst. Statist. Math.* **28A**, 491-497.
5. Franco, M. and Ruiz, J. M. (1995). On characterization of continuous distributions with adjacent order statistics. *Statistics* **26**, 375-385.
6. Franco, M. and Ruiz, J. M. (1996). On characterization of continuous distributions by conditional expectation of record values. *Sankhyā A* **58**, 135-141.
7. Galambos, J. and Hagwood, C. (1992). The characterization of a distribution function by the second moment of the residual life. *Commun. Statist. Theory Meth.* **21**(5), 1463-1468.
8. Ghitany, M. E., El-Saidi, M. A. and Khalil, Z. (1995). Characterization of a general class of life-testing models. *J. Appl. Prob.* **32**, 548-553.
9. Huang, W. J., Li, S. H. and Su, J. C. (1993). Some characterizations of the Poisson process and geometric renewal process. *J. Appl. Prob.* **30**, 121-130.
10. Laurent, A. G. (1974). On characterization of some distributions by truncation properties. *J. Amer. Statist. Assoc.* **69**, 823-827.
11. Marín, J. M., Ruiz, J. M. and Zoroa, P. (1996). Characterization of discrete random vectors by conditional expectations. *J. Multivariate Anal.* **58**, 82-95.
12. Nair, N. U. and Sankaran, P. G. (1991). Characterization of the Pearson family of distributions. *IEEE Trans. Reliability* **40**, 75-77.

13. Osaki, S. and Li, X. (1988). Characterizations of gamma and negative binomial distributions. *IEEE Trans. Reliability* **37**, 379-382.
14. Rao, C. R. and Shanbhag, D. N. (1994). Choquet-Deny Type Functional Equations with Application to Stochastic Models. John Wiley & Sons, New York.
15. Reinhardt, H. E. (1968). Characterizing the exponential distribution. *Biometrics* **24**, 437-438.
16. Ruiz, J. M., Marín, J. and Zoroa, P. (1993). A characterization of continuous multivariate distribution by conditional expectations. *J. Statist. Plann. Inference* **37**, 13-21.
17. Ruiz, J. M. and Navarro, J. (1994). Characterization of distributions by relationships between failure rate and mean residual life. *IEEE Trans. Reliability*. **34**, 640-644.
18. Ruiz, J. M. and Navarro, J. (1995). Characterization of discrete distributions using expected values. *Statist. Papers* **36**, 237-252.
19. Ruiz, J. M. and Navarro, J. (1996). Characterizations based on conditional expectations of the doubled truncated distribution. *Ann. Inst. Statist. Math.* **48**, 563-572.
20. Shanbhag, D. N. (1970). Characterizations for exponential and geometric distributions. *J. Amer. Statist. Assoc.* **65**, 1256-1259.
21. Wu, J. W. and Ouyang, L. Y. (1996). On characterizing distributions by conditional expectations of functions of order statistics. *Metrika* **43**, 135-147.

Jyh-Cherng Su
Department of Mathematics
Chinese Military Academy
FengShan, Kaohsiung
Taiwan, 830, R.O.C.

Wen-Jang Huang
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan, 804, R.O.C.

