國立高雄大學統計學研究所

碩士論文

Asymptotic Distribution of the EPMS Estimator for Option Pricing

EPMS 方法對選擇權價格估計之漸近分佈

研究生：涂雅婷 撰

指導教授：黃士峰

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涂雅婷 撰
中華民國一百零一年七月

Asymptotic Distribution of the EPMS Estimator for Option Pricing

by

Ya-Ting Tu

Advisor

Shih-Feng Huang

Institute of Statistics

National University of Kaohsiung

Kaohsiung, Taiwan 811, R.O.C.

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指導教授：黃士峰 博士
國立高雄大學應用數學系

學生：涂雅婷
國立高雄大學統計學研究所

摘要

本文推導出 Empirical *P*-martingale Simulation (EPMS) 方法對金融衍生性產品價格估計量的漸近常態分佈。當風險中立測度模型不容易得到時， EPMS 是一個容易執行且有效率的方法。文中考慮在 Black-Scholes 和 GARCH 模型假設下，蒙地卡羅法， Empirical Martingale Simulation (EMS) 以及 EPMS 在計算歐式買權的有效性。模擬結果顯示本文所推導之漸近分布在樣本路徑達到500時，即可給出令人滿意的逼近。

關鍵字：*P* 測度下的平賭過程配適模擬法；蒙地卡羅模擬法；選擇權定價。

Asymptotic Distribution of the EPMS Estimator for Option Pricing

Advisor: Shih-Feng Huang
Department of Applied Mathematics
National University of Kaohsiung

Student: Ya-Ting Tu
Institute of Statistics
National University of Kaohsiung

ABSTRACT

The asymptotic normality of the empirical P-martingale simulation (EPMS) estimator for financial derivative pricing is established in this study. The EPMS is an easily implemented and efficient method to compute derivative prices if a risk-neutral model is not convenient to be obtained. The efficiency of the Monte Carlo, empirical martingale simulation (EMS) and EPMS estimators for European call option pricing are compared under the Black-Scholes and GARCH models. Simulation results indicate that the asymptotic distribution serves as a persuasive approximation for samples consisting of as few as 500 simulation paths.

Keywords and phrases: empirical P-martingale simulation; Monte Carlo simulation; option pricing.

1 Introduction

For financial derivative pricing in complex models such as GARCH and stochastic volatility models, the Monte Carlo simulation (MCS) has been commonly used by practitioners due to the absence of closed-form solution (Boyle, P., 1977; Boyle, P., Broadie, M., and Glasserman, P, 1997; Duan, J. C., 1995). However, when a higher degree of pricing accuracy in option pricing is required, MCS usually encounters heavy computational burden. To handle this problem, many numerical and simulation algorithms are proposed to approximate the derivative price (Duan and Simonato, 1998; Glasserman, 2004; Huang and Guo, 2009; Huang, 2012). Duan and Simonato (1998) proposed an empirical martingale simulation (EMS) to improve the efficiency by generating random paths of the underlying assets from the risk-neutral model. They proved that the EMS price estimator is consistent. In addition, the asymptotic distribution of the EMS price estimator is developed by Duan and Simonato (2001).

Note that the EMS can only proceed when a risk-neutral model can be expressed explicitly. In practice, to obtain a risk-neutral model explicitly is not a trivial task when dealing with a complex model. Huang (2012), therefore, proposed an empirical P-martingale simulation (EPMS) to compute the derivative prices under the physical (or dynamic) measure. The strong consistency of the EPMS is established in the same paper. Simulation results indicate that the EPMS is comparable to the EMS if the risk-neutral model can be expressed explicitly, and is more efficient than the MCS whether the risk-neutral model can be obtained.

However, it is time-consuming to obtain the standard deviation for investigating the efficiency of the EPMS by repeatedly generating derivative prices with independent random copies. One statistical mechanism to saving the computational burden is to use the asymptotic distribution of the price estimator to approximate its finite sample standard deviation. However, like many variance reduction methods, the EPMS adjustment creates dependency among sample paths. Due to the lack of independence, the central limit theorem can not be applied directly. This study overcomes this difficulty and establishes the asymptotic normality of the EPMS price estimator if contingent payoffs are piecewise linear continuous functions. In addition, the asymptotic distribution of the EMS derived by Duan and Simonato (2001) is shown to be a special case of our result if the physical and risk-neutral models coincide. By using the asymptotic distribution, the confidence intervals for the EPMS price estimator are obtained. Numerical results show that the confidence intervals derived by the asymptotic distribution match the nominal coverage probabilities even for samples consisting of as few as 500 simulation paths. The EPMS is shown to be more asymptotically efficient than the MCS in our scenarios.

The rest of this study is organized as follows. In Section 2, the EMS and EPMS are briefly introduced. The asymptotic normality of the EPMS price estimator are derived in Section 3. In Section 4, the numerical results for European call options under the Black-Scholes and GARCH models are presented. Conclusions are in Section 5. All the theoretical proofs are given in the Appendix.

2 Literature Review

2.1 The empirical martingale simulation (EMS)

In this section, we present a simple modification to the standard Monte Carlo simulation (MCS) procedure which is proposed by Duan and Simonato (1998). This modification for yielding a substantial reduction in MC errors is called the empirical martingale simulation (EMS) which prescribe the martingale property on the collection of the simulated sample paths. To ensure the martingale property, The basic idea and the procedure of this simulation method are introduced in the following :

1. Let $t_0 = 0$ as the current time, and $t_1, t_2, ..., t_m$ are a sequence of future time points. Generate the paths of stock prices $\hat{S}_{i,t}$, for $i = 1, ..., n$ and $t = 1, ..., T$, by the MCS.

2. Let $S^*_{i,0} = \hat{S}_{i,0} = S_0$, for all $i = 1, ..., n$, and define the empirical martingale

stock prices $S^*_{i,t}$ for $i = 1, ..., n$ and $t \geqslant 1$, iteratively by the following system :

$$S^*_{i,t} = S_0 \frac{Z_i(t,n)}{Z_0(t,n)},$$

where $Z_i(t,n) = S^*_{i,t-1} \frac{\hat{S}_{i,t}}{\hat{S}_{i,t-1}}$ and $Z_0(t,n) = \frac{1}{n} e^{-rt} \sum_{i=1}^{n} Z_i(t,n)$. After the EMS correction, the simulation moves on to the next time point, and repeats the whole process again.

After the adjustment, the independence of paths is destroyed. Using simulations, Duan and Simonato (1998) have shown the EMS price estimator has a smaller variance than the MCS price estimator. They have also proved the EMS for derivatives pricing also preserves consistency under fairly general conditions. Besides, they have offered a characterization of the asymptotic distribution of the EMS price estimator in 2001. The EMS price estimator for the contingent claim with the payoff function $f(S_T)$ is defined by

$$C^{(n)}_{EMS} = e^{-rT} \frac{1}{n} \sum_{i=1}^{n} f(S^*_{i,T}),$$

where $S^*_{i,T}$ is generated from the EMS. They proved under some assumptions,

$$\sqrt{n}(C^{(n)}_{EMS} - C) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V), \quad \text{as } n \to \infty,$$

where C is the true derivative price, $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution and

$$V = e^{-2rT} \Big(\mathrm{Var}[f(S_T)] + \mathrm{Var}[S_T]\Phi^2 - 2\Phi \mathrm{Cov}[f(S_T), S_T] \Big),$$

where $\Phi = \mathrm{E}[\varphi'(S_T) S_T / (S_0 e^{rT})]$ and $\varphi'(\cdot)$ are explained in Remark 3.1 below.

2.2 The empirical P-martingale simulation (EPMS)

Although the EMS can obtain consistent estimator of financial derivative prices in a more efficient method than the MCS, the EMS method can only proceed under the risk-neutral model, which may not be expressed in an explicit form under the complex models. Theoretically, it is not a necessary step in option pricing to obtain the explicit expression of the risk-neutral model. If the change of measure process is given, compute the option price under the dynamic P measure is an alternative way. Therefore, Huang (2012) proposed an empirical P-martingale simulation (EPMS) method for option pricing by generating empirical martingale underlying asset prices under the dynamic P measure. The basic idea and the procedure of this simulation method are introduced in the following :

Let Y_T denote a Radon-Nikodým derivative of a risk-neutral measure Q with respect to the dynamic P measure and $\mathrm{E}(Y_T) = 1$. Q is transformed from the dynamic measure P by a change of measure process $Y_T = dQ/dP$. Define $Y_t = \mathrm{E_t}(Y_T) = \mathrm{E}(Y_T|\mathcal{F}_t)$, $0 \leqslant t < T$, where $\mathcal{F}_t$ denotes the information set up to time t. Then Y_t is a change of measure process and is a martingale process under the P measure (abbreviated as P-martingale). The main idea of the proposed EPMS method is to guarantee both the simulated processes of the discounted underlying asset prices and the change of measure values are empirical P-martingale. This new simulation procedure which generates the EPMS asset prices is illustrated in the following :

1. Let $t_0 = 0$ as the current time, and $t_1, t_2, ..., t_m$ are a sequence of future time points. Generate the paths of stock prices $\hat{S}_{i,t}$, for $i = 1, ..., n$ and $t = 1, ..., T$, by the MCS.

2. Let $Y^*_{i,0} = \hat{Y}_{i,0} = Y_0 = 1$, for all $i = 1, ..., n$, and define the empirical martingale change of measure process $Y^*_{i,t}$, for $i = 1, ..., n$ and $t \geqslant 1$, recursively by the following system :

$$Y^*_{i,t} = \frac{W_i(t,n)}{W_0(t,n)},$$

where $W_i(t,n) = Y^*_{i,t-1}\frac{\hat{Y}_{i,t}}{\hat{Y}_{i,t-1}}$, $W_0(t,n) = \frac{1}{n}\sum_{i=1}^{n} W_i(t,n)$ and $\hat{Y}_{i,t} = Y_t(\hat{S}_{i,u},\, 0 \leqslant u \leqslant t)$.

3. Let $S^*_{i,0} = \hat{S}_{i,0} = S_0$, for all $i = 1, ..., n$, and define the empirical martingale stock prices $S^*_{i,t}$ for $i = 1, ..., n$ and $t \geqslant 1$, recursively by the following system :

$$S^*_{i,t} = S_0\frac{Z_i(t,n)}{Z_0(t,n)},$$

where $Z_i(t,n) = S^*_{i,t-1}\frac{\hat{S}_{i,t}}{\hat{S}_{i,t-1}}$ and $Z_0(t,n) = \frac{1}{n}e^{-rt}\sum_{i=1}^n Z_i(t,n)Y^*_{i,t}$. After the EPMS correction, the simulation moves on to the next time point, and repeats the whole process again.

Note that the two processes $Y^*_{i,t}$ and $S^*_{i,t}$ are functions of the sample size n which satisfy the "empirical P-martingale properties",that is,

$$\frac{1}{n}\sum_{i=1}^{n} Y^*_{i,t} = Y_0 = 1 \text{ and } \frac{1}{n}e^{-rt}\sum_{i=1}^{n} S^*_{i,t}Y^*_{i,t} = S_0,$$

almost surely, for any integer n and $t = 1, ..., T$. Using simulations, Huang (2012) has shown the attractive property of the EPMS estimator is its ability to reduce the MCS error, and he also proved the strong consistency of the EPMS estimator when the payoff function has some good property.

3 Main Results

Let $f : \Re \to \Re$ be a piecewise linear continuous function defined by

$$f(x) = \sum_{j=1}^{m}(a_j x + b_j)\delta_{A_j}(x) \tag{1}$$

where A_j's form a partition of $\Re$, that is, $A_1 = (-\infty, k_1)$, $A_j = [k_{j-1}, k_j)$, for $j = 2, 3, \ldots, m$ and $k_m = \infty$. $\delta_A(\cdot)$ is an indicator function which defined by $\delta_A(x) = 1$, if $x \in A$ and $\delta_A(x) = 0$, if $x \notin A$, and a_j, $b_j \in \Re$ satisfy

$$a_j k_j + b_j = a_{j+1}k_j + b_{j+1}, \quad j = 1, \ldots, m-1, \tag{2}$$

to ensure the continuity of f.

Let S_t denote the price of an underlying asset at time t and $f(S_T)$ be the payoff function of a contingent claim that matures at time T, where $f(\cdot)$ satisfies (1) and (2). To simplify the illustration, assume the riskless interest rate r is constant. Then, the noarbitrage price of the contingent claim with payoff $f(S_T)$ is defined by $e^{-rT}\mathrm{E}[f(S_T)Y_T]$, where Y_T is a Radon-Nikodým derivative of a risk-neutral probability measure Q with respect to the physical (or dynamic) measure P. In incomplete market model such as GARCH models or stochastic volatility models, $\mathrm{E}[f(S_T)Y_T]$ might not have closed-form representation. Many numerical and simulation algorithms are proposed to approximate the derivative price (Duan and Simonato, 1998; Glasserman, 2004; Huang and Guo, 2009; Huang, 2012).

In this study, we focus on the EPMS price estimator proposed by Huang (2012). The EPMS price estimator for the contingent claim with the payoff function $f(S_T)$ is defined by

$$C^{(n)}_{EPMS} = e^{-rT}\frac{1}{n}\sum_{i=1}^{n} f(S^*_{i,T})Y^*_{i,T},$$

where $S^*_{i,T}$ and $Y^*_{i,T}$ are generated from the EPMS. In the following, we derive the asymptotic distribution of $C^{(n)}_{EPMS}$. First, we make some assumptions :

(A1) $\mathrm{E}(S_T^2 Y_T) = \mathrm{E}^Q(S_T^2) < \infty$, where $\mathrm{E}^Q(\cdot)$ denotes the expectation under a risk-neutral measure Q.

(A2) $\mathrm{E}(Y_T^{1+\delta}) < \infty$, for some $\delta > 0$.

Theorem 3.1. *Let the asset price S_T be a positive random variable with a continuous distribution, Y_T be a Radon-Nikodým derivative, and the payoff function $f(S_T)$ be piecewise linear continuous as defined in (1) and (2).*

1. *If* (A1) *and* (A2) *hold, then*

$$C^{(n)}_{MCS} - C^{(n)}_{EPMS} = e^{-rT}\Big[(\overline{S}_{n,T} - S_0 e^{rT})\Phi + (\overline{Y}_{n,T} - 1)\Psi\Big] + o_p(n^{-\frac{1}{2}}) \qquad (3)$$

where $C^{(n)}_{MCS}$ is the derivative value computed by standard Monte Carlo method, $\overline{S}_{n,T} = n^{-1}\sum_{i=1}^{n} S_{i,T}Y_{i,T}$, $\overline{Y}_{n,T} = n^{-1}\sum_{i=1}^{n} Y_{i,T}$, $\Phi = e^{-rT}\mathrm{E}[\varphi'(S_T)S_T Y_T]/S_0$ with $\varphi'(x) = \sum_{j=1}^{m} a_j \delta_{A_j}(x)$, $\Psi = \sum_{j=1}^{m} b_j \mathrm{E}[\delta_{A_j}(S_T)Y_T]$, and $o_p(n^{-k})$ denotes some random variable H_n that satisfies $\lim_{n\to\infty} n^k H_n = 0$ in probability.

2. *Moreover, if* (A1) *and* (A2) *are strengthened to $\mathrm{E}(S_T^2 Y_T^2) < \infty$ and $\mathrm{E}(Y_T^2) < \infty$, then*

$$\sqrt{n}(C^{(n)}_{EPMS} - C) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V), \quad \text{as } n \to \infty, \qquad (4)$$

where C is the true derivative price, $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution and

$$\begin{aligned} V &= e^{-2rT}\Big(\mathrm{Var}[f(S_T)Y_T] + \mathrm{Var}[S_TY_T]\Phi^2 + \mathrm{Var}[Y_T]\Psi^2 \\ &\quad -2\{\Phi\mathrm{Cov}[f(S_T)Y_T, S_TY_T] + \Psi\mathrm{Cov}[f(S_T)Y_T, Y_T] \\ &\quad -\Phi\Psi\mathrm{Cov}[S_TY_T, Y_T]\}\Big). \end{aligned} \tag{5}$$

Remark 3.1.

(i) *$f(x)$ is nondifferentiable only at $\{k_j : j = 1, 2, ..., m-1\}$. In general, $\varphi'(x) = f'(x)$ for $x \notin \{k_j : j = 1, 2, ..., m-1\}$. At the nondifferentiable points, $\varphi'(x)$ is taken as the right-hand derivative of $f(x)$, i.e., $\varphi'(k_j) = f'(k_j^+)$ for $j = 1, 2, ..., m-1$.*

(ii) *The asymptotic distribution of the EMS derived by Duan and Simonato (2001) is a special case of (4) if $Y_T = 1$.*

For further examining (3), let

$$D(n) := \sqrt{n}\Big\{C^{(n)}_{MCS} - C^{(n)}_{EPMS} - e^{-rT}\Big[(\overline{S}_{n,T} - S_0e^{rT})\Phi + (\overline{Y}_{n,T} - 1)\Psi\Big]\Big\}, \tag{6}$$

Figure 1 plots $|D(n)|$ with various expiration date T and S_0/K, where $T = 30$ days in (a) and (b), $T = 90$ days in (c) and (d), $T = 270$ days in (e) and (f), $S_0/K = 1$ in (a),(c), and (e), and $S_0/K = 0.9$ in (b),(d), and (f). We can see that $|D(n)|$ decreases when the sample size n increases.

4 Numerical Study

To investigate the performance of the asymptotic result derived in Theorem 3.1, we conduct several simulation scenarios in this section. Section 4.1 presents the application of the EPMS in pricing European call options under the Black and Scholes (BS) model while Section 4.2 presents the results of GARCH models. In Section 4.3, a comparison study between EPMS and EMS is performed. The superiority of the EPMS estimator over the MCS estimator is shown by comparing their pricing variances in Section 4.4. Furthermore, Section 4.5 presents the coverage probabilities obtained from the asymptotic distribution of the EPMS estimator in pricing digital options to investigate the necessary of the continuity assumption of f in (2).

A European call option with maturity T and strike price K has the payoff function $\max(S_T - K, 0)$. The EPMS price estimate at time 0 using a sample size n is

$$C^{(n)}_{EPMS} = e^{-rT}\frac{1}{n}\sum_{i=1}^{n}(S^*_{i,T} - K)\delta_{[K,\infty)}(S^*_{i,T})Y^*_{i,T},$$

where $S^*_{i,T}$ and $Y^*_{i,T}$ are generated from the EPMS. By Theorem 3.1, we have the following confidence interval of confidence level $1-\alpha$ for the EPMS price estimate,

$$\left[C^{(n)}_{EPMS} - z_{\alpha/2}\sqrt{\mathrm{Var}}(C^{(n)}_{EPMS}),\ C^{(n)}_{EPMS} + z_{\alpha/2}\sqrt{\mathrm{Var}}(C^{(n))}_{EPMS})\right], \tag{7}$$

where $z_{\alpha/2}$ is the $1-\alpha/2$ quantile of a standard normal distribution.

For each model, we consider three maturities 30, 90, and 270 days and three asset-to-strike price ratios 1.1, 1, and 0.9. In our numerical analyses, one year is assumed to have 365 days for the annualizing purpose. We compare the pricing efficiency of the EPMS, MCS and EMS estimators by computing their standard errors of option pricing with sample sizes 500 and 10,000, and 1,000 replications. We also compare the pricing efficiency of the EPMS estimator with other variance reduction scheme such as control-variate simulation.

4.1 The Black and Scholes option pricing framework

In the Black and Scholes (1973), the underlying asset processes S_t is assumed to satisfy

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where μ is the expected return of the underlying asset, σ is the volatility and W_t is a Brownian motion. We set $S_0 = 100$, the riskless interest rate $r = 0.1$ (annualized), $\mu = 0.15$ (annualized) and $\sigma = 0.2$ (annualized). Table 1 reports the standard deviations of the MCS, EMS and EPMS estimators and two other estimators, MCS_CV1 and MCS_CV2, with sample size $n = 500,\ 10,000$ and 1,000 replications, where MCS_CV1 and MCS_CV2 are defined by

$$\text{MCS\_CV1} = C_{MCS}^{(n)} - \Big(\frac{1}{n}e^{-rT}\sum_{i=1}^{n} S_{i,T} - S_0\Big) \text{ and}$$
$$\text{MCS\_CV2} = C_{MCS}^{(n)} - \beta_n\Big(\frac{1}{n}e^{-rT}\sum_{i=1}^{n} S_{i,T} - S_0\Big),$$

in which β_n is the sample regression coefficient from regressing $f(S_{i,T})$ on $(e^{-rT}S_{i,T} - S_0)$. Table 1 also presents various coverage rates obtained from the asymptotic distribution of the EPMS estimator. Numerical results indicate that $\mathrm{C}_{\mathrm{CV2}}^{(\mathrm{n})}$, EMS and EPMS are comparable in most cases, and the simulation error reduction delivered by EPMS is particularly large for in-the-money options. In addition, the four coverage rates are very close to their nominal values.

4.2 The GARCH option pricing framework

Similar to Section 4.1, we examine the efficiency of MCS, EMS, EPMS, MCS_CV1 and MCS_CV2 in pricing European call options under the GARCH (Bollerslev, 1986) framework. In this section, we consider the following GARCH(1,1) model,

$$\begin{cases} \ln\dfrac{S_{t+1}}{S_t} = r + \lambda\sigma_{t+1} - 0.5\sigma_{t+1}^2 + \sigma_{t+1}\varepsilon_{t+1}, \\ \sigma_{t+1}^2 = \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2\varepsilon_t^2, \\ \varepsilon_{t+1} \sim N(0,1), \end{cases}$$

where $\beta_0,\ \beta_1,\ \beta_2$ are nonnegative, $\beta_1 + \beta_2 < 1$, r is the (continuously compounded) one-period risk-free interest rate, and λ denotes the unit risk premium. In the following numerical study, we set the initial conditional variance $\sigma_1^2 = \beta_0(1 - \beta_1 - \beta_2)^{-1}$. We use the same settings of parameters as in Duan and Simonato (2001), that is, $S_0 = 100$, $r = 0.1$ (annual), $\beta_0 = 0.00001$, $\beta_1 = 0.7$, $\beta_2 = 0.2$, and $\lambda = 0.01$. The calculation is repeated 1,000 times to each of two sample sizes 500 and 10,000. Table 2 reports the simulation results. In addition, two more control-variate estimators based on MCS and EPMS price estimators are considered and defined as follows:

$$\text{MCS\_CV3} = C^{(n)}_{MCS} - \Big(\frac{1}{n}e^{-rT}\sum_{i=1}^{n}\max(\hat{S}_{i,T}-K;0) - C_{BS}\Big) \text{ and}$$
$$\text{EPMS\_CV3} = C^{(n)}_{EPMS} - \Big(\frac{1}{n}e^{-rT}\sum_{i=1}^{n}\max(\hat{S}_{i,T}-K;0) - C_{BS}\Big),$$

where C_{BS} is the Black-Scholes formula price and $\hat{S}_{i,T}$ is simulated from Black-Scholes model. In Table 2, EPMS_CV3 performs better than others and similar to Table 1, the four coverage rates presented in Table 2 are very close to their nominal values.

4.3 EPMS vs. EMS

Since the asymptotic variances of the EMS and EPMS estimators are derived by Duan and Simonato (2001) and Theorem 3.1, respectively, we conduct some simulation scenarios to compare the efficiency of these two estimators by computing the values of their theoretical formulae. The parameters are set the same as in Duan and Simonato (2001). Table 3 presents the 2.5% and 97.5% empirical quantiles, denoted by $q_{0.025}$ and $q_{0.975}$, respectively, of the ratios of the variance of the EPMS with respect to the variance of the EMS with 500 and 10,000 sample paths and 1,000 replications under the Black-Scholes and GARCH model. For out-of-the-money options, the ratio shows that the variance of the EPMS is smaller than the EMS while the EMS performs better than the EPMS for at-the-money and in-the-money options.

4.4 Comparison study of the asymptotic variances of the EPMS and MCS estimators

Tables 1 and 2 demonstrate the superiority of the EPMS estimator over the MCS estimator in finite samples. In this section, we further investigate the difference between their asymptotic variances. Let V_{MCS} denote the asymptotic variance of the MCS estimator. By Central Limit Theorem, $V_{\text{MCS}} = e^{-2rT}\text{Var}[f(S_T)Y_T]$. On the other hand, Theorem 3.1 derives an analytic form of the variance of the EPMS estimator, denoted by V in (5), which is expected to be smaller than V_{MCS}. For example, the payoff of a European call option, $(S_T - K)^+$, can be represented as the piecewise linear continuous form in (1), where $m = 2$, $a_1 = 0$, $b_1 = 0$, $A_1 = [0, K)$, $a_2 = 1$, $b_2 = -K$, and $A_2 = [K, \infty)$. Hence,

$$\begin{aligned} V - V_{\mathrm{MCS}} &= e^{-2rT}\Big(\mathrm{Var}[S_T Y_T]\Phi^2 + \mathrm{Var}[Y_T]\Psi^2 \\ &\quad -2\{\Phi\mathrm{Cov}[f(S_T)Y_T, S_T Y_T] + \Psi\mathrm{Cov}[f(S_T)Y_T, Y_T] \\ &\quad -\Phi\Psi\mathrm{Cov}[S_T Y_T, Y_T]\}\Big), \end{aligned} \tag{8}$$

where $\Phi = e^{-rT}\mathrm{E}[\varphi'(S_T)S_T Y_T]/S_0 = e^{-rT}\mathrm{E}[\delta_{[k,\infty)}(S_T)S_T Y_T]/S_0 > 0$ and $\Psi = \sum_{j=1}^{m} b_j \mathrm{E}[\delta_{A_j}(S_T)Y_T] = -K\mathrm{E}[\delta_{[k,\infty)}(S_T)Y_T] < 0$. However, it is not trivial to prove the conjecture of that (8) is negative theoretically even for the standard European options. Thus, we conduct some simulation scenarios to check the conjecture. In Table 4, numerical results indicate that the conjecture seems to be true in both Black-scholes and GARCH models in pricing European call and put options.

4.5 The exotic option pricing framework

In this section, we conduct a simulation study to investigate the necessary of the continuity assumption of f in (2). We compute the digital option prices by the EPMS method under the GARCH model. The payoff of a digital option is defined by

$$V_T = \delta_{\{S_T \le K\}},$$

where δ_A is an indicator function. The simulation results are given in Table 5 under 10,000 sample paths and 1,000 replications. From examining the coverage probabilities, the asymptotic normality of the EPMS derived in Theorem 3.1 can not be applied directly to the digital options. This is caused by the discontinuity of the payoff of a digital option.

5 Conclusion

This study establishes the asymptotic distribution of the EPMS price estimator to save the computational burden by estimating the simulation error using just one realization. In particular, the asymptotic distribution of the EMS is shown to be a special case of the EPMS if the physical and risk-neutral models are the same. Simulation results show that the asymptotic normal distribution serves a persuasive approximation for samples consisting of as few as 500 simulation paths. The results in Section 4.4 indicate that the EPMS is more asymptotically efficient than the MCS. Interestingly, Section 4.5 reveals that Theorem 3.1 can not be applied directly to a financial derivative with discontinuous payoff. Further investigation is needed to extend our results to handle the discontinuous case. In addition, the extension of the high-dimensional domain of the payoff function f is also an interesting topic. We refer these extensions to our future study.

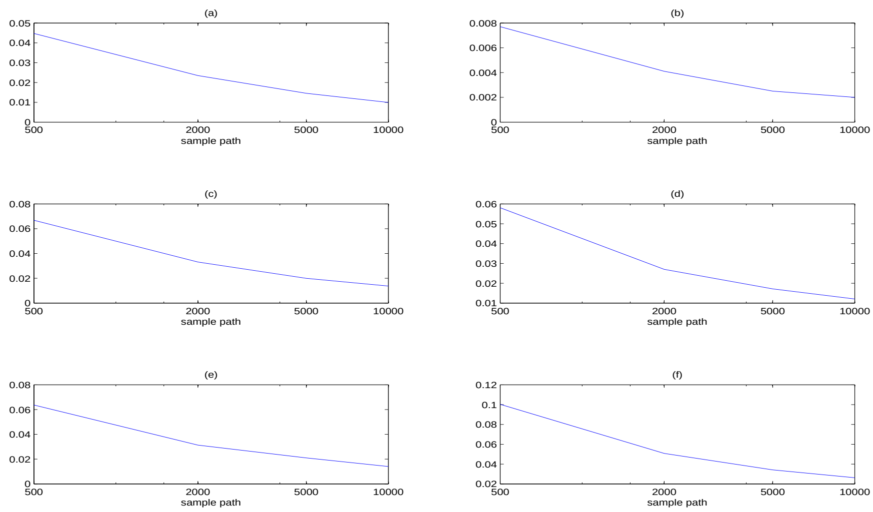

Figure 1: The plot of $|D(n)|$ defined in (6) with various expiration date T and S_0/K, where $T = 30$ days in (a) and (b), $T = 90$ days in (c) and (d), $T = 270$ days in (e) and (f), $S_0/K = 1$ in (a),(c), and (e), and $S_0/K = 0.9$ in (b),(d), and (f).

Appendix

PROOF OF THEOREM 3.1. The proof of Theorem is based on a sequence of smooth approximations to the piecewise linear continuous function. Recall the piecewise linear continuous function is

$$f(x) \quad = \sum_{j=1}^{m} (a_j x + b_j)\delta_{A_j}(x),$$

where $a_j k_j + b_j = a_{j+1} k_j + b_{j+1}$, for any $j \in 1, 2, ..., m-1$. Consider a sequence $\{\varphi_n : n \in N\}$ of differentiable functions:

$$\varphi_n(x) \equiv \begin{cases} a_1 x + b_1 & \text{if } x < k_1 - \varepsilon_n, \\ a_j x + b_j & \text{if } k_{j-1} + \varepsilon_n < x < k_j - \varepsilon_n, \quad j \in 2, 3, ..., m-1, \\ a_m x + b_m & \text{if } x > k_{m-1} + \varepsilon_n, \\ p_n^{(j)}(x) & \text{if } k_j - \varepsilon_n \leq x \leq k_j + \varepsilon_n, \qquad j \in 1, 2, ..., m-1, \end{cases}$$

where $\varepsilon_n = o(n^{-\frac{1}{2}})$ is a positive decreasing function, and

$$p_n^{(j)}(x) \equiv b_j + \Big(\frac{a_{j+1} - a_j}{4\varepsilon_n}\Big)(k_j - \varepsilon_n)^2 + \Big(a_j - \frac{a_{j+1} - a_j}{2\varepsilon_n}(k_j - \varepsilon_n)\Big)x + \frac{a_{j+1} - a_j}{4\varepsilon_n} x^2,$$

for any $j \in 1, 2, ..., m-1$,

Let $P_n = \alpha X^2 + \beta X + \gamma$, and $P_n^{'} = 2\alpha X + \beta$. We obtain the solution by the following four equations :

$$\begin{aligned} P_n(k_j - \varepsilon_n) &= f(k_j - \varepsilon_n) &&= a_j(k_j - \varepsilon_n) + b_j, \\ P_n(k_j + \varepsilon_n) &= f(k_j + \varepsilon_n) &&= a_{j+1}(k_j + \varepsilon_n) + b_{j+1}, \\ P_n^{'}(k_j - \varepsilon_n) &= f^{'}(k_j - \varepsilon_n) &&= a_j, \\ P_n^{'}(k_j + \varepsilon_n) &= f^{'}(k_j + \varepsilon_n) &&= a_{j+1}. \end{aligned}$$

We also have

$$\|f - \varphi_n\|_\infty \equiv \sup_{x \in R} |f(x) - \varphi_n(x)| = o(n^{-\frac{1}{2}}). \tag{9}$$

Recall that $\varphi^{'}(x) \equiv \sum_{j=1}^{m} a_j \delta_{A_j}(x)$. Since

$$
\varphi_n^{'}(x) \equiv \begin{cases} a_1 & \text{if } x < k_1 - \varepsilon_n, \\ a_j & \text{if } k_{j-1} + \varepsilon_n < x < k_j - \varepsilon_n, \\ & \quad j = 2, 3, ..., m-1, \\ a_j + \dfrac{a_{j+1} - a_j}{2\varepsilon_n}(x - k_j + \varepsilon_n) & \text{if } k_j - \varepsilon_n \le x \le k_j + \varepsilon_n, \\ & \quad j = 1, 2, ..., m-1, \\ a_m & \text{if } x > k_{m-1} + \varepsilon_n, \end{cases}
$$

thus φ_n is continuous and has a continuous first derivative $\varphi_n^{'}$. Moreover, we have the following upper bound of the difference between ϕ_n' and ϕ',

$$
\| \varphi_n^{'} - \varphi^{'} \|_\infty \equiv \sup_{x \in R} |\varphi_n^{'}(x) - \varphi^{'}(x)| \le \kappa,
$$

where $\kappa = \max_{j=1,...,m-1} |a_j|$ is independent of n.

In addition, by the definition of $C_{EPMS}^{(n)}$, (9) and the fact of

$$
\frac{1}{n}\sum_{i=1}^{n} Y_{i,t}^* = Y_0 = 1,
$$

we have

$$
\begin{aligned}
e^{rT}\sqrt{n}\, C_{EPMS}^{(n)} \quad &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varphi_n(S_{i,T}^*) Y_{i,T}^* + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_{i,T}^*\, o(n^{-\frac{1}{2}}) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varphi_n(S_{i,T}^*) Y_{i,T}^* + o(1). \qquad (10)
\end{aligned}
$$

Hence, by (10) and the definition of $C_{MCS}^{(n)}$, we have

$$
\begin{aligned}
& e^{rT}\sqrt{n}\,(C_{EPMS}^{(n)} - C_{MCS}^{(n)}) \\
& = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\varphi_n(S_{i,T}^*) Y_{i,T}^* - \varphi_n(S_{i,T}) Y_{i,T} \Big) \\
& + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\varphi_n(S_{i,T}) - f(S_{i,T}) \Big) Y_{i,T} + o(1) \\
& = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\varphi_n(S_{i,T}^*) Y_{i,T}^* - \varphi_n(S_{i,T}) Y_{i,T} \Big) + o_p(1), \qquad (11)
\end{aligned}
$$

where the second equality holds by (9) and the fact of

$$\frac{1}{n}\sum_{i=1}^{n} Y_{i,t} = 1, \text{a.s.}$$

Furthermore, the rhs of (11) can be decomposed as

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\varphi_n(S^*_{i,T}) - \varphi_n(S_{i,T})\Big)Y_{i,T} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(S_{i,T})(Y^*_{i,T} - Y_{i,T})$$
$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\varphi_n(S^*_{i,T}) - \varphi_n(S_{i,T})\Big)(Y^*_{i,T} - Y_{i,T}) + o_p(1). \qquad (12)$$

In the following, we deal with the first three terms in (12) separately.

Note that for any fixed $s > 0$, the function $v \to \varphi_n(s/v)$ is continuous and differentiable on the interval $(0, \infty)$. Since the asset price is a positive random variable, we can use the mean value theorem to expand $\varphi_n(s/v)$. That is,

$$\begin{aligned}\varphi_n(S^*_{i,T}) - \varphi_n(S_{i,T}) &= \varphi_n\Big(S_{i,T}\,\frac{S_0e^{rT}}{\overline{S}^*_{n,T}}\Big) - \varphi_n\Big(S_{i,T}\,\frac{S_0e^{rT}}{S_0e^{rT}}\Big) \\ &= -\varphi_n^{'}\Big(S_{i,T}\,\frac{S_0e^{rT}}{V^*_{i,n}}\Big)S_{i,T}\,\frac{S_0e^{rT}}{V^{*2}_{i,n}}\Big(\overline{S}^*_{n,T} - S_0e^{rT}\Big), \qquad (13)\end{aligned}$$

where $\overline{S}^*_{n,T} = n^{-1}\sum_{i=1}^{n} S_{i,T}Y^*_{i,T}$, $V^*_{i,n} \in [\min(S_0e^{rT}, \overline{S}^*_{n,T}), \max(S_0e^{rT}, \overline{S}^*_{n,T})]$, the 1st equality is due to $S^*_{i,T} = S_{i,T}\; S_0e^{rT}/\,\overline{S}^*_{n,T}$, and the 2nd equality follows by the mean value theorem. By similar arguments used in (13) and recall that $Y^*_{i,T} = Y_{i,T}/\,\overline{Y}_{n,T}$, we also have

$$Y^*_{i,T} - Y_{i,T} = -\frac{Y_{i,T}}{U^{*2}_{i,n}}(\,\overline{Y}_{n,T} - 1), \qquad (14)$$

where $U^*_{i,n} \in [\min(\,\overline{Y}_{n,T}, 1), \max(\,\overline{Y}_{n,T}, 1)]$.

By substituting (13) and (14) to (12), we have

$$-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\varphi_n^{'}(S_{i,T}\,\frac{S_0e^{rT}}{V^*_{i,n}})S_{i,T}\,\frac{S_0e^{rT}}{V^{*2}_{i,n}}(\overline{S}^*_{n,T} - S_0e^{rT})\Big)Y_{i,T}$$
$$-\frac{1}{\sqrt{n}}\sum_{i=1}^{n} f(S_{i,T})\Big(\frac{Y_{i,T}}{U^{*2}_{i,n}}(\,\overline{Y}_{n,T} - 1)\Big)$$
$$+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\varphi_n(S^*_{i,T}) - \varphi_n(S_{i,T})\Big)(Y^*_{i,T} - Y_{i,T}) + o_p(1).$$
$$:= \text{(I)+(II)+(III)} + o_p(1). \qquad (15)$$

Let $\Phi = \mathrm{E}[\varphi'(S_T)\frac{S_T}{S_0 e^{rT}} Y_T]$. Lemmas A.1, A.2 and A.3 below, respectively, yield

$$
\begin{aligned}
\text{(I)} &= -\sqrt{n}(\overline{S}_{n,T} - S_0 e^{rT}\overline{Y}_{n,T})\Phi + o_p(1). && (16)\\
\text{(II)} &= -\sqrt{n}(\overline{Y}_{n,T} - 1)\mathrm{E}[f(S_T)Y_T] + o_p(1). && (17)\\
\text{(III)} &= o_p(1). && (18)
\end{aligned}
$$

Finally, by (1), (11), (12), (15)-(18), the definition of $\varphi'(x)$ and let $\Psi = \mathrm{E}[f(S_T)Y_T] - S_0 e^{rT}\Phi = \sum_{j=1}^{m} b_j \mathrm{E}(Y_T \delta_{A_j})$, we have

$$
C_{MCS}^{(n)} - C_{EPMS}^{(n)} = e^{-rT}[(\overline{S}_{n,T} - S_0 e^{rT})\Phi + (\overline{Y}_{n,T} - 1)\Psi] + o_p(n^{-\frac{1}{2}}).
$$

Hence, Theorem 3.1 (i) holds.

Next, we derive the result of Theorem 3.1 (ii). By Theorem 3.1 (i), we have

$$
\begin{aligned}
& \sqrt{n}(C_{EPMS}^{(n)} - C)\\
= {} & \sqrt{n}(C_{MCS}^{(n)} - C) - \sqrt{n}e^{-rT}\Big[(\overline{S}_{n,T} - S_0 e^{rT})\Phi + (\overline{Y}_{n,T} - 1)\Psi\Big] + o_p(1)\\
= {} & \frac{e^{-rT}}{\sqrt{n}} \sum_{i=1}^{n} \Big(\{f(S_{i,T})Y_{i,T} - \mathrm{E}[f(S_T)Y_T]\} - [S_{i,T}Y_{i,T} - \mathrm{E}(S_T Y_T)]\Phi\\
& \qquad - [Y_{i,T} - \mathrm{E}(Y_T)]\Psi \Big) + o_p(1).
\end{aligned}
$$

By Central Limit Theorem, we have

$$
\sqrt{n}(C_{EPMS}^{(n)} - C) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V), \quad \text{as } n \to \infty,
$$

where

$$
\begin{aligned}
V = {} & e^{-2rT}\Big(\mathrm{Var}[f(S_T)Y_T] + \mathrm{Var}(S_T Y_T)\Phi^2 + \mathrm{Var}(Y_T)\Psi^2\\
& -2\{\Phi \mathrm{Cov}[f(S_T)Y_T, S_T Y_T] + \Psi \mathrm{Cov}[f(S_T)Y_T, Y_T] - \Phi\Psi \mathrm{Cov}[S_T Y_T, Y_T]\}\Big).
\end{aligned}
$$

□

Lemma A.1. *By using the same assumptions as in Theorem 3.1 (i), (16) holds.*

Proof :

$$
\begin{aligned}
\text{(I)} &:= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(-\varphi_n'(S_{i,T}\frac{S_0 e^{rT}}{V_{i,n}^*}) S_{i,T} \frac{S_0 e^{rT}}{V_{i,n}^{*2}} (\overline{S}_{n,T}^* - S_0 e^{rT}) \Big) Y_{i,T}\\
&= -\sqrt{n}(\overline{S}_{n,T}^* - S_0 e^{rT}) \Big(\mathrm{E}\Big[\varphi_n'(S_T)\frac{S_T Y_T}{S_0 e^{rT}}\Big] - \Lambda_{n,T} \Big), \qquad (19)
\end{aligned}
$$

where $\Lambda_{n,T} = \mathrm{E}[\varphi_n^{'}(S_T)\frac{S_T Y_T}{S_0 e^{rT}}] - \frac{1}{n}\sum_{i=1}^n \varphi_n^{'}(S_{i,T}\ \frac{S_0 e^{rT}}{V_{i,n}^*})S_{i,T}\ \frac{S_0 e^{rT} Y_{i,T}}{V_{i,n}^{*2}}$.

Since $\sup_n \sup_s |\varphi_n^{'}(s)| \le w$ for some finite w and $\varphi_n^{'}(S_T)$ converges almost surely (a.s.) to $\varphi^{'}(S_T)$ (because $\Pr[S_T \in \{k_1, ..., k_{m-1}\}] = 0$), the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty} \mathrm{E}\Big[\varphi_n^{'}(S_T)\frac{S_T}{S_0 e^{rT}} Y_T\Big] = \Phi, \tag{20}$$

where $\Phi = \mathrm{E}[\varphi^{'}(S_T)\frac{S_T Y_T}{S_0 e^{rT}}]$ is defined the same as in Theorem 3.1 (i). Recall that $\overline{S}_{n,T} = \frac{1}{n}\sum_{i=1}^n S_{i,T} Y_{i,T}$ and $\overline{S}^*_{n,T} = \frac{1}{n}\sum_{i=1}^n S_{i,T} Y^*_{i,T}$ in Theorem 3.1 (i). We then have

$$\begin{aligned}
\sqrt{n}(\overline{S}^*_{n,T} - S_0 e^{rT}) &= \overline{Y}_{n,T}^{\,-1}[\sqrt{n}(\overline{S}_{n,T} - S_0 e^{rT}) - \sqrt{n} S_0 e^{rT}(\overline{Y}_{n,T} - 1)] \\
&= [\sqrt{n}(\overline{S}_{n,T} - S_0 e^{rT}) - \sqrt{n} S_0 e^{rT}(\overline{Y}_{n,T} - 1)] \\
&\quad + (\overline{Y}_{n,T}^{\,-1} - 1)[\sqrt{n}(\overline{S}_{n,T} - S_0 e^{rT}) - \sqrt{n} S_0 e^{rT}(\overline{Y}_{n,T} - 1)].
\end{aligned} \tag{21}$$

By Central Limit Theorem, $\sqrt{n}(\overline{S}_{n,T} - S_0 e^{rT})$ and $\sqrt{n} S_0 e^{rT}(\ \overline{Y}_{n,T} - 1)$ converge weakly to proper normal random variables. In addition, since $\overline{Y}_{n,T} \to 1$ a.s., thus (21) can be rewritten as

$$\sqrt{n}(\overline{S}^*_{n,T} - S_0 e^{rT}) = \sqrt{n}(\overline{S}_{n,T} - S_0 e^{rT}\overline{Y}_{n,T}) + o_p(1). \tag{22}$$

By (19), (20) and (22), we have

$$\text{(I)} \quad = \quad -\sqrt{n}(\overline{S}_{n,T} - S_0 e^{rT}\overline{Y}_{n,T})(\Phi - \Lambda_{n,T}) + o_p(1).$$

To obtain (16), it remains to show that

$$\Lambda_{n,T} = o_p(1). \tag{23}$$

Note that

$$\Lambda_{n,T} = \Lambda^{(1)}_{n,T} + \Lambda^{(2)}_{n,T}, \tag{24}$$

where

$$\Lambda^{(1)}_{n,T} = \mathrm{E}\Big[\varphi_n^{'}(S_T)\frac{S_T}{S_0 e^{rT}} Y_T\Big] - \frac{1}{n}\sum_{i=1}^n \varphi_n^{'}(S_{i,T})\ \frac{S_{i,T}}{S_0 e^{rT}} Y_{i,T} = o_p(1) \tag{25}$$

by the law of large numbers, and

$$\Lambda^{(2)}_{n,T} = \frac{1}{n}\sum_{i=1}^n \Big[\varphi_n'(S_{i,T})\ \frac{S_{i,T}}{S_0 e^{rT}} - \varphi_n'(S_{i,T}\ \frac{S_0 e^{rT}}{V^*_{i,n}})S_{i,T}\ \frac{S_0 e^{rT}}{V^{*2}_{i,n}}\Big] Y_{i,T}. \tag{26}$$

In addition,

$$\Lambda_{n,T}^{(2)} \quad \leq \quad \Lambda_{n,T}^{(2,1)} + \Lambda_{n,T}^{(2,2)}, \tag{27}$$

where

$$\Lambda_{n,T}^{(2,1)} = \frac{1}{n}\sum_{i=1}^{n}\left|\varphi_n^{'}(S_{i,T}) - \varphi_n^{'}(\frac{S_{i,T}S_0e^{rT}}{V_{i,n}^{*}})\right| \times \frac{S_{i,T}Y_{i,T}}{S_0e^{rT}}$$

and

$$\Lambda_{n,T}^{(2,2)} = \parallel \varphi_n^{'} \parallel_\infty \frac{1}{n}\sum_{i=1}^{n}\left|\frac{S_{i,T}}{S_0e^{rT}} - \frac{S_{i,T}S_0e^{rT}}{V_{i,n}^{*2}}\right| Y_{i,T}.$$

By the Hölder's inequality, for $\delta > 0$

$$\begin{aligned}
&\mathrm{E}(\Lambda_{n,T}^{(2,1)}) \\
\leq \quad & \max_{i=1,\dots n}\left\{\mathrm{E}\left[\left|\varphi_n^{'}(S_{i,T}) - \varphi_n^{'}\Big(\frac{S_{i,T}S_0e^{rT}}{V_{i,n}^{*}}\Big)\right|^{\frac{2+2\delta}{\delta}}\right]\right\}^{\frac{\delta}{2+2\delta}}\left\{\mathrm{E}\left[\Big(\frac{S_TY_T}{S_0e^{rT}}\Big)^{\frac{2+2\delta}{2+\delta}}\right]\right\}^{\frac{2+\delta}{2+2\delta}}.
\end{aligned}$$

Since $\sup_{n,s}|\varphi_n^{'}(s)| \leq w$ for some finite w and $V_{i,n}^{*}$ converges almost surely to S_0e^{rT} as $n \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\max_{i=1,\dots n}\mathrm{E}\left[\left|\varphi_n^{'}(S_{i,T}) - \varphi_n^{'}\Big(\frac{S_{i,T}S_0e^{rT}}{V_{i,n}^{*}}\Big)\right|^{\frac{2+2\delta}{\delta}}\right] = 0.$$

This together with $\mathrm{E}[(S_T^2Y_T^2)^{\frac{1+\delta}{2+\delta}}] \leq [\mathrm{E}(S_T^2Y_T)]^{\frac{1+\delta}{2+\delta}}[\mathrm{E}(Y_T^{1+\delta})]^{\frac{1}{2+\delta}} = O(1)$, provided by (A1), (A2) and the Hölder's inequality, we thus have

$$\mathrm{E}(\Lambda_{n,T}^{(2,1)}) = o(1). \tag{28}$$

Furthermore, define

$$q_{n,T} \equiv \sup_{x\in[\min(S_0e^{rT},\overline{S}_{n,T}^{*}),\max(S_0e^{rT},\overline{S}_{n,T}^{*})]}\left|\frac{1}{(S_0e^{rT})^2} - \frac{1}{x^2}\right|.$$

Note that $q_{n,T}$ converges to zero a.s. as n goes to infinity. Thus,

$$\begin{aligned}
\Lambda_{n,T}^{(2,2)} \quad &= \quad \|\varphi_n^{'}\|_\infty \frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{(S_0e^{rT})^2} - \frac{1}{(V_{i,n}^{*})^2}\right| S_{i,T}S_0e^{rT}Y_{i,T} \\
&\leq \quad \|\varphi_n^{'}\|_\infty\, q_{n,T}\, S_0e^{rT}\,\frac{1}{n}\sum_{i=1}^{n}S_{i,T}Y_{i,T} = o_p(1). \qquad (29)
\end{aligned}$$

Therefore, by (24)-(29), (23) holds. $\square$

Lemma A.2. *By using the same assumptions as in Theorem 3.1 (i), (17) holds.*

Proof : By (15),

$$
\begin{aligned}
\text{(II)} &= -\sqrt{n}(\overline{Y}_{n,T}-1)\frac{1}{n}\sum_{i=1}^{n} f(S_{i,T})\frac{Y_{i,T}}{U_{i,n}^{*2}} \\
&= -\sqrt{n}(\overline{Y}_{n,T}-1)\mathrm{E}[f(S_T)Y_T] \\
&\quad +\sqrt{n}(\overline{Y}_{n,T}-1)\Big(\mathrm{E}[f(S_T)Y_T]-\frac{1}{n}\sum_{i=1}^{n} f(S_{i,T})\frac{Y_{i,T}}{U_{i,n}^{*2}}\Big). \qquad (30)
\end{aligned}
$$

By Central Limit Theorem, $\sqrt{n}(\overline{Y}_{n,T}-1)$ converges weakly to a normal random variable. By (30) and using similar arguments in (24)-(29),

$$
\Lambda_{n,T} := \mathrm{E}[f(S_T)Y_T]-\frac{1}{n}\sum_{i=1}^{n} f(S_{i,T})\frac{Y_{i,T}}{U_{i,n}^{*2}} = o_p(1),
$$

thereby (17) holds. □

Lemma A.3. *By using the same assumptions as in Theorem 3.1 (i), (18) holds.*

Proof : Note that

$$
\begin{aligned}
\text{(III)} &:= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\varphi_n(S_{i,T}^{*})-\varphi_n(S_{i,T})\Big)(Y_{i,T}^{*}-Y_{i,T}) \\
&\le \max_{i=1,\dots,n}\Big|\frac{Y_{i,T}^{*}}{Y_{i,T}}-1\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\varphi_n(S_{i,T}^{*})-\varphi_n(S_{i,T})\Big)Y_{i,T}, \qquad (31)
\end{aligned}
$$

using (12), (13), (15) and (16),

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\varphi_n(S_{i,T}^{*})-\varphi_n(S_{i,T})\Big)Y_{i,T} = -\sqrt{n}(\overline{S}_{n,T}-S_0e^{rT}\overline{Y}_{n,T})\Phi + o_p(1). \qquad (32)
$$

By Central Limit Theorem, $\sqrt{n}(\overline{S}_{n,T}-S_0e^{rT}\overline{Y}_{n,T})$ converges weakly to proper normal random variables. In addition, by Huang (2012, eqn(12)),

$$
\frac{Y_{i,T}^{*}}{Y_{i,T}} \stackrel{n\to\infty}{\longrightarrow} 1 \ \text{ a.s., for } \ i=1.\dots,n. \qquad (33)
$$

Hence, by (31) - (33), (18) holds. □

Table 1: The standard deviations of computing European calls by various methods and the coverage rates of the EPMS obtained by (7) in the Black-Scholes model.

	$n = 500$ sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1,000)									
MCS	0.2403	0.1622	0.0278	0.3813	0.2851	0.1351	0.6666	0.5736	0.4264
MCS_CV1	0.0209	0.1259	0.2365	0.0730	0.1937	0.3429	0.1569	0.2803	0.4455
MCS_CV2	0.0352	0.0750	0.0256	0.0723	0.1203	0.1043	0.1490	0.2013	0.2310
EMS	0.0197	0.0743	0.0256	0.0623	0.1176	0.1034	0.1299	0.1942	0.2290
EPMS	0.0219	0.0750	0.0233	0.0705	0.1212	0.0953	0.1547	0.2116	0.2249
Cov.rates									
25% cov.rate epms	0.2230	0.2660	0.2390	0.2610	0.2610	0.2580	0.2090	0.2740	0.2580
50% cov.rate epms	0.4490	0.5240	0.4780	0.4950	0.4910	0.5070	0.4990	0.5080	0.5190
75% cov.rate epms	0.7260	0.7750	0.7360	0.7570	0.7640	0.7400	0.7360	0.7460	0.7680
95% cov.rate epms	0.9480	0.9610	0.9440	0.9650	0.9440	0.9400	0.9510	0.9550	0.9500

	$n = 10,000$ sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1,000)									
MCS	0.0525	0.0347	0.0062	0.0881	0.0681	0.0325	0.1475	0.1274	0.0953
MCS_CV1	0.0043	0.0282	0.0518	0.0168	0.0428	0.0756	0.0356	0.0629	0.0979
MCS_CV2	0.0043	0.0163	0.0057	0.0149	0.0276	0.0230	0.0318	0.0447	0.0505
EMS	0.0041	0.0162	0.0057	0.0147	0.0278	0.0232	0.0291	0.0436	0.0504
EPMS	0.0045	0.0163	0.0052	0.0167	0.0290	0.0215	0.0342	0.0473	0.0499
Cov.rates									
25% cov.rate epms	0.2450	0.2410	0.2690	0.2540	0.2440	0.2350	0.2670	0.2660	0.2600
50% cov.rate epms	0.5010	0.4800	0.5090	0.4740	0.4920	0.4820	0.5230	0.5230	0.5220
75% cov.rate epms	0.7390	0.7380	0.7560	0.7350	0.7440	0.7390	0.7580	0.7360	0.7660
95% cov.rate epms	0.9530	0.9540	0.9490	0.9510	0.9450	0.9360	0.9490	0.9560	0.9560

Table 2: The standard deviations of computing European calls by various methods and the coverage rates of the EPMS obtained by (7) in the GARCH model.

	$n = 500$ sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1,000)									
MCS	0.2321	0.1550	0.0423	0.3882	0.2954	0.1509	0.6444	0.5549	0.4124
MCS_CV1	0.0305	0.1322	0.2305	0.0823	0.1988	0.3428	0.1560	0.2679	0.4300
MCS_CV2	0.0411	0.0768	0.0384	0.0808	0.1263	0.1136	0.1479	0.1962	0.2292
MCS_CV3	0.0605	0.0483	0.0270	0.1090	0.0948	0.0698	0.1954	0.1827	0.1585
EMS	0.0284	0.0762	0.0394	0.0710	0.1242	0.1138	0.1335	0.1903	0.2260
EPMS	0.0304	0.0769	0.0366	0.0774	0.1265	0.1068	0.1541	0.2032	0.2202
EPMS_CV3	0.0219	0.0325	0.0253	0.0447	0.0614	0.0567	0.0802	0.1016	0.1114
Cov.rates									
25% cov.rate epms	0.2370	0.2640	0.2520	0.2680	0.2530	0.2560	0.2380	0.2660	0.2760
50% cov.rate epms	0.4850	0.5050	0.5250	0.5320	0.5230	0.4970	0.4730	0.4860	0.5190
75% cov.rate epms	0.7520	0.7700	0.7460	0.7730	0.7630	0.7450	0.7300	0.7420	0.7510
95% cov.rate epms	0.9470	0.9540	0.9620	0.9490	0.9430	0.9470	0.9450	0.9550	0.9530

	$n = 10,000$ sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
STD(1,000)									
MCS	0.0526	0.0355	0.0092	0.0877	0.0667	0.0336	0.1474	0.1289	0.0973
MCS_CV1	0.0069	0.0283	0.0512	0.0186	0.0444	0.0768	0.0359	0.0623	0.0968
MCS_CV2	0.0068	0.0167	0.0085	0.0163	0.0275	0.0247	0.0323	0.0460	0.0525
MCS_CV3	0.0133	0.0107	0.0062	0.0246	0.0210	0.0151	0.0438	0.0399	0.0345
EMS	0.0066	0.0167	0.0086	0.0161	0.0275	0.0249	0.0304	0.0455	0.0528
EPMS	0.0071	0.0168	0.0080	0.0176	0.0281	0.0235	0.0344	0.0479	0.0520
EPMS_CV3	0.0048	0.0073	0.0054	0.0098	0.0130	0.0123	0.0182	0.0237	0.0253
Cov.rates									
25% cov.rate epms	0.2350	0.2470	0.2590	0.2310	0.2340	0.2360	0.2690	0.2580	0.2520
50% cov.rate epms	0.4870	0.4830	0.5050	0.4910	0.5150	0.5070	0.5130	0.5230	0.5090
75% cov.rate epms	0.7290	0.7330	0.7420	0.7490	0.7490	0.7650	0.7720	0.7590	0.7500
95% cov.rate epms	0.9520	0.9410	0.9480	0.9530	0.9560	0.9570	0.9630	0.9530	0.9460

Table 3: The 2.5% and 97.5% empirical quantiles, denoted by $q_{0.025}$ and $q_{0.975}$, respectively, of the ratios of the variance of the EPMS with respect to the variance of the EMS.

	$n = 10,000$ sample paths (repeat = 1000)								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
Model	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
BS($q_{0.025}$)	1.1560	1.0083	0.7957	1.2205	1.0388	0.8301	1.3439	1.1414	0.9284
BS($q_{0.975}$)	1.2810	1.0324	0.8680	1.3392	1.0850	0.8828	1.5395	1.2486	0.9996
GARCH($q_{0.025}$)	1.0975	1.0036	0.8338	1.1533	1.0239	0.8449	1.2247	1.0867	0.9232
GARCH($q_{0.975}$)	1.2090	1.0296	0.9147	1.2479	1.0630	0.9017	1.3448	1.1548	0.9770

Table 4: Difference of the variances between the EPMS and the MCS.

	European call option $n = 10,000$ sample paths (repeat = 1000)								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
Model	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
BS	-20.122	-7.540	-0.025	-46.838	-22.958	-2.252	-98.640	-67.574	-26.341
GARCH	-20.543	-7.736	-0.058	-51.189	-25.395	-2.875	-123.089	-83.987	-34.801

	European put option $n = 10,000$ sample paths (repeat = 1000)								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
Model	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
BS	-0.015	-5.952	-35.122	-0.863	-16.277	-76.417	-7.209	-37.661	-130.887
GARCH	-0.033	-5.464	-30.680	-0.874	-14.090	-66.410	-5.709	-29.924	-106.453

Table 5: The coverage rates obtained by (7) for the digital options in the GARCH model.

	$n = 10,000$ sample paths (repeat = 1000)								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
Cov.rates	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
25% cov.rate epms	0.2720	0.3510	0.2520	0.2880	0.3720	0.3320	0.3540	0.3970	0.3970
50% cov.rate epms	0.5530	0.6970	0.5170	0.5840	0.7040	0.6460	0.6400	0.7070	0.7230
75% cov.rate epms	0.8000	0.9330	0.7890	0.8320	0.9250	0.8770	0.8850	0.9260	0.9260
95% cov.rate epms	0.9680	0.9990	0.9620	0.9820	0.9970	0.9960	0.9850	0.9970	0.9990

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