國立高雄大學統計學研究所

碩士論文

Pricing Catastrophe Options in GARCH-Jump with Stochastic Intensity Models

考慮有隨機跳躍頻率強度的 GARCH 模型之巨災選擇權定價

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Pricing Catastrophe Options in GARCH-Jump with Stochastic Intensity Models

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巨災選擇權定價

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摘要

本文利用 Esscher 轉換法推導出在有隨機跳躍頻率強度的 GARCH 的風險中立測度模型，並且用來定價巨災權益賣權(CatEPut)。CatEPut 會處理巨災所造成的損失進而保護公司的股東權益。模擬研究指出當巨災發生頻率，跳躍項的分佈其平均數、變異數和到期日增加時巨災權益賣權的定價上升。

關鍵字：巨災選擇權; Esscher 轉換法; 跳躍 GARCH 模型

Pricing Catastrophe Options in GARCH-Jump with Stochastic Intensity Models

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ABSTRACT

A risk-neutral GARCH-Jump model with stochastic intensity is established by the Esscher transform and is applied to evaluate the price of a catastrophe equity put (CatEPut) option. The CatEPut contract provides protection for the shareholders of the underlying company to handle catastrophic losses. Numerical results indicate that the CatEPut prices increase as the intensity rate, the expected value of the jump size, the variance of the jump size or the time to maturity increasing.

Keywords and phrases: Catastrophe options; Esscher transform; GARCH-Jump model.

1 Introduction

A catastrophe equity put (CatEPut) option was first issued by the RLI corporation in 1996 to cover the potential losses caused by catastrophe such as typhoons, hurricanes, storms, floods, waves and earthquakes. Recently, the frequency of the occurrence of catastrophe tends to be increasing and leads to enormous financial losses. For example, the Sumatra-Andaman earthquake (2004) in Indonesia, the Hurricane Katrina (2005) in the Gulf of Mexico, the Wenchuan earthquake (2008) in Sicuan, the Haiti earthquake (2010) in Haiti, the New Zealand earthquake (2011) in Christchurch and the Great East Japan earthquake (2011). The total amount of loss caused by these catastrophes is over 457 billion U.S. dollars (around 3.03% of U.S. GDP in 2011). In Taiwan, recent catastrophes like the 921 earthquake (1999) in Nantou, the Jiaxian earthquake (2010) in Kaohsiung, and several typhoons such as the Mindulle (2004), Morakot (2009), Fanapi (2010) and Nanmado (2011) also lead to more than 450 billion loss in N.T. dollars (around 32.74% of Taiwan's GDP in 2011). To hedge these enormous catastrophic losses, CatEPut options provide protection for the corporation and shareholders. Consequently, the pricing scheme of CatEPut options attracts more attention from practitioners and researchers (Cox, et al., 2004; Jaimungal and Wang, 2006; Lin, et al., 2009).

In financial derivative pricing, the GARCH models and jump-diffusion models are commonly used to describe the dynamics of the underlying processes (Merton, 1976; Engle, 1982; Bollerslev, 1986; Duan, 1995; Kou, 2002; Lin, et al., 2009; Huang and Guo, 2009; Huang, 2012). For CatEPut option pricing, Cox, et al. (2004) derived a closed-form solution under a jump-diffusion model with constant jump size and constant intensity rate. Jaimungal and Wang (2006) extended Cox, et al.'s result to the case of random riskless interest and random jump size. Lin, et al. (2009) further considered the valuation of CatEPut options under stochastic intensity rate of the occurrence of catastrophe. While considerable attention has been given in the literature to the valuation of CatEPut options in jump-diffusion model, less attention has been devoted to GARCH model, which is capable of successfully depicting heteroskedasticity of financial data. Therefore, this study combines the ideas of the GARCH and jump-diffusion model, denoted by GARCH-Jump, to depict the underlying dynamics. In particular, the intensity rate is assumed to follow a geometric Brownian motion and the jump size is constant, gamma or normally distributed.

Since there is usually no closed-form solution in GARCH option pricing, practitioners can only compute the derivative prices by simulation or numerical methods (Duan, 1995; Duan and Simonato, 1998; Huang and Guo, 2009) based on a risk-neutral model. To obtain a risk-neutral GARCH-Jump model, we use the Esscher transform to derive a change of measure process (Gerber and Shiu, 1994; Siu, et al., 2004; Huang, et al., 2012). In the literature, two popular economic considerations on the jump part are proposed and yield different risk-neutral measures. Merton, (1976), Cox, et al. (2004) and Jaimungal and Wang (2006) assumed that the risk caused by the jump is non-diversifiable while Amin (1993) took the opposite point of view that the jump risk is diversifiable. In stead of jumping into the debate of which measure is appropriate, this study proposes a general approach of deriving risk-neutral GARCH-Jump models to accommodate both economic assumptions. Moreover, to investigate the influence of the different risk-neutral measures, the intensity rate, the mean and variance of jump size and time to maturity on the CatEPut prices, we conduct several simulation scenarios. Numerical results indicate that the CatEPut prices increase as the intensity rate, the expected value of the jump size, the variance of the jump size or the time to maturity increasing.

The remainder of the paper is organized as follows. Section 2 briefly introduces the Esscher transform and several methods of pricing CatEput options in jump-diffusion models. Section 3 establishes the corresponding risk-neutral GARCH-Jump model by the Esscher transform. Sensitivity analyses are given in Section 4. Conclusions are in Section 5. All the tables, figures and theoretical proofs are in the Appendix.

2 Literature Review

2.1 Esscher transform (Gerber and Shiu, 1994; Siu, et al., 2004; Huang, et al., 2012)

Let $R_t = \log S_t/S_{t-1}$ denote the log return process of a stock price process S_t at time t, for $t = 1, 2, \ldots, T$. A papular approach to derive an equivalent probability measure of R_t for financial derivative pricing is the Esscher transform (Gerber and Shiu, 1994). We sketch the scheme of it in the following : Let $f_t(\cdot)$ denote the conditional probability density function (pdf) of $R_t|\mathcal{F}_{t-1}$ and $\mathrm{M}_{R_t|\mathcal{F}_{t-1}}(z) = \mathrm{E}_{t-1}(e^{zR_t})$ denote the conditional moment generating function (mgf) of $R_t|\mathcal{F}_{t-1}$ under the physical (or dynamic) measure P, where $\mathrm{E}_{t-1}(X)$ denotes the conditional expectation of X given $\mathcal{F}_{t-1}$ under P and $\mathcal{F}_{t-1}$ is an information set including the prices of the underlying asset and the riskless bond prior to time $t-1$. To simplify the illustration, we assume that the riskless interest rate r is constant herein.

1. Define a new conditional pdf as

$$f_t(x_t; \delta_t) = \frac{e^{\delta_t x_t}}{\mathrm{M}_{R_t|\mathcal{F}_{t-1}}(\delta_t)} f_t(x_t) \tag{1}$$

   with an extra parameter δ_t.

2. Define an equivalent probability measure Q by $dQ = \Lambda_T dP$ with a Radon-Nikodým derivative

$$\Lambda_T = \prod_{t=1}^{T} \frac{e^{\delta_t^* R_t}}{\mathrm{M}_{R_t|\mathcal{F}_{t-1}}(\delta_t^*)}, \tag{2}$$

   where δ_t^*'s are obtained in the next step.

3. By (2), define a change of measure process Λ_t by $\Lambda_t = \mathrm{E}_t(\Lambda_T)$, $t = 0, 1, \ldots, T$. Then, the conditional pdf of $R_t|\mathcal{F}_{t-1}$ under Q is $f_t(x_t; \delta_t^*)$ defined in (1) and the parameter δ_t^* is determined by solving the following equation:

$$\mathrm{E}_{t-1}^{Q}(e^{R_t}) = e^r, \tag{3}$$

   where $\mathrm{E}_{t-1}^{Q}(X)$ denotes the conditional expectation of X under Q.

Siu, et al. (2004) and Huang, et al. (2012) applied the Esscher transform to GARCH option pricing with non-normal innovations.

2.2 CatEPut pricing (Cox, et al. 2004; Jaimungal and Wang, 2006; Lin, et al. 2009)

Let V_T be the payoff of a CatEput option with maturity time T,

$$V_T = I_{\{L_T - L_{t_0} > L\}}(K - S_T)^+, \tag{4}$$

where S_T denotes the price of the underlying asset at time T, $L_T - L_{t_0}$ denotes the total loss-percentage rate process from time t_0 to T, L is the pre-determined limit of loss on the contract, K is the strike price, and $I_{\{B\}}$ is the indicator function of an event B.

Cox, et al. (2004) considered the following jump-diffusion model to depict the dynamics of the underlying processes $\{S_t,\ t \geq 0\}$,

$$S_t = S_0 \exp\{-L_t + \sigma W_t + (\mu - 0.5\sigma^2)t\}, \tag{5}$$

where S_0 is the initial price of the underlying asset, W_t is a standard Brownian motion, and $L_t = AN_t$ with a constant jump size A and a pure Poisson process N_t. Note that L_t in (5) causes a drop in underlying asset prices. Cox, et al. (2004) derived the closed-form solution for CatEPut options of Model (5). To extend Cox's result to more general settings, Jaimungal and Wang (2006) considered the case of random riskless interest and assuming L_t being a compound Poisson process with Gamma distributed jump size. The explicit closed-form formulae for the price of the option is derived and the hedging parameters such as Delta, Gamma and Rho are obtained. They found that accounting for stochastic interest rates, the Rho hedging can significantly reduce the expected conditional loss of the hedged portfolio. Moreover, Lin, et al. (2009) considered the case of compound Poisson process L_t with N_t having stochastic intensity rate, called the doubly stochastic Poisson process. They also established the pricing formulae of CatEPut options by the Merton measure.

3 Pricing CatEPut in GARCH-Jump Model

Assume that the log returns R_t, $t = 1, 2, \ldots, T$, are generated from the following GARCH-Jump model under the physical P measure:

$$\begin{cases} R_t = R_t^G - L_t, \\ R_t^G = r - \frac{1}{2}\sigma_t^2 + \lambda\sigma_t + \sigma_t\varepsilon_t, \quad \varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0,1) \\ \sigma_{t+1}^2 = \alpha_0 + \alpha_1\sigma_t^2(\varepsilon_t - \gamma)^2 + \beta_1\sigma_t^2, \\ L_t = \sum_{n=0}^{\Delta N_t} Y_{t,n}, \quad Y_{t,n} \overset{\text{i.i.d.}}{\sim} f_{Y_{t,n}}(x_t), \end{cases} \tag{6}$$

where r is the one-period risk-free interest rate, λ denotes the risk premium, γ is a nonnegative constant to describe asymmetric effects between positive and negative stock returns, $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$ in the GARCH part R_t^G. For the jump part L_t, $\Delta N_t := N_t - N_{t-1}$ denotes the total number of catastrophe during $[t-1, t)$. N_t is a Poisson process with stochastic intensity rate θ_t, which is assumed to be governed by the diffusion,

$$d\theta_t = \mu_\theta\theta_t dt + \sigma_\theta\theta_t dW_t, \tag{7}$$

where μ_θ and σ_θ are constants and W_t is a Brownian motion. The jump size $Y_{t,n}$, $n = 1, 2, \ldots$ are assumed to be independent and identically distributed and $Y_{t,0}$ is set to be zero for all t. Moreover, assume that ε_t, W_t, N_t and $Y_{t,n}$ are independent.

A risk-neutral counterpart of Model (6) is derived by the Esscher transform. Before illustrate the main result, we first give the following useful lemma.

Lemma 3.1. *Let $\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h)$ denotes the conditional moment generating function of θ_t^* given $\mathcal{F}_{t-1}$, where $\theta_t^* = \int_{t-1}^t \theta_s ds$ with θ_s satisfying (7). Then,*

$$\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h) = \mathrm{E}_{t-1}(e^{h\theta_t^*}) < \infty.$$

Furthermore, $\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h)$ can be approximated by

$$\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h) \approx e^{h\mu_{\theta_t^*}}(1 + \frac{h^2}{2}\sigma_{\theta_t^*}^2),$$

where $\mu_{\theta_t^*} = \theta_{t-1}(e^{\mu_\theta} - 1)/\mu_\theta$ *is the conditional expectation of* θ_t^* *given* $\mathcal{F}_{t-1}$ *and*

$$\sigma^2_{\theta_t^*} = \frac{2\theta^2_{t-1}}{\mu_\theta}\Big[e^{2\mu_\theta+\sigma^2_\theta}\Big(\frac{1}{\mu_\theta+\sigma^2_\theta} - \frac{1}{2\mu_\theta+\sigma^2_\theta}\Big) + \Big(\frac{1}{2\mu_\theta+\sigma^2_\theta} - \frac{e^{\mu_\theta}}{\mu_\theta+\sigma^2_\theta}\Big)\Big] - \mu^2_{\theta_t^*}$$

is the conditional variance of θ_t^* *given* $\mathcal{F}_{t-1}$.

By Lemma 3.1, a change of measure process derived from the Ecccher transform is obtained in the following proposition.

Proposition 3.1. *For Model (6), a change of measure process derived by the Esscher transform is*

$$\frac{d\mathrm{Q}}{d\mathrm{P}} = \Lambda_T^{R^G} \times \Lambda_T^{L|\theta^*} \times \Lambda_T^{\theta^*}, \tag{8}$$

where

$$\begin{aligned}
\Lambda_T^{R^G} &= \prod_{t=1}^{T} \frac{e^{\delta_t^G R_t^G}}{\mathrm{M}_{R_t^G|\mathcal{F}_{t-1}}(\delta_t^G)}, \\
&= \prod_{t=1}^{T} \exp\{-\frac{1}{2\sigma_t^2}[(c+\lambda\sigma_t)^2 + 2(c+\lambda\sigma_t)(R_t^G - r - \lambda\sigma_t + \frac{1}{2}\sigma_t^2)]\}, \qquad (9)\\
\Lambda_T^{L|\theta^*} &= \prod_{t=1}^{T} \frac{e^{\delta_t^L L_t}}{e^{h(\delta_t^L)\,\theta_t^*}}, \qquad \textit{and} \quad \Lambda_T^{\theta^*} = \prod_{t=1}^{T} \frac{e^{h(\delta_t^L)\,\theta_t^*}}{\mathrm{E}_{t-1}[e^{h(\delta_t^L)\,\theta_t^*}]},
\end{aligned}$$

in which

$$\delta_t^G = -\frac{c+\lambda\,\sigma_t}{\sigma_t^2}, \tag{10}$$

c *is a constant,* $h(\delta_t^L) = \mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L) - 1$ *and* δ_t^L *is the solution of*

$$\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L - 1) - \mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L) = \frac{c}{\theta_t^*}. \tag{11}$$

Let $\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)$ *denotes the conditional mgf of* $Y_{t,n}$ *given* $\mathcal{F}_{t-1}$ *and* $\mathrm{E}_{t-1}[e^{h(\delta_t^L)\,\theta_t^*}]$ *can be obtained either by numerical methods or by the approximation in Lemma 3.1.*

Corollary 3.1. *In Proposition 3.1, if* θ_t *is a deterministic function of* t*, then the change of measure process becomes*

$$\frac{d\mathrm{Q}}{d\mathrm{P}} = \Lambda_T^{R^G} \times \Lambda_T^{L},$$

where $\Lambda_T^{R^G}$ is defined the same as in Proposition 3.1 and

$$\Lambda_T^L = \prod_{t=1}^{T} \frac{e^{\delta_t^L L_t}}{e^{h(\delta_t^L)\,\theta_t^*}},$$

where δ_t^L and $h(\delta_t^L)$ are defined the same as in Proposition 3.1.

Corollary 3.2. *If the jump risk is assumed to be non-diversifiable, then the jump activity rate and distribution are not altered by the measure changed (Merton, 1976; Cox et al., 2004; Jaimungal and Wang, 2006). Therefore, the constant c in (9), (10) and (11) is set to be*

$$c = \theta_t^*[\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(-1) - 1] \tag{12}$$

such that $\delta_t^L = 0$.

By the change of measure process obtained in (8), the following proposition establishes the corresponding risk neutral counterpart of Model (6).

Proposition 3.2. *The risk neutral counterpart of Model (6) corresponding to the change of measure process obtained in Proposition 3.1 is:*

$$\begin{cases} R_t = R_t^G - L_t, \\ R_t^G = r - c - \dfrac{1}{2}\sigma_t^2 + \sigma_t \xi_t, \quad \xi_t \overset{\text{i.i.d}}{\sim} \mathrm{N}(0,1) \\ \sigma_{t+1}^2 = \alpha_0 + \alpha_1 \sigma_t^2 (\xi_t - \lambda - c\sigma_t^{-1} - \gamma)^2 + \beta_1 \sigma_t^2, \\ L_t = \displaystyle\sum_{n=0}^{\Delta N_t} Y_{t,n}, \quad Y_{t,n} \overset{\text{i.i.d}}{\sim} f_t(x_t; \delta_t^L), \end{cases} \tag{13}$$

where $\xi_t = \varepsilon_t + \lambda + c\sigma_{t-1}^{-1}$, N_t is a Poisson process with stochastic intensity rate $\tilde{\theta}_t = \mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)\theta_t$, which satisfies

$$d\tilde{\theta}_t = \mu_\theta \tilde{\theta}_t dt + \sigma_\theta \tilde{\theta}_t dW_t.$$

Moreover, ξ_t, W_t, N_t and $Y_{t,n}$ are independent under the risk-neutral measure Q.

Note that if θ_t is a deterministic function of t, then the intensity rate under Q is $\tilde{\theta}_t = \mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)\theta_t$, which is still a deterministic function of t.

4 Sensitivity Analysis

In this section, several simulation scenarios are conducted to investigate the sensitivity of the parameters in the GARCH-Jump model (6). Under the Merton measure, Sections 4.1, 4.2 and 4.3 investigate the influence of the constant intensity rate θ, the mean μ_y and the variance σ_y^2 of the jump size, and the time to maturity T, respectively, on pricing CatEPut options. Section 4.4 presents a sensitivity study of μ_θ and σ_θ in the stochastic intensity rate model (7). Moreover, to investigate the influence of different choices of the risk-neutral measures, Section 4.5 conducts a comparison study by choosing different c such that (23) and (24) hold in the GARCH-Jump model, where the intensity rate is a constant and the jump-size is $N(\mu_y, \sigma_y^2)$ distributed. Throughout this section, let $L = \mathrm{E}[L_T - L_{t_0}]$, which is the same as in Jaimungal and Wang (2006), $S_0 = 25$, $K = 80$, $L_{t_0} = 0$, $r = 0.05$, $\alpha_0 = 3.3 \times 10^{-3}$, $\alpha_1 = 0.2$, $\beta_1 = 0.7$, $\lambda = 0.01$, $\gamma = 0.3$, initial conditional variance $\sigma_1^2 = \alpha_0[1 - \alpha_1(1 + \gamma^2) - \beta_1]^{-1}$, and 10,000 sample paths.

4.1 The influence of the magnitude of intensity rate

In this section, the following GARCH-Jump model with constant intensity rate is considered,

$$\begin{cases} R_t = R_t^G - L_t, \\ R_t^G = r - c - \dfrac{1}{2}\sigma_t^2 + \sigma_t \xi_t, \quad \xi_t \overset{\text{i.i.d}}{\sim} \mathrm{N}(0,1), \\ \sigma_{t+1}^2 = \alpha_0 + \alpha_1 \sigma_t^2 (\xi_t - \lambda - c\sigma_t^{-1} - \gamma)^2 + \beta_1 \sigma_t^2, \\ L_t = \displaystyle\sum_{n=0}^{\Delta N_t} Y_{t,n}, \quad Y_{t,n} \overset{\text{i.i.d}}{\sim} f_t(x_t), \end{cases} \tag{14}$$

where $Y_{t,n}$ is assumed to be a constant A, $Ga(\alpha,\beta)$ or $N(\mu_y, \sigma_y^2)$ distributed, and the corresponding c is $\theta[e^{-A} - 1]$, $\theta[e^{-\mu_y + 0.5\sigma_y^2} - 1]$ or $\theta[(1+\beta)^{-\alpha} - 1]$, respectively.

For the case of constant jump size, which is considered by Cox, et al. (2004), Figure 1 plots the CatEPut prices under various initial stock price S_0 and intensity rate θ with $A = 0.02$ and $T = 1$. Figure 1 has the same shape as that in Figure 2 of Cox, et al. (2004) because the GARCH-Jump model with $T = 1$ is the same as the jump-diffusion Model (5) considered in Cox, et al. (2004).

To investigate the impacts of the magnitude of intensity rate and the above three distributions, constant, normal and gamma of the jump size on the CatEPut prices, we compute the value of $P(L_T > L)$, which is proportional to the CatEPut price and has a simpler analytic representation than the CatEPut price, under various settings of θ, (α, β) and (μ_y, σ_y^2). Table 1 shows the simulation results of $\theta = 0.2, 0.4, \ldots, 1.2$. In Table 1, the CatEPut prices increase when $P(L_1 > L)$ (in parentheses) increases in each column. We also plot the CatEPut prices versus the value of $P(L_1 > L)$ in Figure 3, which shows that the CatEPut prices are positively correlated to $P(L_1 > L)$.

More interesting findings are illustrated as follows.

1. If the jump size is assumed to be a constant A, then $L_T = N_T A$, where N_T is a Poisson(θT) distribution. Hence, we have

$$P(L_T > L) = 1 - \sum_{j=0}^{[\theta T]} \frac{e^{-\theta T}(\theta T)^j}{j!}, \tag{15}$$

   where $[x]$ denotes the largest integer less and equal to x. Figure 2 plots $P(L_1 > L)$ with $\theta = 0, 0.1, 0.2, \ldots, 5$ and shows that $P(L_1 > L)$ increases in $\theta \in [n, n+1)$, $n = 0, 1, 2, \ldots, 5$ but drops when $\theta = 1, 2, \ldots, 5$. Since the CatEPut option is proportional to $P(L_T > L)$ for a fixed T, thus the values of the CatEPut prices in Table 1 increases when $\theta \in [0.2, 0.8]$ and $[1, 1.2]$ but drops at $\theta = 1$.

2. If the jump size is assumed to be $Ga(\alpha,\beta)$ or $N(\mu_y, \sigma_y^2)$, then

$$P(L_T > L) = \sum_{k=0}^{\infty} \frac{e^{-\theta T}(\theta T)^k}{k!} \int_L^{\infty} f_{Y|N_T}(y|k)\mathrm{d}y, \tag{16}$$

   which is a continuous function of θ, where $f_{Y|N_T}(y|k)$ is the conditional density of the jump size Y conditional on $N_T = k$. Figure 4 plots $P(L_1 > L)$ defined in (16) with $\theta = 0, 0.1, 0.2, \ldots, 5$, and three different settings of the distribution parameters, where the values of the $Ga(\alpha,\beta)$ case are denoted in red dots and the values of the $N(\mu_y, \sigma_y^2)$ case are in blue circles. Note that the means of the gamma and normally distributions considered in Figure 4 are all equal to 0.02, which is the same as the previous constant jump size, and (α,β) are set to be satisfying $\alpha\beta = \mu_y$ and $\alpha\beta^2 = \sigma_y^2$.

The middle and lower panels of Figure 4 show that the values of $P(L_1 > L)$ defined in (16) increase in θ, but does not increase in θ in the upper panel. In particular, if $Y_{t,n}$ is gamma distributed, the values of $P(L_1 > L)$ drops dramatically when σ_y^2 increases. However, if $Y_{t,n}$ is normally distributed, then the three patterns of $P(L_1 > L)$ of our settings are more stable than the gamma case.

From the above discussion, we find that the CatEPut prices are proportional to the values of $P(L_T > L)$, $P(L_T > L)$ does not necessary increase as θ increasing if the jump size $Y_{t,n}$ is a constant or has relatively small variance, and $P(L_T > L)$ decreases more dramatically as the variance of $Y_{t,n}$ increasing when $Y_{t,n}$ is gamma distributed than $Y_{t,n}$ is normally distributed.

4.2 The influence of the noise ratio

In this section, the sensitivity analysis of μ_y and σ_y^2/μ_y on pricing CatEPut is presented under Model (14). Figure 5 plots the CatEPut prices under various θ, μ_y and σ_y^2/μ_y with $T = 1$. The jump size is assumed to be $Ga(\alpha,\beta)$ or $N(\mu_y, \sigma_y^2)$ distributed, where the ratio is set to be $\sigma_y^2/\mu_y = 1, 2, \ldots, 10$ with $\mu_y = 0.01, 0.02, 0.05, 0.5$. Similar to Section 4.1, let $\alpha = \mu_y^2/\sigma_y^2$ and $\beta = \sigma_y^2/\mu_y$ such that $\alpha\beta = \mu_y$ and $\alpha\beta^2 = \sigma_y^2$. Figure 5 indicates that

1. the CatEPut prices increase when μ_y increases,
2. if $Y_{t,n}$ is gamma distributed, the CatEPut prices decrease in σ_y^2/μ_y; which is consistent with the results reported in Figure 5 in Jaimungal and Wang (2006);
3. if $Y_{t,n}$ is normally distributed, the CatEPut prices does not have significant changes in σ_y^2/μ_y;

4. the CatEPut prices increases as θ increasing.

4.3 The influence of maturity

In this section, the sensitivity analysis of T on pricing CatEPut is presented. In addition to Model (14), we also consider the following jump-diffusion model under the Merton measure :

$$\begin{aligned} S_t &= S_0 \exp\{-\tilde{L}_t - ct + \sigma \tilde{W}_t + (r - 0.5\sigma^2)t\}, \\ \widetilde{W_t} &= \frac{\mu + c - r}{\sigma} t + W_t, \end{aligned} \tag{17}$$

where S_0 is the initial stock price, W_t is a standard Brownian motion under the physical measure, $\widetilde{W_t}$ is a new Brownian motion under the Merton measure, c is defined the same as in (14), and $\tilde{L}_t = \sum_{n=0}^{N_t} Y_{t,n}$, in which $\{N_t : t \geqslant 0\}$ is a Poisson process with intensity rate θ.

Figure 6 plots the CatEPut prices under various θ and T, where the jump size is constant. In Figure 6, there is no significant difference between the GARCH-Jump model and jump-diffusion model for CatEPut prices. Interestingly, the CatEPut prices shown on the left panel of Figure 6 do not increase as T increasing. Nevertheless, recall that the CatEPut prices are proportional to $P(L_T > L)$. In Figure 6, the blue, yellow and green points are used to denote the CatEPut prices (left panel) and $P(L_T > L)$ (right panel) of $T = 1, 3$ and 5, respectively. In particular, note that the green points on the right panel are lower than most of the blue and yellow points, which explains why the CatEPut prices do not increase as T increasing since the corresponding values of $P(L_T > L)$ do not increasing with θT.

Figure 7 plots the CatEPut prices under various θ, T and σ_y^2. The jump size is assumed to be $Ga(\alpha,\beta)$ or $N(\mu_y, \sigma_y^2)$ distributed, where the variance of jump size is set to be $\sigma_y^2 = 4 \times 10^{-3}$, 4×10^{-1} with fixed $\mu_y = 0.02$. The results shown in Figure 7 are consistent with those in Figures 4 and 6, except for $T = 3, 5$ in the normally distributed case, where the CatEPut prices increase as σ_y^2 increasing. Moreover, comparing to the case of constant jump size shown in Figure 6, the CatEPut prices in Figure 7 increase as T increasing.

4.4 The influence of stochastic intensity rate

In this section, we consider Model (14) with stochastic intensity rate and normally distributed jump sizes. The solution of the stochastic intensity rate model (7) is

$$\theta_t = \theta_0 \exp[(\mu_\theta - 0.5\sigma_\theta^2)t + \sigma_\theta W_t], \tag{18}$$

where θ_0 is the initial intensity rate. Table 2 presents the CatEPut prices under various θ_0, μ_θ and σ_θ with $T = 1, 3, 5$. Numerical results in Table 2 indicate that the CatEPut prices increase in θ_0, $\mu_\theta - 0.5\sigma_\theta^2$ for fixed μ_θ, and T. Moreover, if $\mu_\theta - 0.5\sigma_\theta^2$ is fixed, then different combinations of $(\mu_\theta, \sigma_\theta)$ yield different CatEPut prices.

4.5 The influence of risk-neutral measures

Recall Model (6) and let the intensity rate be a constant θ. By Proposition 3.2, if the jump-size is $N(\mu_y, \sigma_y^2)$ distributed under the P measure, then $Y_{t,n} \overset{\text{i.i.d}}{\sim} N(\tilde{\mu}_y, \sigma_y^2)$ under the Q_c measure, where Q_c denotes the risk-neutral measure corresponding to a specific constant c satisfying (23) and (24), $\tilde{\mu}_y = \mu_y + \delta_t^L \sigma_y^2$, and δ_t^L is obtained from solving (11) with a fixed c. In addition, the intensity rate of the Poisson process N_t under Q_c becomes $\tilde{\theta} = \mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)\theta$. Since different choices of c correspond to different risk-neutral measures Q_c, we investigate the influence of the choices of c on the CatEPut prices. Note that the parameters δ_t^G and δ_t^L defined in (10) and (11), respectively, when proceeding of the the Esscher transform in the GARCH and Jump parts, are determined by c.

1. If $c = \theta[\exp(-\mu_y + 0.5\sigma_y^2) - 1]$, which corresponds to the Merton measure, then $\delta_t^L = 0$ and $\delta_t^G = -\{\theta[\exp(-\mu_y + 0.5\sigma_y^2) - 1] + \lambda\sigma_t\}/\sigma_t^2$.

2. If $c = 0$, which corresponds to the risk-neutral measure in Amin (1993) when $T = 1$, then $\delta_t^L = 0.5 - \mu_y/\sigma_t^2$ and $\delta_t^G = -\lambda/\sigma_t$, which is the same as the results of Siu, et al. (2004) and Huang, et al. (2012) in the GARCH framework.

3. In particular, if $\delta_t^L = \delta_t^G = \delta_t^*$, then by (10) and (11) δ_t^* satisfies $\exp[\mu_y(\delta_t^* - 1) + 0.5\sigma_y^2(\delta_t^* - 1)^2] - \exp[\mu_y\delta_t^* + 0.5\sigma_y^2(\delta_t^*)^2] = -(\lambda\sigma_t + \delta_t^*\sigma_t^2)/\theta$ and $c = -(\lambda\sigma_t + \delta_t^*\sigma_t^2)$.

Table 3 presents the CatEPut prices under various $k = c\{\theta[\exp(-\mu_y + 0.5\sigma_y^2) - 1]\}^{-1}$ with $\theta = 0.4$, $\mu_y = 0.02, 0.04, 0.08$, $\sigma_y^2 = 0.4$ and $T = 1, 3, 5$. For the case of $\delta_t^L = \delta_t^G = \delta_t^*$, we report the values of k with $c = -(\lambda\sigma_1 + \delta_1^*\sigma_1^2)$, that is, $k = -0.3322, -0.3401$ and -0.3608 if $\mu_y = 0.02, 0.04$ and 0.08, respectively. The CatEPut prices presented in Table 3 versus k are shown in Figure 8, which indicates that the CatEPut prices decrease as k (or c, since c is proportional to k) increasing. That is, different Q_c measure computes different CatEPut prices. From Table 3, we find that the CatEPut prices increase in T for all the different Q_c measures, which is the same as the phenomenon observed in Section 4.3, where only Merton measure is considered. Interestingly, as μ_y increasing, the CatEPut prices increase for all the different Q_c measures if $T = 1$ while the CatEPut prices decrease for all Q_c if $T = 5$.

5 Conclusion

This study establishes a general approach to derive a risk-neutral GARCH-Jump model with stochastic intensity rate by the Esscher transform. The CatEPut option prices are computed under the GARCH-Jump model. Sensitivity analysis of the influence of the model parameters on the CatEPut prices are conducted by simulation. Numerical results indicate that there is no significant difference between the GARCH-Jump model and jump-diffusion model for CatEPut prices. Table 4 summarizes the influence of the intensity rate of the occurrence of catastrophes, the distribution assumption of the jump size, the time to maturity, and the selection of risk-neutral measure on pricing CatEPut options in GARCH-Jump model. In general, the CatEPut prices increase as the intensity rate, the expected value of the jump size, the variance of the jump size or the time to maturity increasing.

Appendix

Proof of Lemma 3.1. Let $\eta_t = (\log\theta_t)/\sigma_\theta$ and $\alpha = (\mu_\theta - 0.5\sigma_\theta^2)/\sigma_\theta$. By (7), we have

$$d\eta_t = \alpha dt + dW_t. \tag{19}$$

Further let

$$M_t = \max\{\eta_s : \ s \in [t-1, t]\}. \tag{20}$$

By (19), (20) and Corollary 7.2.2 in Shreve (2004), the pdf of M_t is

$$f_{M_t}(m) = \frac{2}{\sqrt{2\pi}} e^{-0.5(m-\alpha)^2} - 2\alpha\, e^{2\alpha\, m} \Phi(-m-\alpha), \tag{21}$$

where $m \geq 0$ and $\Phi(\cdot)$ denotes the cdf of $N(0,1)$. By Taylor's expansion and (20), we have

$$\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h) = \sum_{j=0}^{\infty} \frac{h^j}{j\,!} \mathrm{E}_{t-1}[(\theta_t^*)^j] \leq \sum_{j=0}^{\infty} \frac{h^j}{j\,!} \mathrm{E}_{t-1}[e^{j\sigma_\theta M_t}]. \tag{22}$$

By (21) and noting that

$$\int_0^\infty e^{j\sigma_\theta m - 0.5(m-\alpha)^2} dm = O(e^{j\alpha\sigma_\theta + 0.5j^2\sigma_\theta^2})$$

and

$$\int_0^\infty e^{j\sigma_\theta m + 2\alpha m} \Phi(-m-\alpha) dm = O(j^{-1} e^{j\alpha\sigma_\theta + 0.5j^2\sigma_\theta^2}),$$

the right-hand-side of (22) is finite. Consequently, $\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h)$ exists.

However, it is difficult to obtain closed-form representation of $\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h)$. In this lemma, we approximate it by the following second order Taylor's expansion:

$$\mathrm{M}_{\theta_t^*|\mathcal{F}_{t-1}}(h) \ \ \approx \ \ e^{h\mu_{\theta_t^*}}(1 + \frac{h^2}{2}\sigma^2_{\theta_t^*}),$$

where $\mu_{\theta_t^*} = \mathrm{E}_{t-1}(\theta_t^*)$ and $\sigma^2_{\theta_t^*} = \mathrm{Var}_{t-1}(\theta_t^*)$ are the conditional mean and variance of θ_t^* given $\mathcal{F}_{t-1}$, respectively. In the following, we derive the closed-form representations for $\mu_{\theta_t^*}$ and $\sigma^2_{\theta_t^*}$.

Let $X_n = n^{-1}\sum_{i=0}^{n-1}\theta_{t-1+i/n}$. First note that X_n converges to θ_t^* in L^2 sicne

$$
\begin{aligned}
& \mathrm{E}_{t-1}|X_n - \theta_t^*|^2 \\
= \; & \mathrm{E}_{t-1}\Big|\sum_{i=0}^{n-1}\int_{t-1+i/n}^{t-1+(i+1)/n}(\theta_{t-1+\frac{i}{n}} - \theta_s)ds\Big|^2 \\
\leq \; & \sum_{i=0}^{n-1}\int_{t-1+i/n}^{t-1+(i+1)/n}\mathrm{E}_{t-1}|\theta_{t-1+\frac{i}{n}} - \theta_s|^2 ds \\
= \; & \sum_{i=0}^{n-1}\int_{t-1+i/n}^{t-1+(i+1)/n}\mathrm{E}_{t-1}\mathrm{E}_{t-1+\frac{i}{n}}|\theta_{t-1+\frac{i}{n}} - \theta_s|^2 ds \\
= \; & \theta_{t-1}^2\frac{1}{n}\sum_{i=0}^{n-1}e^{(2\mu_\theta+\sigma_\theta^2)\frac{i}{n}}\Big[1 - 2\frac{n}{\mu_\theta}\Big(e^{\frac{\mu_\theta}{n}} - 1\Big) + \frac{n}{2\mu_\theta + \sigma_\theta^2}\Big(e^{(2\mu_\theta+\sigma_\theta^2)\frac{1}{n}} - 1\Big)\Big] \\
\to \; & 0 \quad \text{as}\, n \to \infty,
\end{aligned}
$$

where the inequality holds by Cauchy-Schwarz inequality. By Theorem 4.5.4 in Chung (2001), $\mathrm{E}_{t-1}(\theta_t^*)^r = \lim_{n\to\infty}\mathrm{E}_{t-1}(|X_n|^r)$, for $0 < r \leq 2$. Therefore, the conditional mean and conditional variance of θ_t^* given $\mathcal{F}_{t-1}$ can be computed as $\mu_{\theta_t^*} = \mathrm{E}_{t-1}(\theta_t^*) = \lim_{n\to\infty}\mathrm{E}_{t-1}(X_n) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\theta_{t-1}e^{\mu_\theta\frac{i}{n}} = \frac{\theta_{t-1}}{\mu_\theta}(e^{\mu_\theta} - 1)$, and $\sigma_{\theta_t^*}^2 = \mathrm{E}_{t-1}(\theta_t^*)^2 - \mu_{\theta_t^*}^2$, where

$$
\begin{aligned}
\mathrm{E}_{t-1}(\theta_t^*)^2 &= \lim_{n\to\infty}\mathrm{E}_{t-1}(|X_n|^2) \\
&= \lim_{n\to\infty}\Big[\frac{1}{n^2}\sum_{i=0}^{n-1}\theta_{t-1}^2e^{(2\mu_\theta+\sigma_\theta^2)\frac{i}{n}} + \frac{2}{n^2}\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\theta_{t-1}^2e^{(\mu_\theta+\sigma_\theta^2)\frac{i}{n}}e^{\mu_\theta\frac{j}{n}}\Big] \\
&= \frac{2\theta_{t-1}^2}{\mu_\theta}\Big[e^{2\mu_\theta+\sigma_\theta^2}\Big(\frac{1}{\mu_\theta+\sigma_\theta^2} - \frac{1}{2\mu_\theta+\sigma_\theta^2}\Big) + \Big(\frac{1}{2\mu_\theta+\sigma_\theta^2} - \frac{e^{\mu_\theta}}{\mu_\theta+\sigma_\theta^2}\Big)\Big].
\end{aligned}
$$

□

Proof of Proposition 3.1. Let Λ_T be a Radon-Nikodým derivative and the corresponding risk-neutral measure Q be defined by $dQ = \Lambda_T dP$. Recall the scheme of the Esscher transform introduced in Section 2.1 and the equation, $R_t = R_t^G - L_t$, in Model (6). By using similar arguments as in Shreve (2004), Λ_T is decomposed by

$$\Lambda_T = \Lambda_T^{R^G} \times \Lambda_T^L,$$

where $\Lambda_T^{R^G}$ and Λ_T^L are two independent Radon-Nikodým derivatives such that

$$\mathrm{E}_{t-1}^Q(e^{R_t^G}) = \mathrm{E}_{t-1}\Big(\frac{\Lambda_t^{R^G}}{\Lambda_{t-1}^{R^G}}e^{R_t^G}\Big) = e^{r-c} \tag{23}$$

and

$$\mathrm{E}^{Q}_{t-1}(e^{-L_t}) = \mathrm{E}_{t-1}\Big(\frac{\Lambda^L_t}{\Lambda^L_{t-1}}\, e^{-L}\Big) = e^{c}, \tag{24}$$

in which $\Lambda^{R^G}_t = \mathrm{E}_t(\Lambda^{R^G}_T)$ and $\Lambda^L_t = \mathrm{E}_t(\Lambda^L_T)$, $t = 0, 1, \ldots, T$.

In Model (6), the conditional distribution of $R^G_t|\mathcal{F}_{t-1}$ under P is $\mathrm{N}(r - 0.5\sigma^2_t + \sigma_t, \sigma^2_t)$ and the corresponding conditional mgf is

$$\mathrm{M}_{R^G_t|\mathcal{F}_{t-1}}(z) = e^{(r-0.5\sigma^2_t+\sigma_t)z+0.5\sigma^2_t z^2}. \tag{25}$$

Define a new conditional pdf of $R^G_t|\mathcal{F}_{t-1}$ with a parameter δ_t

$$f_{R^G_t|\mathcal{F}_{t-1}}(x_t;\delta_t) = \frac{e^{x_t\delta_t}}{\mathrm{M}_{R^G_t|\mathcal{F}_{t-1}}(\delta_t)} f_{R^G_t|\mathcal{F}_{t-1}}(x_t). \tag{26}$$

Then, choose a $\delta_t = \delta^G_t$ in (26) for $0 \le t \le T$ such that (23) holds, that is,

$$\delta^G_t = -\frac{c + \lambda\sigma_t}{\sigma^2_t}. \tag{27}$$

Consequently, the Radon-Nikodým derivative $\Lambda^{R^G}_T$ derived by the Esscher transform is

$$\begin{aligned}
\Lambda^{R^G}_T &= \prod_{t=1}^{T} \frac{e^{\delta^G_t R^G_t}}{\mathrm{M}_{R^G_t|\mathcal{F}_{t-1}}(\delta^G_t)},\\
&= \prod_{t=1}^{T} \exp\{-\frac{1}{2\sigma^2_t}[(c+\lambda\sigma_t)^2 + 2(c+\lambda\sigma_t)(R^G_t - r - \lambda\sigma_t + \frac{1}{2}\sigma^2_t)]\}.
\end{aligned} \tag{28}$$

In Model (6), the conditional density of $L_t|\mathcal{F}_{t-1}$ under P is

$$f_{L_t|\mathcal{F}_{t-1}}(x_t) = \sum_{k=0}^{\infty} \frac{\mathrm{E}_{t-1}[e^{-\theta^*_t}\theta^{*k}_t]}{k!} f_{Y_{t,n}}(x_t|\Delta N_t = k), \tag{29}$$

and the conditional mgf of $L_t|\mathcal{F}_{t-1}$, denoted by $\mathrm{M}_{L_t|\mathcal{F}_{t-1}}(z)$, is

$$\mathrm{M}_{L_t|\mathcal{F}_{t-1}}(z) = \mathrm{E}_{t-1}\{\exp[\theta^*_t(M_{Y_{t,n}|\mathcal{F}_{t-1}}(z) - 1)]\}. \tag{30}$$

Define a new conditional pdf of $L_t|\mathcal{F}_{t-1}$ with a parameter δ_t

$$f_{L_t|\mathcal{F}_{t-1}}(x_t;\delta_t) = \frac{e^{x_t\delta_t}}{\mathrm{M}_{L_t|\mathcal{F}_{t-1}}(\delta_t)} f_{L_t|\mathcal{F}_{t-1}}(x_t), \tag{31}$$

Then, choose a $\delta_t = \delta_t^L$ in (31) for $0 \le t \le T$ such that (24) holds where δ_t^L is the solution of (11). Consequently, the Radon-Nikodým derivative Λ_T^L derived by the Esscher transform is

$$\Lambda_T^L = \prod_{t=1}^{T} \frac{e^{\delta_t^L L_t}}{\mathrm{E}_{t-1}[e^{h(\delta_t^L)\,\theta_t^*}]}, \tag{32}$$

where $h(\delta_t^L) = \mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L) - 1$ and $\mathrm{E}_{\mathrm{t}-1}[e^{h(\delta_t^L)\,\theta_t^*}]$ can be approximated either by Lemma 3.1 or Monte Carlo simulation. In addition, if Λ_T^L in (32) is further decomposed by

$$\Lambda_T^L = \Lambda_T^{L|\theta^*} \times \Lambda_T^{\theta^*}, \tag{33}$$

where $\Lambda_T^{L|\theta^*}$ is the Radon-Nikodým derivative of Q with respect to P for L_T conditional on θ_T^*, and $\Lambda_T^{\theta^*}$ is the Radon-Nikodým derivative for θ_T^*. Note that $\Lambda_t^{L|\theta^*}$ can be obtained by similar arguments used in (29)-(32), that is,

$$\Lambda_T^{L|\theta^*} = \prod_{t=1}^{T} \frac{e^{\delta_t^L L_t}}{e^{h(\delta_t^L)\,\theta_t^*}}. \tag{34}$$

By (32)-(34), we have

$$\Lambda_T^{\theta^*} = \prod_{t=1}^{T} \frac{e^{h(\delta_t^L)\,\theta_t^*}}{\mathrm{E}_{t-1}[e^{h(\delta_t^L)\,\theta_t^*}]}. \tag{35}$$

Finally, the desire result holds by (33)-(35). □

Proof of Proposition 3.2. By (28) in the proof of Proposition 3.1, the change of measure processes $\Lambda_t^{R^G}$ be defined by $\Lambda_t^{R^G} = \mathrm{E}_t(\Lambda_T^{R^G})$ for $t = 1, \ldots, T$. Choosing δ_t^G in (10) the conditional mgf of $R_t^G|\mathcal{F}_{t-1}$ under measure Q is

$$\begin{aligned} \mathrm{M}^Q_{R_t^G|\mathcal{F}_{t-1}}(z;\delta_t^G) &= \mathrm{E}^Q_{t-1}(e^{z\,R_t^G}) = \mathrm{E}_{t-1}\Big(\frac{\Lambda_t^{R^G}}{\Lambda_{t-1}^{R^G}} e^{z\,R_t^G}\Big) \\ &= e^{(r-c-0.5\sigma_t^2)z + 0.5\sigma_t^2 z^2}, \end{aligned} \tag{36}$$

which has the same form as the conditional mgf of $R_t^G|\mathcal{F}_{t-1}$ under P given in (25). That is, $R_t^G|\mathcal{F}_{t-1}$ is normally distributed with mean $r - c + \sigma_t^2/2$ and variance $\sigma_t^2/2$ under Q.

By (34) and (35) in the proof of Proposition 3.1, the change of measure processes $\Lambda_t^{L|\theta^*}$ and $\Lambda_t^{\theta^*}$ are defined by

$$\Lambda_t^{L|\theta^*} = \mathrm{E}_t(\Lambda_T^{L|\theta^*}) \quad \text{and} \quad \Lambda_t^{\theta^*} = \mathrm{E}_t(\Lambda_T^{\theta^*}), \tag{37}$$

for $t = 1, \ldots, T$. By (33), the conditional mgf of $L_t|\mathcal{F}_{t-1}$ under measure Q is

$$\begin{aligned}
\mathrm{M}^Q_{L_t|\mathcal{F}_{t-1}}(z;\delta_t^L) &= \mathrm{E}^Q_{t-1}(e^{z\,L_t}) \\
&= \mathrm{E}_{t-1}(\frac{\Lambda_t^L}{\Lambda_{t-1}^L} e^{z\,L_t}) = \mathrm{E}_{t-1}[\frac{\Lambda_t^{\theta^*}}{\Lambda_{t-1}^{\theta^*}} \mathrm{E}_{t-1}(e^{z\,L_t} \frac{\Lambda_t^{L|\theta^*}}{\Lambda_{t-1}^{L|\theta^*}} |\theta_t^*)] \\
&= \mathrm{E}_{\mathrm{t-1}}\Big\{ \frac{e^{h(\delta_t^L)\,\theta_t^*}}{\mathrm{E}_{\mathrm{t-1}}(e^{h(\delta_t^L)\,\theta_t^*})} \exp\{\theta_t^*\,\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)[\frac{\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(z+\delta_t^L)}{\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)} - 1]\} \Big\}
\end{aligned} \tag{38}$$

where the last equality holds by (34), (35) and (37).

In addition, let $\psi_{\theta_t^*|\mathcal{F}_{t-1}}(x)$ denote the conditional pdf of $\theta_t^*|\mathcal{F}_{t-1}$ under P. By (35) and (37), the conditional density of $\theta_t^*|\mathcal{F}_{t-1}$ under Q can be represented as

$$\tilde{\psi}_{\theta_t^*|\mathcal{F}_{t-1}}(x) = \frac{e^{h(\delta_t^L)\,x}}{\mathrm{E}_{\mathrm{t-1}}(e^{h(\delta_t^L)\,\theta_t^*})} \psi_{\theta_t^*|\mathcal{F}_{t-1}}(x). \tag{39}$$

By (39) and let $\tilde{\theta}_t^* = \theta_t^*\,\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)$, the conditional mgf obtained in (38) can be rewritten as

$$\mathrm{M}^Q_{L_t|\mathcal{F}_{t-1}}(z;\delta^L) = \mathrm{E}^Q_{t-1}\{\exp[\tilde{\theta}_t^*(\frac{\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(z+\delta_t^L)}{\mathrm{M}_{Y_{t,n}|\mathcal{F}_{t-1}}(\delta_t^L)} - 1)]\}. \tag{40}$$

□

Table 1: The CatEPut prices and $P(L_1 > L)$ (in parentheses) versus intensity rates under the GARCH-Jump model with constant, gamma and normally distributed jump sizes in the Merton measure with $T = 1$.

	Case 1			Case 2		Case 3	
θ	$A = 0.02$	$Ga(10, 0.002)$	$N(0.02, 4 \times 10^{-5})$	$Ga(0.1, 0.2)$	$N(0.02, 4 \times 10^{-3})$	$Ga(10^{-3}, 20)$	$N(0.02, 4 \times 10^{-1})$
0.2	9.1009	9.1013	9.0808	3.1014	5.9534	0.0663	5.6801
	(0.1813)	(0.1813)	(0.1803)	(0.0566)	(0.1098)	(0.0016)	(0.0926)
0.4	17.4452	17.3270	17.0452	4.9773	10.2147	0.1987	10.0838
	(0.3297)	(0.3275)	(0.3219)	(0.0924)	(0.194)	(0.0029)	(0.1678)
0.6	23.1722	21.8047	21.6306	6.4104	13.9292	0.3020	14.3979
	(0.4512)	(0.4236)	(0.4172)	(0.1202)	(0.2584)	(0.0041)	(0.2289)
0.8	28.3179	23.1340	23.4905	7.6496	16.1245	0.3949	17.6613
	(0.5507)	(0.4488)	(0.4554)	(0.1428)	(0.3074)	(0.0052)	(0.2786)
1	13.5460	22.1135	22.6213	8.7856	17.9866	0.4018	20.8239
	(0.2642)	(0.4321)	(0.4458)	(0.162)	(0.3447)	(0.0063)	(0.319)
1.2	17.7409	21.9585	21.9910	9.5113	19.7644	0.4333	22.5657
	(0.3374)	(0.4204)	(0.4246)	(0.1784)	(0.3729)	(0.0073)	(0.3518)

Table 2: The CatEPut prices under GARCH-Jump model with stochastic intensity rate and jump sizes is $N(0.02,\ 0.4)$ distributed in the Merton measure.

			θ_0			
T	$(\mu_\theta, \sigma_\theta)$	$\mu_\theta - 0.5\sigma_\theta^2$	0.2	0.4	0.6	0.8
1	(0.1, 0)	0.1	6.2771	11.3545	15.4044	18.9942
	(0.1, 0.3)	0.055	6.2392	11.1265	15.1625	18.5459
	(0.1, 0.4)	0.02	6.0868	10.9545	15.0098	18.3812
	(0.2, 0.6)	0.02	6.5768	11.5850	15.4185	18.2758
3	(0.1, 0)	0.1	14.7899	22.6858	26.8348	29.3594
	(0.1, 0.3)	0.055	14.3457	21.6676	25.6937	28.5816
	(0.1, 0.4)	0.02	13.9540	21.0800	25.2308	27.6481
	(0.2, 0.6)	0.02	14.5843	21.1960	24.8104	27.2773
5	(0.1, 0)	0.1	19.1988	25.8191	28.1968	29.1458
	(0.1, 0.3)	0.055	18.1266	24.2215	26.9488	28.4127
	(0.1, 0.4)	0.02	17.4332	23.4287	26.2077	27.5207
	(0.2, 0.6)	0.02	17.7179	23.0994	25.7644	27.1952

Table 3: The CatEPut prices versus k under normally distributed jump sizes, where $k = c\{\theta[\exp(-\mu_y + 0.5\sigma_y^2) - 1]\}^{-1}$ with $\theta = 0.4$ and $\sigma_y^2 = 0.4$.

$\mu_y = 0.02$	k	-0.3322	0	0.25	0.5	0.75	1
	$T = 1$	14.4340	13.3650	12.5233	11.6947	11.0095	10.3423
	$T = 3$	28.0865	25.8917	24.3212	22.7853	21.3443	20.1021
	$T = 5$	31.8047	29.4976	28.0237	26.1909	24.6224	23.2531
$\mu_y = 0.04$	k	-0.3401	0	0.25	0.5	0.75	1
	$T = 1$	14.3354	13.2432	12.4245	11.7216	11.1213	10.5368
	$T = 3$	27.6199	25.5841	24.1402	22.7616	21.6054	20.4345
	$T = 5$	31.2082	29.1659	27.3588	26.1471	24.6395	23.2393
$\mu_y = 0.08$	k	-0.3608	0	0.25	0.5	0.75	1
	$T = 1$	13.9434	13.1623	12.4645	11.9243	11.5145	10.9894
	$T = 3$	26.2002	24.6243	23.6016	22.6098	21.6747	20.7056
	$T = 5$	29.5389	27.5814	26.5712	25.3519	24.2673	23.0902

Table 4: Summary of sensitivity analysis on the CatEPut prices.

		$Y_{t,n}$		
		Constant	$Ga(\alpha, \beta)$	$N(\mu_y, \sigma_y^2)$
Merton measure and constant intensity	θ	↑↓	↑	↑
	μ_y (if $T = 1$)	↑	↑	↑
	σ_y^2	–	↓	↑
	T	↑↓	↑	↑
Merton measure and stochastic intensity	θ_0	–	–	↑
	$\mu_\theta - 0.5\sigma_\theta^2$	–	–	↑
Q_c and constant intensity	c	–	–	↓

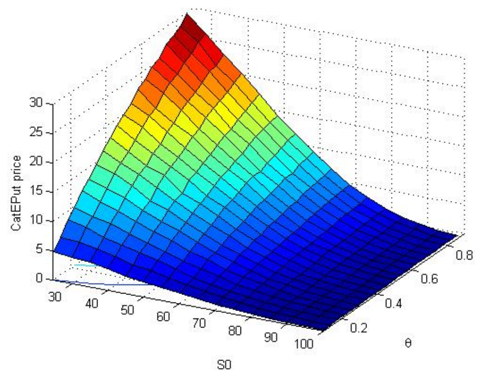

Figure 1: The CatEPut prices under various initial stock price S_0 and intensity rate θ in the GARCH-Jump model, where $T = 1$.

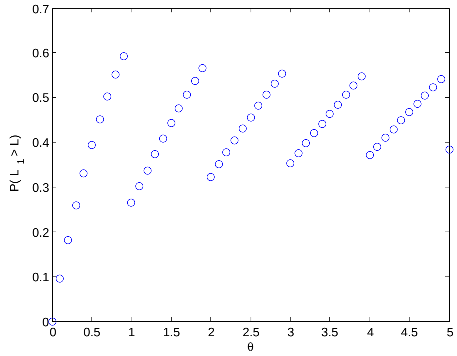

Figure 2: The value of $P(L_1 > L)$ versus the intensity rate θ in the case of constant jump size.

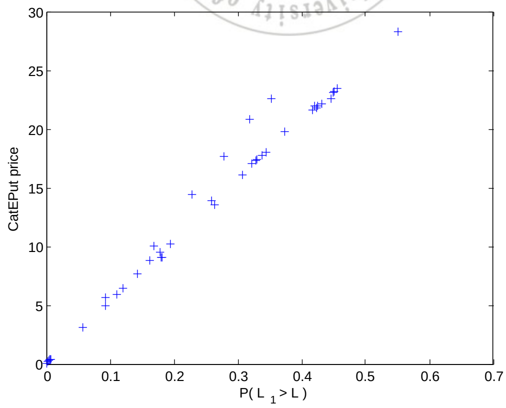

Figure 3: The CatEPut prices given in Table 1versus $P(L_1 > L)$.

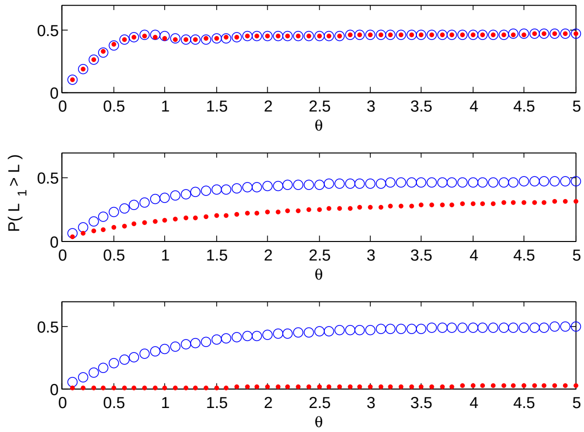

Figure 4: The probability of $P(L_1 > L)$ versus the intensity rate θ in the cases of normally (blue circle) and gamma (red dot) distributed jump size. The upper panel is the case of Ga(10,0.0002) and N(0.02,4×10^{-5}). The middle panel is the case of Ga(0.1,0.2) and N(0.02,4×10^{-3}). The lower panel is the case of Ga(10^{-3},20) and N(0.02,4×10^{-1}).

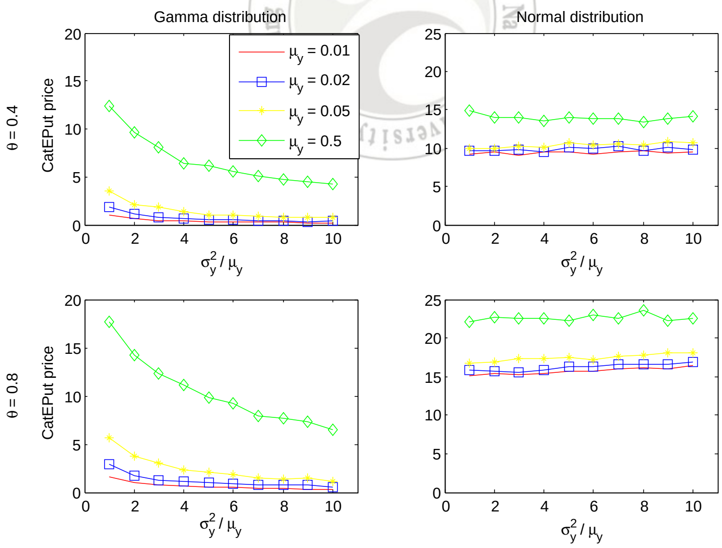

Figure 5: The CatEPut prices versus the ratio $\sigma_y^2/\mu_y = 1, 2, \cdots, 10$ under GARCH-Jump model with gamma and normally distributed jump sizes in the Merton measure, where $\mu_y = 0.01, 0.02, 0.05, 0.5$, $\theta = 0.4, 0.8$ and $T = 1$.

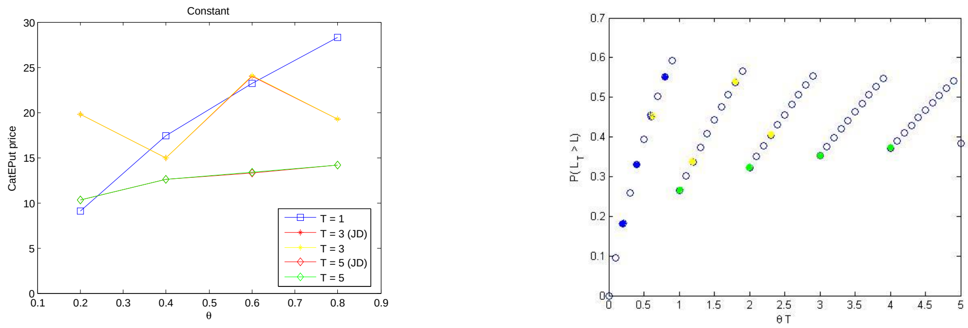

Figure 6: The CatEPut prices versus the intensity rate $\theta = 0.2, 0.4, 0.6, 0.8$ under GARCH-jump model and jump-diffusion (JD) model with constant jump size in the Merton measure, where $T = 1, 3, 5$.

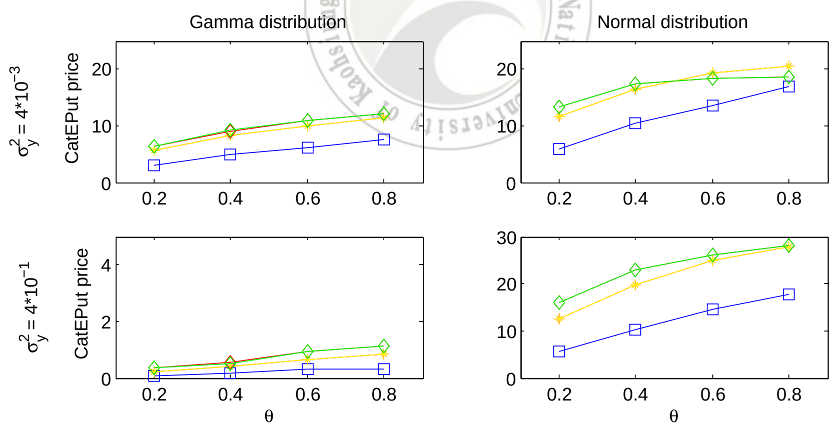

Figure 7: The CatEPut prices versus the intensity rate θ under GARCH-Jump model and jump-diffusion (JD) model in the Merton measure, where the notations for $T = 1, 3, 5$ are the same as those in Figure 6.

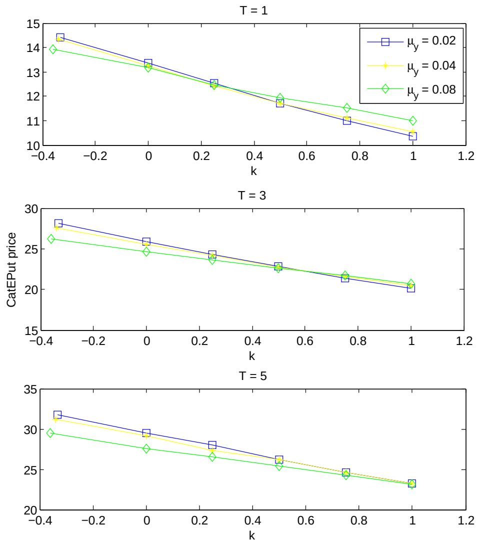

Figure 8: The CatEPut prices given in Table 4 versus k.

References

[1] Amin, K. I. (1993). Jump diffusion option valuation in discrete time. *Journal of Financial*, **48**, 1833-1863.

[2] Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometics*, **31**, 307-327.

[3] Cox, H., Fairchild, J., and Pedersen, H. (2004). Valuation of structured risk management preducts. *Insurance: Mathematics and Economics*, **34**, 259-272.

[4] Duan, J. C. (1995). The GARCH option pricing model. *Mathematical Finance*, **43**, 13-32.

[5] Duan, J. C. and Simonato, J. G. (1998). Empirical martingale simulation for asset prices. *Management Science*, **44**, 1218-1233.

[6] Engle, R. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica*, **50**, 987-1008.

[7] Gerber, H. U. and Shiu, S. W. (1994). Option pricing by Esscher transform. *Transactions of Society of Actuaries*, **46**, 99-191.

[8] Huang, S. F. and Guo, M. H. (2009) Financial derivative valution - A dynamic semiparametric approach. *Statistica*, **19**, 1037-1054.

[9] Huang, S. F. (2012). A modified empirical martingale simulation for financial derivative pricing. *Communications in Statistics - Theory and Methods.* Accepted.

[10] Huang, S. F., Liu, Y. C., and Wu, J. Y. (2012). An empirical study on implied GARCH models. *Journal of Data Science*, **10**, 87-105.

[11] Jaimungal, S. and Wang, T. (2006). Catastrophe options with stochastic interest rates and compound Poisson losses. *Insurance: Mathematics and Economics*, **38**, 469-483.

[12] Kou, S. G. (2002). A jump-diffusion model for option priceing. *Management Science*, **50**, 1178-1192.

[13] Lin, S. K., Chang, C. C., and Michael, R. P. (2009). The valuation of contingent capital with catastrophe risks. *Insurance: Mathematics and Economics*, **45**, 65-73.

[14] Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, **3**, 125-144.

[15] Shreve, S. E. (2004). *Stochastic Calculus for Finance II: Continuous Time Models*, (Springer, New York).

[16] Siu, T. K., Tong, H., and Yang, H. (2004). On pricing derivatives under GARCH models: A dynamic Gerber-Shiu approach. *North American Actuarial Journal*, **8**, 17-31.