1. Suppose $X_1, \cdots, X_n$ is a sample from $U(0, \theta)$. The maximum likelihood estimate of θ is $X_{(n)}$, the maximum of the sample. Since $X_{(n)} < \theta$ with probability 1, we might use $\frac{n+c}{n} X_{(n)}$ to be an estimator of θ for some positive number c. (a) What is the asymptotic distribution of $\frac{n+c}{n} X_{(n)}$? What value of c should be used if we measure the accuracy of the estimate by (b) squared error loss? (c) absolute error loss?

2. Let $X_1, \cdots, X_n$ be a sample from the geometric distribution with pdf, $P(X = x) = (1-\theta)\theta^x$ for $x = 0, 1, \cdots$, where $0 < \theta < 1$ is a success probability. Let $S_n = \sum_{i=1}^n X_i$ denote the total number of successes, and $T_n = \sum_{i=1}^n I(X_i > 0)$ denote the number of trials that had at least one success. (a) Find the joint asymptotic distribution of (S_n, T_n). (b) Find the joint asymptotic distribution of (U_n, V_n), where $U_n = S_n/T_n$ and $V_n = n - T_n$.

3. Let $X_1, X_2, \cdots$, be i.i.d. double exponential random variables with density,

$$f(x) = \exp(-|x|/\tau)/(2\tau),$$

where τ is a positive parameter that represents the mean deviation, i.e. $\tau = E|X|$. Let $\bar{X}_n = \sum_{i=1}^n X_i/n$ and $\bar{Y}_n = \sum_{i=1}^n |X_i|/n$. (a) Find the joint asymptotic distribution of $\bar{X}_n$ and $\bar{Y}_n$. (b) Find the asymptotic distribution of $(\bar{Y}_n - \tau)/\bar{X}_n$.

4. Let $X_1, X_2, \cdots$, be i.i.d. random variables with mean 0 and variance 1. Let $Y_i = a_i X_i, 1 \leq a_i \leq 2$ for all i. Find the asymptotic distribution of $\bar{Y}_n$ by the Lindeberg-Feller Theorem.

5. $X_1, X_2, \cdots$, be i.i.d. sequence of Bernoulli random variables with probability p of success. Then

$$\sqrt{n}(\bar{X}_n - p) \xrightarrow{\mathcal{L}} N(0, p(1-p))$$

as $n \to \infty$. Suppose we want to estimate the variance $\sigma^2 = p(1-p)$. The maximum likelihood estimate is $\hat{\sigma}^2 = \bar{X}_n(1 - \bar{X}_n)$. (a) Find the asymptotic distribution of $\hat{\sigma}^2$ if $p \neq 1/2$. (b) Find the asymptotic distribution of $\hat{\sigma}^2$ if $p = 1/2$.

6. If $Y_1, Y_2, \cdots$, are i.i.d random variables with mean μ and variance σ^2, and $X_i = Y_i Y_{i+1}$, what is the asymptotic distribution of $\bar{X}_n$?

7. (a) One measure of the homogeneity of a multinomial population with k cells and probabilities, $p = (p_1, \cdots, p_k)$, is the sum of the squares of the probabilities, $S(p) = \sum_{i=1}^k p_i^2$. Given a sample of size n from this population (with replacement), we may estimate $S(p)$ by $S(\hat{p})$, where $\hat{p} = (\hat{p}_1, \cdots, \hat{p}_k)$, and $\hat{p}_i$ is the proportion of the observations that fall in cell i. What is the asymptotic distribution of $S(\hat{p})$? (b) Let

$$H(p) = -\sum_{i=1}^{k} p_i \log p_i$$

is Shannon entropy. What is the asymptotic distribution of $H(\hat{p})$?

8. Suppose $X_1, X_2, \cdots,$ are i.i.d. unobservable (latent) random variables with an unknown distribution having finite mean μ and variance σ_x^2. The observations are $Y_t = \beta_0 X_t + \beta_1 X_{t+1} + \cdots + \beta_m X_{t+m}$ for $t = 1, 2, \cdots$, where the β_i are constants. Such a process $Y_1, Y_2, \cdots,$ is said to be a moving average process of order m. Find the asymptotic distribution of $\bar{Y}_n$.

9. Suppose $z_j = j$ and $a_j = 1/\sqrt{j}$ for all $j = 1, 2, \cdots$. Find the asymptotic distribution of $S_N = \sum_{i=1}^N z_j a(R_j)$, where $(R_1, \cdots, R_N)$ is a random permutation of $(1, \cdots, N)$ with each permutation having probability $1/n!$.