1. (10 points) Please write down the five major assumptions in Regression Analysis.

2. (35 points) Consider a no-intercept model, i.e. $y = \beta_1 x + \beta_2 x^2 + \varepsilon$, where $\varepsilon$ is a normal distribution with mean 0 and variance $\sigma^2$. Additionally we also assume that the errors are uncorrelated. Given $n$ observations, $(y_i, x_i), i = 1, 2, \ldots, n, n > 2$. Let $Y = (y_1, \ldots, y_n)'$ and $X = \begin{pmatrix} x_1 & x_1^2 \\ \vdots & \vdots \\ x_n & x_n^2 \end{pmatrix}$.

a. Find the least-square estimator of $\beta = (\beta_1, \beta_2)'$, $\hat{\beta}$.

b. Show $\hat{\beta}$ is an unbiased estimator of $\beta$ and find the covariance matrix of $\hat{\beta}$.

c. Give a geometrical interpretation of this least-square estimator.

d. Please write down $SS_{Res}$ in terms of $Y$ and $X$, and find the unbiased estimator of $\sigma^2$.

e. Find the MLE of $\beta$ and compare this MLE with the least-square estimator.

3. (15 points) Assume the model is $Y = X\beta + \varepsilon,$

where $Y = (y_1, \ldots, y_n)'$ is the vector of the observations; $X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix}$ is the design matrix; $\beta = (\beta_1, \ldots, \beta_p)'$ is the vector of the unknown parameters, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$ is a vector of errors with mean vector, 0, and covariance matrix, $\sigma^2 I_n$. Let $\hat{y}_{(i)}$ be the fitted value of the $i$th response based on all observations except the $i$th one, and define $e_{(i)} = y_i - \hat{y}_{(i)}$ to be the $i$th prediction errors. Show that $e_{(i)} = \frac{e_i}{1 - h_{ii}}$, where $e_i$ is the original $i$th residual and $h_{ii}$ is the $i$th diagonal elements of hat matrix.

(Hits: Let $X_{(i)}$ represent the original $X$ matrix with the $i$th row $x_i$ withheld. $h_{ii} = x_i'(X'X)^{-1} x_i$ and $[X_{(i)}'X_{(i)}]^{-1} = (X'X)^{-1} + \frac{(X'x_i)(x_i'x_i)(X'X)^{-1}}{1 - h_{ii}}$.)

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4. (15 points) Consider the normal probability plot.
   
   a. How to construct this plot?
   
   b. What is the purpose of this plot?
   
   c. If the observations come from a heavy-tailed distribution, then please show the corresponding normal probability plot.

5. (15 points) Give 32 observations, $y_1, \ldots, y_{32}$, and assume the model is $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$. Please fill the following ANOVA table up.

<table>
<thead>
<tr>
<th>Source Variation</th>
<th>Sum of Square</th>
<th>Degree of Freedom</th>
<th>Mean Square</th>
<th>$F_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Residual</td>
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<td></td>
</tr>
<tr>
<td>Total</td>
<td>1237.54</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6. (10 points) Consider the following two models where $E(\epsilon) = 0$ and $\text{Var}(\epsilon) = \sigma^2 I$:
   
   Model A: $y = X_1 \beta_1 + \epsilon$
   
   Model B: $y = X_1 \beta_1 + X_2 \beta_2 + \epsilon$.
   
   Show that $R^2_A \leq R^2_B$. 

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