



國立高雄大學統計學研究所

碩士論文

The Sample Size Study of Moderated Mediation Effect

調節式中介效果之樣本大小研究

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The Sample Size Study of Moderated Mediation Effect

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調節式中介效果之樣本大小研究

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摘要

在研究社會科學、教育學和心理學等時，常探討變數間是如何互相影響，因而發展出中介變數 (Mediator) 和調節變數 (Moderator)。調節式中介效果 (Moderated Mediation Effect) 是其中一種中介效果和調節效果的組合，指的是一中介變數影響解釋變數和相依變數之關係，而此中介效果會隨著調節變數的值而改變。檢定調節式中介效果的方法有很多種，許多研究者偏好使用以迴歸為基礎的檢定方法，其中常見的為係數的乘積 (the Product of Coefficients)，假設此乘積服從常態性，以 Sobel 所提的一階標準誤較為被廣泛使用 (First-Order Multivariate Delta Method)，但事實上此乘積並非為常態分佈，因此當樣本數不夠大時建議使用 Bootstrap 方法。

本文除了介紹中介效果模型和調節效果模型之外，我們延伸 Preacher 等人在 2007 年所列舉的五種調節式中介效果模型，探討在不同係數組合與五種檢定方法下，其檢定力之表現；更進一步提供研究者在研究調節式中介效果模型，決定樣本數大小時，為達到適當檢定力所需樣本數大小，給予一個可參考的依據。

關鍵字：中介變數、調節變數、調節式中介效果

The Sample Size Study of Moderated Mediation

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ABSTRACT

When researchers study social science, education, psychology and so on, they often study how and why variables are related. Hence, mediator and moderator are developed. Moderated mediation effect is one of the combinations of mediation and moderation effects. It refers to a mediator affecting the relation between an independent variable and a dependent variable, and then the mediation effect depends on the value of the moderator. There are many test methods of moderated mediation effects. Many researchers prefer to use regression-based tests. The most common method is the product of coefficients and is assumed the product is normally distributed. First-order multivariate delta method (Sobel, 1982, 1986) was commonly used. In fact, moderated mediation effect is not normally distributed. We recommend using the bootstrapping method when sample size is not large enough.

Except for introducing mediation and moderation models, we extended five moderated mediation models which were studied by Preacher et al. (2007). Using five testing methods to detect how many sample size required for different coefficients combination with appropriate statistical power. Moreover, we provide guidelines for researchers to study moderated mediation model in determining sample size.

Keywords: mediator, moderator, moderated mediation effect

1 Introduction

Mediation effect is a causal effect which the effect of an independent variable (X) on a dependent variable (Y) is transmitted by a mediator (M). Sometimes a mediation effect is also called an *indirect effect*. Testing the mediation effect became popular in psychology after publications by Judd and Kenny (1981) and Baron and Kenny (1986). Then researchers were interested in whether the mediation effect depends on other variables, moderators. There were two kinds of effect combined by mediation and moderation. Baron and Kenny (1986) referred to those effects as *mediated moderation* or *moderated mediation*. Preacher et al. (2007) gathered such effects as the *conditional indirect effect*. They defined a conditional effect as “the magnitude of an indirect effect at a particular value of a moderator (or more than one moderator).” Mediation and moderation are commonly used in social science, health, psychological, educational, and sociological research.

When researchers plan a study, they often think how many sample size they need. MacKinnon et al. (2002) studied empirical power for common sample size of mediation for many of testing methods. But it would more useful to know the sample size required for 0.8 power to detect an effect. Fritz and MacKinnon’s (2007) studied present the necessary sample size for six of the most common (according to the literature survey) and the most recommended tests of mediation for various combinations of parameters. The most common and the most recommended tests of mediation are *Baron and Kenny’s causal-steps test*, *joint significance test*, *Sobel first-order test*, *PRODCLIN*, *percentile bootstrap*, and *bias-corrected bootstrap*. They offered guidelines for researchers in determining sample size with statistical power 0.8 to conduct mediation effect.

Preacher et al. (2007) listed five specific ways to think about moderated mediation effects and performed a simulation study to examine empirical power and Type I error rate to provide some guidance on approximately sample sizes. However, they only studied simple conditions. They also did not clearly point out how many sample size were necessary when detecting moderated mediation effect. Hence, we extend their study and then investigate other conditions. In other words, we will investigate empirical power for various combinations of relevant path coefficients and for different methods of testing moderated mediation effect. In attempt to find an approximate sample sizes with appropriate statistic power. To provide guidelines for researchers when detecting moderated mediation effect.

At the beginning of this article, we reviewed the simple mediation model, moderation model and the moderated mediation model. In addition, this article's purpose is about the sample size. We will briefly discuss why sample size is important and discuss the relation between sample size and statistical power. Second, we will introduce several methods of testing moderated mediation effect, including multivariate delta methods and bootstrapping methods. Those testing methods will be used in our simulation study. Thirdly, we study empirical power and Type I error rate for moderated mediation effects under unequal relevant path coefficients in each moderated mediation model. We attempt to find some information from this simulation. Further, we try to find an approximately sample size required for 0.8 power to detect moderated mediation effect. At last, we provide guidelines of sample size for researchers to study moderated mediation model.

2 Literature Reviews

Moderated mediation is one of the combinations of mediation and moderation. We will review mediation and moderation in advance. There were five possibility formal path models of moderated mediation model described by Preacher et al. (2007). We will introduce those moderated mediation models respectively. In addition, the purpose of this article is to provide guidelines for researchers in determining the sample size for statistical power 0.8. Many people often have an idea of the better the larger sample size because it is easily to reject null hypothesis. Hence, we will briefly discuss the relation between power and sample size.

2.1 Mediation

Mediation effect or indirect effect is a casual effect which the effect of an independent variable on a dependent variable is transmitted by mediator. Mediator is a variable which is intermediate in the relation between an independent variable and a dependent variable. In other words, the independent variable causes the mediator, and in turn, the mediator causes the dependent variable then the relation of these variables form a mediation model. Mediation model not only addresses the question of how two variables are related but also why two variables are related. Hence, mediation model is commonly used in social science, education, psychology and so on. For example, how do drug abuse prevention programs reduce drug offer? Researchers are not only interested in how drug abuse prevention programs reduced drug offer but interested in why these two variables were related. Because a drug abuse prevention program may cause a participant's resistance to drugs, which in turn causes the outcome of a drug offer. And a drug abuse prevention program is the independent variable (treatment or control), resistance to drug use is the mediator and the outcome of a drug offer is the

dependent variable (acceptance or refusal) (MacKinnon, 2008; Preacher et al., 2007).

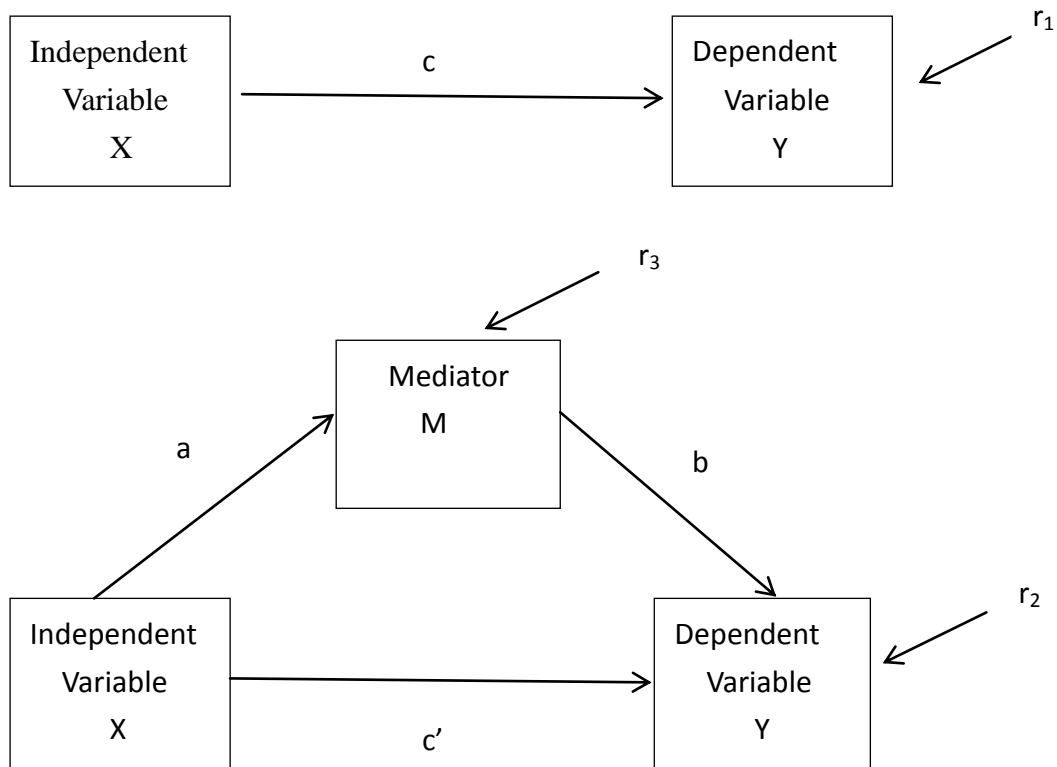


Figure 1 Simple Mediation path diagrams

Figure 1 is the path diagram of the simple mediation model. At the top of figure 1, the independent variable (X) directly leads to the dependent variable (Y) without consideration of other variables and it represents an overall effect model (or total effect model). At the bottom of figure 1, it represents the mediation model, the independent variable causes the mediator (M) which in turn causes the dependent variable. In addition, there is a relation of X to Y that is not through M and that is the direct effect of X on Y. Following three regression equations are used to investigate mediation.

$$Y = i_1 + cX + r_1 \quad (2.1)$$

$$Y = i_2 + c'X + bM + r_2 \quad (2.2)$$

$$M = i_3 + aX + r_3 \quad (2.3)$$

where c is the overall effect of the independent variable on the dependent variable; c' is the effect of the independent variable on the dependent variable controlling for the mediator; b is the effect of the mediator on the dependent variable controlling for the independent variable; a is the effect of the independent variable on the mediator; i_1 , i_2 and i_3 are the intercepts; r_1 , r_2 and r_3 are the residuals.

There have been some assumptions of the mediation regression equations. Each mediation regression equation requires the usual assumptions for regression analysis. There are four assumptions: correct function form, no omitted influences, accurate measurement, and well-behaved residuals. *Correct function form* refers to each mediation regression assumes linear relations among variables. Another aspect of the correct function form assumption is that relations among variables are additive, in other words, variables do not interact. *No omitted influences*, it is assumed that the mediation regression equations reflect the correct underlying model. No important variables or other influences are omitted from the regression model. *Accurate measurement* is that X, M, and Y are reliable and valid measures. *Well-behaved residuals*, the residuals in each equation are independent of each other, and the residuals are assumed to have constant variance at each value of the predictor variable (MacKinnon, 2008).

The coefficient relating the mediator to the dependent variable is estimated (\hat{b}) in equation 2.2. The coefficient relating the independent variable to the mediator is estimated (\hat{a}) in equation 2.3. The mediation effect or indirect effect equals the product of two estimates ($\hat{a}\hat{b}$). *The product of coefficients* \hat{a} and \hat{b} are used to estimate mediation effect in simple mediation model and this method is most easily

applied to complex models. Under the assumptions of maximum likelihood and ordinary least squares, \hat{a} and \hat{b} are asymptotically independent and normally distributed. It is further assumed that the product $\hat{a}\hat{b}$ is normally distributed. Sobel (1982,1986) derived the asymptotic standard error of $\hat{a}\hat{b}$ product based on first derivatives using the multivariate delta method. This is the most commonly used formulas for the standard error of the mediation effect.

$$SE_{\hat{a}\hat{b}} = \sqrt{\hat{a}^2 s_b^2 + \hat{b}^2 s_a^2}, \quad (2.4)$$

where s_a^2 and s_b^2 correspond to the square standard error of \hat{a} and \hat{b} , respectively. However, there are other standard error estimators for the mediation effect based on the product of coefficients.

$$SE_{\hat{a}\hat{b}} = \sqrt{\hat{a}^2 s_b^2 + \hat{b}^2 s_a^2 + s_a^2 s_b^2} \quad (2.5)$$

The standard error (2.5) is based on the second-order derivatives, although it is negligible that using second-order standard error to improves accuracy. But both standard errors are routinely reported in literates on simple mediation. It was found evidence that the first- and second-order standard errors had the least bias of several formulas for the mediation effect.

The null hypothesis is $ab = 0$, in other words, there is no mediation effect and the alternative hypothesis is $ab \neq 0$. To test for significance, the product is divided by the standard error of the product then the resulting ratio is compared to the standard normal distribution. For example, using first-order standard error, the test statistic is

$$Z \approx \frac{\hat{a}\hat{b}}{\sqrt{\hat{a}^2 s_b^2 + \hat{b}^2 s_a^2}} . \quad (2.6)$$

Then this statistic is compared to standard normal distribution. The $100(1 - \alpha)\%$ confidence limits for the indirect effect are

$$\hat{a}\hat{b} \pm Z_{\alpha/2}SE_{\hat{a}\hat{b}} , \quad (2.7)$$

where $Z_{\alpha/2}$ is the value on the standard normal distribution corresponding to the desired Type I error. The null hypothesis of no mediation effect is rejected at the α level of significance if the CI does not contain 0.

In fact, using the product of coefficients method is required the assumption that $\hat{a}\hat{b}$ is normally distributed. Because $\hat{a}\hat{b}$ is usually positively skewed and kurtotic. MacKinnon et al. (2004) explained that using this method to test for significance of the indirect effect has low statistical power and Type I error rates. In addition, the confidence limits are imbalanced. If the product of coefficients method is used to test for significance and the total sample size is not large, *bootstrapping* is recommended.

The alternative tests, *the distribution of the product*, have been shown to outperform traditional methods (MacKinnon et al. 2002; MacKinnon et al. 2004). A program called “PRODCLIN” (MacKinnon et al. 2007) used the product of two normally distributed variables to compute asymmetric confidence intervals for the mediation effect. In addition, there was the latest version of PRODCLIN program, REMEDIATION program (Tofighi and MacKinnon, 2011). It is an R package and solves several programs with the PRODCLIN package.

2.2 Moderation

Moderation model is testing for the relation between an independent variable and a dependent variable whether depends on another variable, moderator. Figure 2 represents the moderation model but most researchers prefer to use path diagram of Figure 3 now. Moderator is a variable that changes the form (direction) or strength of relation between two variables. For example, a researcher is not only interested in knowing whether a new instructional method leads to a better learning outcome but also interested in knowing whether the new instructional method is equally effective for students with low and high parental involvement (Wu and Zumbo, 2008). A new instructional method is the independent variable, learning outcome is the dependent variable and parental involvement is the moderator. A moderator variable can be continuous or categorical, although a categorical moderator variable will be easier to interpret (MacKinnon, 2008; Wu and Zumbo, 2008). A single regression equation forms the moderation model:

$$Y = a_0 + a_1X + a_2W + a_3XW + r , \quad (2.8)$$

where W is the moderator; a_0 is the intercept; a_1 is the relation between the independent variable and the dependent variable; a_2 is the relation between the moderator and the dependent variable; a_3 is the relation between the independent variable by a moderator and the dependent variable; r is the residual.

The variables X and W are often centered before the product is formed to improve interpretation of effects in the interaction model and to reduce collinearity among the measures, thereby improving the estimation of model parameters. The main reason of centering is that, unless the moderator has a meaningful zero point, the interpretation of the main effects, a_1 and a_2 , are meaningless. The dependent

variable is not necessary centered (Fairchild and MacKinnon, 2009; MacKinnon, 2008).

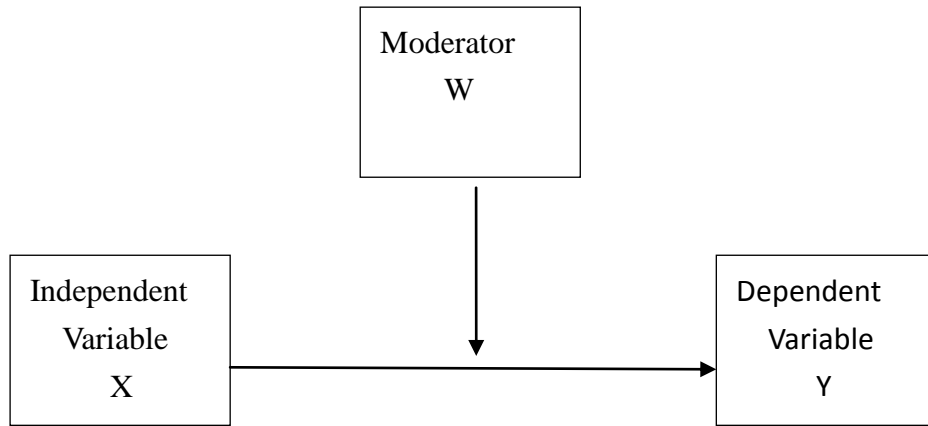


Figure 2 Moderation path diagram-1

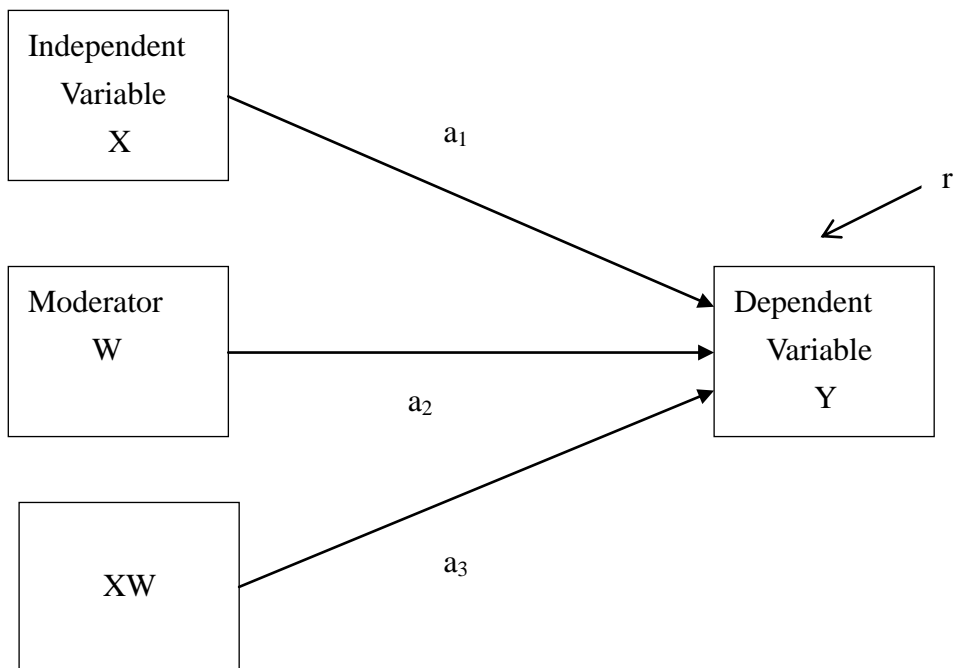


Figure 3 Moderation path diagram-2

The regression coefficient \hat{a}_3 is used to estimate moderation effect. If \hat{a}_3 statistically different from zero, it is say that there is a significance moderation of the

relation between X and Y. The coefficient \hat{a}_1 or \hat{a}_2 may be significant or nonsignificant, but this does not affect the test of moderation. In addition, in moderation, moderator is uncorrelated with independent variable or dependent variable.

The moderation effect is commonly known as an interaction effect. It is important to distinguish the difference between a moderation effect and an interaction effect. Interaction analysis has been extensively applied to correlational data and experimental data, therefore, the interaction term seems not necessarily causal in nature. But, the term of moderation effect needs to have causal hypotheses. In short, an interaction effect is not necessary causal hypothesis for models but a moderation effect is required the causal hypothesis. That is to say, a moderation effect is a special case of an interaction effect (Wu and Zumbo, 2008).

2.3 Moderated Mediation

There are two kinds of effects that combine mediation and moderation: *mediated moderation* and *moderated mediation*. Mediated moderation referred to the effect of an interaction on a dependent variable is mediated. Moderated mediation occurs when the strength of mediated effect depends on the level of a moderator. In other words, moderated mediation happens when a mediator is intermediate in the interaction between the independent variable and moderator to a dependent variable. However, we only limit our attention on moderated mediation.

There are many conflicting definitions about moderated mediation. For example, Morgan-Lopez and MacKinnon (2006) defined moderated mediation as when “the effect of the independent variable on mediator is constant and the effect of the mediator on the dependent variable depends on the level of moderator”. Muller et al.

(2005) defined moderated mediation “happens if the mediating process that is responsible for producing the effect of the treatment on the outcome depending on the value of a moderator variable.” Hence, Preacher et al. (2007) discussed several specific ways to think about moderated mediation effects, which they refer to as Models 1 through 5.

Model 1: When the Independent Variable is Also the Moderator

In this model, the independent variable (X) functions as a moderator of the b path. Figure 4 describes the case in which the effect of M on Y is moderated by the X. This model was described by Judd and Kenny (1981). And this model can also be understood as the effect of X on Y is moderated by M, that is the path of c' is moderated by M. The relevant regression equations for Model 1 are:

$$M = a_0 + a_1X + r_1, \quad (2.9)$$

$$Y = b_0 + c'X + b_1M + b_2XM + r_2. \quad (2.10)$$

Equation 10 can also be written as:

$$Y = b_0 + c'X + (b_1 + b_2X)M + r_2. \quad (2.11)$$

Equation 11 explains the regression of M on Y depends on X.

The point estimation of moderated mediation effect of Model 1 is $f(\hat{\theta}|X) = \hat{a}_1(\hat{b}_1 + \hat{b}_2X)$. The method of obtained point estimation is described by Sobel (1986) and Bollen (1987, 1989). The point estimation may also be obtained by the chain rule from elementary calculus. The detail of point estimation was discussed in the Appendix C.

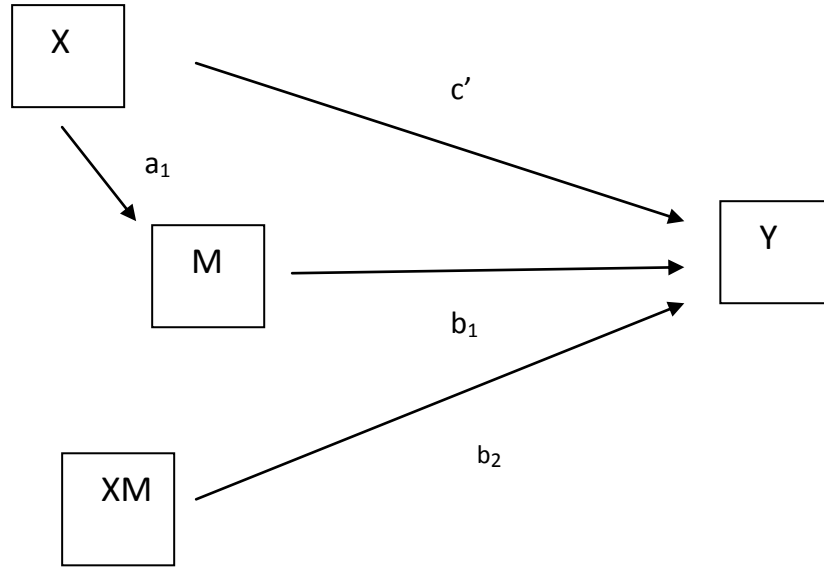


Figure 4 Model 1 path diagram

Model 2: When the a Path is Moderated by W

Figure 5 describes Model 2 of the a path is moderated by W . This model is the traditionally termed moderated mediation. The relevant regression equations for Model 2 are:

$$M = a_0 + a_1X + a_2W + a_3XW + r_1, \quad (2.12)$$

$$Y = b_0 + b_1M + c_1'X + c_2'W + c_3'XW + r_2. \quad (2.13)$$

This model can also be used to investigate another conditions, a moderated effect is mediated by M , mediated moderation. Although Model 2 can be used to discuss two types of effect, but different parameters are emphasized in each. Mediated moderation is addressed on the significance of the product $\hat{a}_3\hat{b}_1$ and moderated mediation is addressed on the point estimation $f(\hat{\theta}|W) = \hat{b}_1(\hat{a}_1 + \hat{a}_3W)$. We focus attention on moderated mediation: the effect of X on Y is mediated by M and whether mediation effect depends on the values of moderator W .

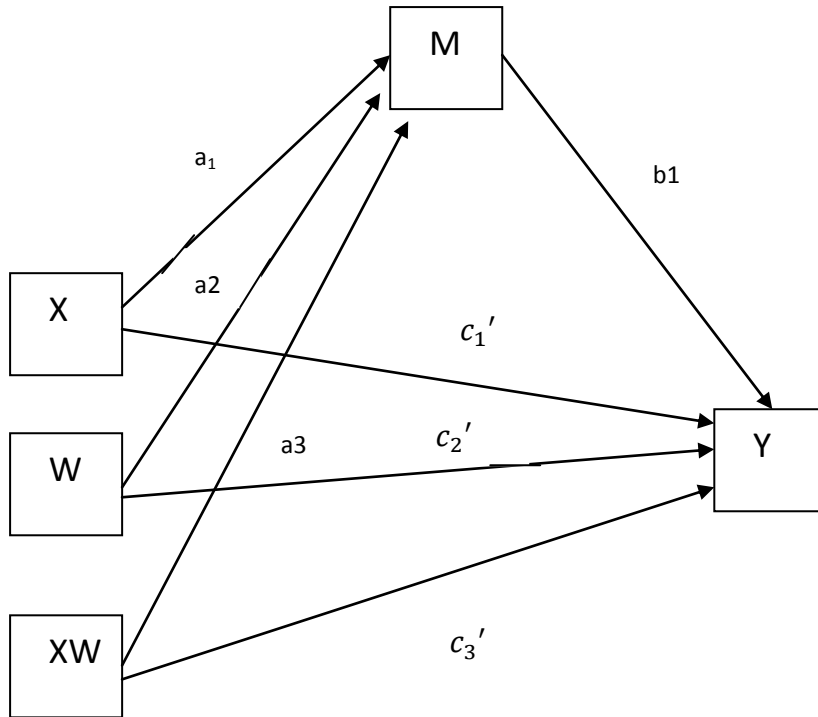


Figure 5 Model 2 path diagram

Model 3: When the b path is moderated by W

Figure 6 describes Model 3 of the b path is moderated by M . The relevant regression equations for Model 3 are:

$$M = a_0 + a_1X + r_1, \quad (2.14)$$

$$Y = b_0 + b_1M + b_2W + b_3MW + c'X + r_2 . \quad (2.15)$$

The moderated mediation effect is expressed as $f(\hat{\theta}|W) = \hat{a}_1(\hat{b}_1 + \hat{b}_3W)$.

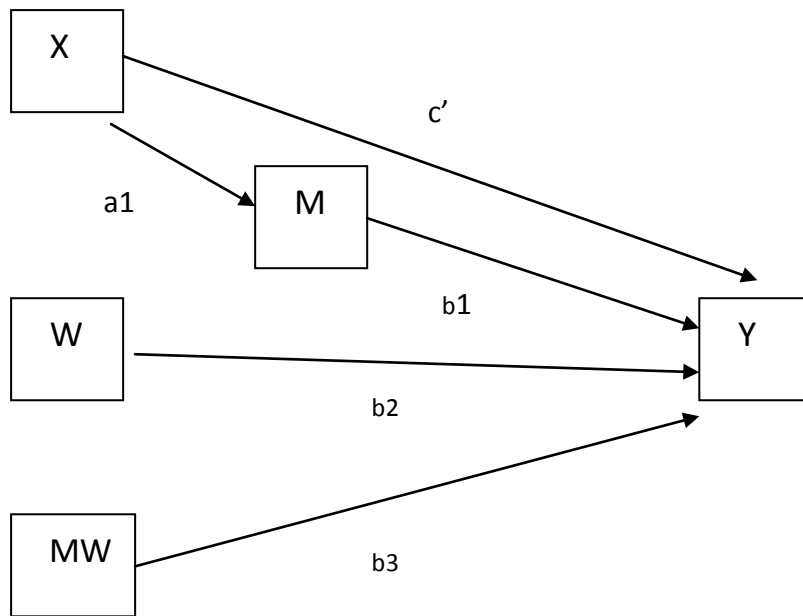


Figure 6 Model 3 path diagram

Model 4: When the *a* Path is Moderated by *W* and the *b* Path is Moderated by *Z*

This model is the extension of Model 2 and 3. The paths *a* and *b* are moderated by different moderators, *W* and *Z*. This model is described in figure 7. The relevant regression equations for Model 4 are:

$$M = a_0 + a_1X + a_2W + a_3XW + r_1, \quad (2.16)$$

$$Y = b_0 + b_1M + b_2Z + b_3MZ + c_1'X + c_2'W + c_3'XW + r_2. \quad (2.17)$$

The moderated mediation effect is expressed as $f(\hat{\theta}|W, Z) = (\hat{a}_1 + \hat{a}_3W)(\hat{b}_1 + \hat{b}_3Z)$.

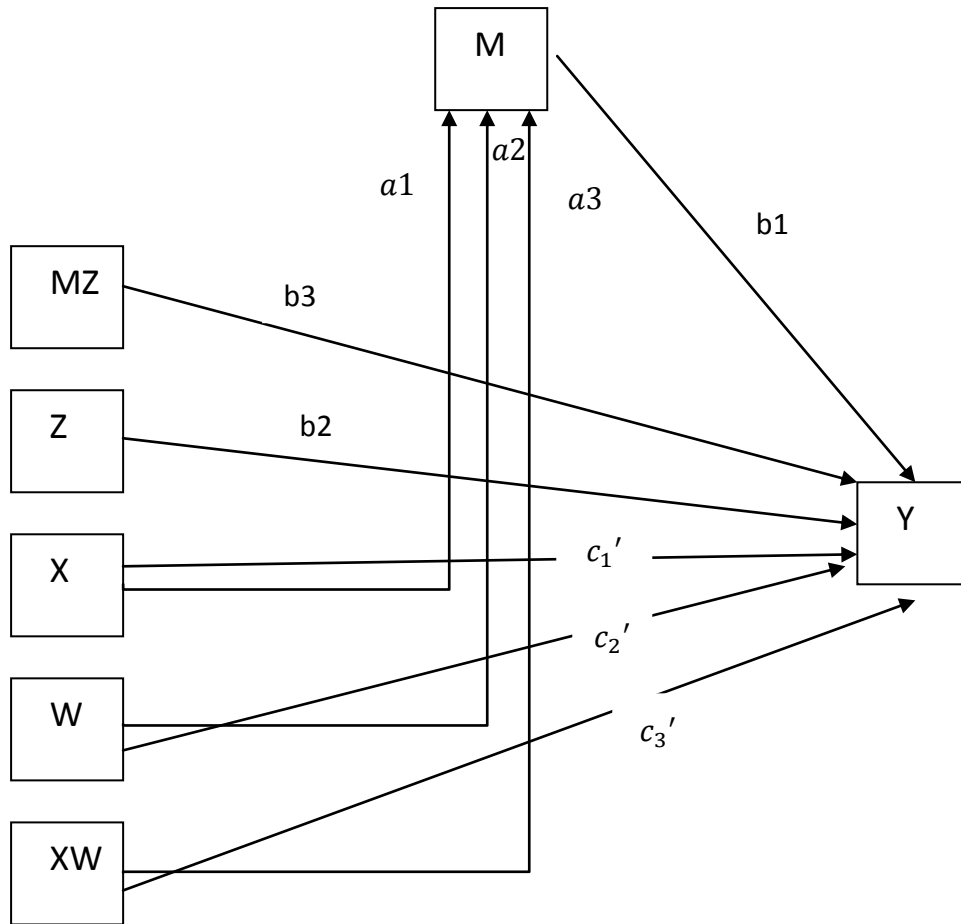


Figure 7 Model 4 path diagram

Model 5: When the a and b Paths are both Moderated by W

The a and b paths are both moderated by the same moderator W . This model describes in figure 8 and this figure is the same as the one Baron and Kenny (1986) described when they discussed moderated mediation and is also the model suggested by Muller et al. (2005) to address both mediated moderation and moderated mediation. The relevant regression equations for Model 5 are:

$$M = a_0 + a_1X + a_2W + a_3XW + r_1, \tag{2.18}$$

$$Y = b_0 + b_1M + b_2MW + c_1'X + c_2'W + c_3'XW + r_2. \tag{2.19}$$

The moderated mediation effect is expressed as $f(\hat{\theta}|W) = (\hat{a}_1 + \hat{a}_3W)(\hat{b}_1 + \hat{b}_2W)$.

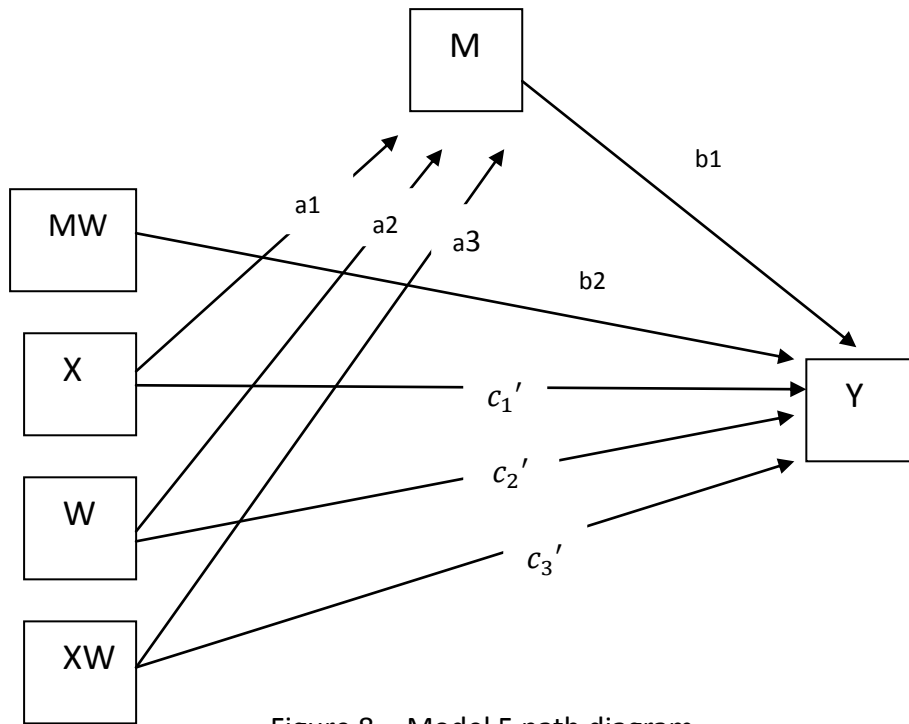


Figure 8 Model 5 path diagram

2.4 Power and Sample Size

When researchers plan a study, how many sample size are appropriate? Many people consider the better the larger sample size because it is easily to have a significant result for large sample size. But this idea is not true. The power of a statistical test depends on three parameters: the significance level, the reliability of the sample results and sample size, and the “effect size,” that is, the degree to which the effect exists. The relation between significance (α), power, sample size (n), and effect size (ES) are complementary. There are four types of power analysis; in each, one of these parameters is determined as a function of the other three (Cohen, 1988).

The role of significance level represents “the standard of the proof that the phenomenon exists, or the risk of mistakenly rejecting the null hypothesis (Cohen,

1988).” The significance level α (Type I error) represents the rate of rejecting a true null hypothesis, it is taken as a relatively small value. The lower the value of α , the poorer the opportunities are that the sample will provide results which meet this standard, that is the lower the power. The power of a statistical test is the probability that it will lead to the significant of results. In other words, statistical power is the correct probability of correctly rejecting a false null hypothesis. Type II error is the complement of the power (1-power). It represents the error probability of failing to reject a false null hypothesis. Generally, we set Type I error equal to 0.05 and Type II error equal to 0.2, that is power equal to 0.8 by Cohen’s (1988) advices.

Cohen (1988) defined effect size (ES) as “the degree to which the phenomenon is present in the population” or “the degree to which the null hypothesis is false.” For example, R^2 effect size measures are presented to assess variance accounted for in mediation models. The measures offer a means to evaluate both component paths and the overall mediated effect in mediation models (Fairchild et al., 2009). Other effect size measures for mediation, such as the partial r^2 and standardized regression coefficients. Cohen (1988) has suggested conventional values for “small”, “medium” and “large” effects in the social sciences. The larger the ES, other conditions (significance, sample size) being equal, the greater the power of the test. The larger the ES, other conditions (significance, power) being equal, the smaller the required sample size.

In our first part of simulation, power as a function of Type I error (α), effect size (ES), and sample size (n). We may decide to change our specifications (effect size or sample size) to increase power when Type I error is fixed. The second part of our simulation, the types of power analysis is “n as a function of ES, α , and power.” We wishes to have power equal to 0.8 under a certain ES and Type I error α .

3 Assessing Methods of Moderated Mediation Effects

3.1 Product of Coefficients

Normal-theory standard error is the most commonly used method of testing moderated mediation effect. Because it is most easily applied to complex models. Sobel (1982) and Bollen (1987, 1989) described the use of the first-order multivariate delta method in determining asymptotic SEs (standard errors) of a broad class of mediation effects. A second-order multivariate delta method can be used to yield the exact SE (Mood et al., 1974). By using these two methods can generate the first- and second-order variances of moderated mediation effect. The details of first- and second-order variance approximations for Models 1-5 were referred to appendix D. Table 1 describes first- and second-order variances of Models 1-5.

For testing the significance of moderated mediation effect, we assume the point estimator normality. The point estimator is divided by the standard error of the point estimator then the resulting ratio is compared to the standard normal distribution. For example of Model 2:

$$Z \approx \frac{\hat{b}_1(\hat{a}_1 + \hat{a}_3 W)}{\sqrt{(\hat{a}_1 + \hat{a}_3 W)^2 s_{\hat{b}_1}^2 + (\hat{b}_1^2 + s_{\hat{b}_1}^2)(s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2)}} \quad (3.1)$$

and comparing Z to the standard normal distribution.

This method is appropriate only if the sample size is large and the researcher has in mind a limited number of key conditional values of moderator(s). Because the test statistic is asymptotically normal, if the sample size is not large, bootstrapping is recommended (Preacher et al., 2007).

Table 1 First- and Second-Order Variances

Model	Second-order variance
1	$(\hat{b}_1 + \hat{b}_2 X)^2 s_{\hat{a}_1}^2 + (\hat{a}_1^2 + \underline{s_{\hat{a}_1}^2})(s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} X + s_{\hat{b}_1}^2 X^2)$
2	$(\hat{a}_1 + \hat{a}_3 W)^2 s_{\hat{b}_1}^2 + (\hat{b}_1^2 + \underline{s_{\hat{b}_1}^2})(s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2)$
3	$(\hat{b}_1 + \hat{b}_3 W)^2 s_{\hat{a}_1}^2 + (\hat{a}_1^2 + \underline{s_{\hat{a}_1}^2})(s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} W + s_{\hat{b}_3}^2 W^2)$
4	$(\hat{a}_1 + \hat{a}_3 W)^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} Z + s_{\hat{b}_3}^2 Z^2)$ $+ (\hat{b}_1 + \hat{b}_3 Z)^2 (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2)$ $+ \underline{(s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} Z + s_{\hat{b}_3}^2 Z^2)} (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2)$
5	$(\hat{b}_1 + \hat{b}_2 W)^2 (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2) +$ $+ (\hat{a}_1 + \hat{a}_3 W)^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} W + s_{\hat{b}_2}^2 W^2) +$ $+ \underline{(s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2)} (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} W + s_{\hat{b}_2}^2 W^2)$

Note. The first-order variance was omitted the part which second-order variance has been underline in each formula.

3.2 Percentile Bootstrap

Bootstrapping is the most recommended method for testing moderated mediation effect. Using bootstrapping, no assumptions needed about the shape of the sampling distribution, no particular formula for the *SE* is required (MacKinnon et al., 2004).

Take a sample size of n units from the original sample of n units. Using this resample, reestimate the point estimate. This process is repeated B times and then sorting the B bootstrap values of moderated mediation effect from low to high. The bootstrap confidence interval lower and upper bounds of a $100(1 - \alpha)\%$ CI for the population moderated mediation effect is defined as $(\alpha/2)B$ th and $(1 - \alpha/2)B$ th values in this sorted distribution, where α is the desired nominal Type I error rate. This confidence interval is called *percentile* bootstrap CI. For example, $\alpha = 0.05$, the 95% CI with $B=1,000$, the lower and upper bounds of the interval would be the 25th and 975th values of the moderated mediation effect in the sorted distribution. The null hypothesis of no moderated mediation effect is rejected at the α level of significance if the CI does not contain 0 (Preacher et al., 2007).

The percentile bootstrap CIs can be asymmetrical because they are based on an empirical estimation of the sampling distribution of the moderated mediation effect. The percentile bootstrap CIs may be inappropriate and can be further improved through *bias-corrected* (BC) and *bias-corrected and accelerated* (BCa) intervals (Preacher et al., 2007; MacKinnon et al., 2004).

3.3 Bias-Corrected Bootstrap

The problem of the percentile bootstrap is the confidence interval possibly will not be centered on the true parameter value. The bias-corrected (BC) bootstrap is adjusted

for bias in the bootstrap distribution. In other words, the bias-corrected bootstrap contains a correction for the bias created by the central tendency of the estimate. It is just like the percentile intervals except that different percentiles of the bootstrap distribution are used. The correction is made under the assumption that there is a monotonically increasing function T such that $T(\hat{\theta})$ is normal distributed with

$$E[T(\hat{\theta})] = T(\theta) - \hat{z}_0 \quad (3.2)$$

$$Var[T(\hat{\theta})] = 1 \quad (3.3)$$

where \hat{z}_0 is the bias, which is the z score of the value obtained from the proportion of bootstrap estimate below the original estimate in the total number of bootstrap. Let $\hat{\theta}^*_{(\alpha)}$ indicate the 100 α th percentile of B bootstrap replications. The BC interval of intended coverage $1 - \alpha$ is given by

$$(\hat{\theta}_{lo}, \hat{\theta}_{up}) = (\hat{\theta}^*_{(\alpha_1)}, \hat{\theta}^*_{(\alpha_2)}), \quad (3.4)$$

where

$$\alpha_1 = \Phi(2\hat{z}_0 + z_{\alpha/2}) \quad (3.5)$$

$$\alpha_2 = \Phi(2\hat{z}_0 + z_{1-\alpha/2}) \quad (3.6)$$

Here $\Phi(\cdot)$ is the standard normal cumulative distribution function and $z_{(\alpha/2)}$ is the (100 $\alpha/2$)th percentile point of a standard normal distribution. For example, $z_{(0.975)} = 1.96$ and $\Phi(1.96) = .975$. The null hypothesis of no moderated mediation effect is rejected at the α level of significance if the CI does not contain 0 (Efron and Tibshirani, 1993; MacKinnon et al., 2004).

3.4 Bias-Corrected and Accelerated Bootstrap

The percentile intervals are less erratic, but have less satisfactory coverage properties. The *bias-corrected* and *accelerated* (BCa) is an improved version of the percentile method. The BCa intervals are substantial improvement over the percentile method in both theory and practice. Same as in bias-corrected bootstrap, let $\hat{\theta}^*_{(\alpha)}$ indicate the 100α th percentile of B bootstrap replications. The BCa interval endpoints are also given by percentiles of the bootstrap distribution. The percentiles used depends on two numbers \hat{a} and \hat{z}_0 , called the *acceleration* and *bias-correction*. The BCa interval of intended coverage $1 - \alpha$ is given by

$$(\hat{\theta}_{lo}, \hat{\theta}_{up}) = (\hat{\theta}^*_{(\alpha_3)}, \hat{\theta}^*_{(\alpha_4)}), \quad (3.7)$$

where

$$\alpha_3 = \Phi \left(\hat{z}_0 + \frac{\hat{z}_0 + z_{(\alpha/2)}}{1 - \hat{a}(\hat{z}_0 + z_{(\alpha/2)})} \right) \quad (3.8)$$

$$\alpha_4 = \Phi \left(\hat{z}_0 + \frac{\hat{z}_0 + z_{(1-\alpha/2)}}{1 - \hat{a}(\hat{z}_0 + z_{(1-\alpha/2)})} \right) \quad (3.9)$$

Here $\Phi(\cdot)$ is the standard normal cumulative distribution function and $z_{(\alpha/2)}$ is the $(100 \alpha/2)$ th percentile point of a standard normal distribution.

The value of the bias-correction \hat{z}_0 is obtained directly from the proportion of bootstrap replications less than the original estimate $\hat{\theta}$,

$$\hat{z}_0 = \Phi^{-1} \left(\frac{\sum_{i=1}^B I(\hat{\theta}_b^* < \hat{\theta})}{B} \right), \quad (3.10)$$

Φ^{-1} indicating the inverse function of a standard normal cumulative distribution function, e.g., $\Phi^{-1}(.975) = 1.96$. \hat{z}_0 measures the median bias of $\hat{\theta}^*$, that is, the

discrepancy between the median of $\hat{\theta}^*$ and $\hat{\theta}$, in normal units.

There are various ways to compute the acceleration \hat{a} . The easiest to explain is given in terms of the *jackknife values* of a statistic $\hat{\theta} = s(\mathbf{x})$. Let \mathbf{x}_i be the original sample with i th point x_i deleted, let $\hat{\theta}_{(i)} = s(\mathbf{x}_i)$ and define $\hat{\theta}_{(\cdot)} = \sum_{i=1}^n \hat{\theta}_{(i)}/n$. A simple expression for the acceleration is

$$\hat{a} = \frac{\sum_{i=1}^n (\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^3}{6 \left\{ \sum_{i=1}^n (\hat{\theta}_{(\cdot)} - \hat{\theta}_{(i)})^2 \right\}^{3/2}} \quad (3.11)$$

The quantity \hat{a} is called the *acceleration* because it refers to the rate of change of the standard error of $\hat{\theta}$ with respect to the true parameter value θ , measured on a normalized scale (Efron and Tibshirani, 1993).

4 Simulation

4.1 Empirical Power

This simulation study was to examine the empirical power for the methods of testing moderated mediation effects across several conditions to provide some guidance on appropriate sample sizes. The data were simulated using moderated mediation models across five models. All variables were generated from a normal distribution with a mean of 0 and a variance of 1. There were five different sample sizes in simulation: 50, 100, 200, 500, and 1,000. For each simulation, the values for relevant coefficients were varied and those relevant coefficients values included 0, 0.14, 0.39, and 0.59, corresponding to zero, small (2% of the variance), medium (13% of the variance), and large (26% of the variance) effect sizes as described in Cohen (1988). For all models, the direct effect (c') was always set to 0 because previous simulation pointed out no difference in power calculations as the direct effect increased (MacKinnon et al., 2004) and other coefficients were also set to be equal to +1 in the population. In addition, all conditional values of moderator(s) were also set to be equal to +1.

The four effect sizes for relevant coefficients, five sample sizes and five methods totally have 1,600 combinations for Models 1-3 and 6,400 combinations for Models 4-5, respectively. The process of generating data sets for a specific combination, running regression analyses and then testing for significance was replicated a total of 1,000 times for each combination. For the bootstrap methods, a total of 1,000 resampled data sets of size N were taken from the original sample of size N with replacement. And the point estimate (moderated mediation effect) was calculated for each bootstrap sample. These 1,000 bootstrap point estimates were used to construct confidence intervals for the moderated mediation effect and these confidence intervals

were tested for significance by examining whether they contained zero. This process was done a total of 1,000 times for each combination. Then power for bootstrap methods was equal to the proportion of that the bootstrap confidence intervals detecting the moderated mediation effect. In addition, the Type I error α was set to be equal to 0.05. According to the literature survey, it would be more useful to know the sample size required for power 0.8 to detect an effect for researchers planning studies (Cohen, 1988).

4.1.1 Results and Discussion

Tables 2-6 represent the Type I error rates for Models 1-5, respectively. Tables 7-11 are empirical power for Models 1-5, respectively. Because the results are too much to display all conditions, we only display some results here. We didn't display the conditions of the product of relevant coefficients equal to zero and only display the results for sample sizes of power over 0.8. For example, in Model 1, we didn't list the results when $a_1=0$ or $b_1=b_2=0$. The cells values are represent proportions of 1,000 trials found significant at $\alpha=0.05$ in two-tailed z -tests. For example of Model 2 in Table 8, $b_1=0.39$, $a_1=0.14$ and $a_3=0.39$, the empirical power is 0.934, 0.930, 0.951, 0.966 and 0.966 for each methods when sample sizes equal to 100.

Here, empirical power for using first-order delta method variances is slightly larger than using second-order variances method. That is, rejection rates using second-order delta method variances slightly lower than first-order variances method. Type I error rates and empirical power for using bootstrap method is larger than using z -tests (first and second-order variances); but Type I error rates still not well for using bootstrap methods; bias-corrected bootstrap and bias-corrected and accelerated bootstrap results showed high power; bias-corrected bootstrap and bias-corrected and

accelerated bootstrap showed almost have same empirical power. These results are the same as the simulations studied by Preacher et al. (2007) which they conduct simple condition. As the relevant coefficients increased, the power was increased; the larger the sample size the better the power. In addition, if the product of population relevant coefficients was equal to zero, the power was very low.

Because Models 1-3 have one path are moderated by moderator and Models 4-5 have two paths are moderated by moderator. We divide Models 1-5 into two groups. Models 1-3 are first group and Models 4-5 are second group. The results of Models 1-3 are similar and Models 4-5 are similar. Statistical power for detecting moderated mediation effects was slight larger for Model 1 than Models 3, which in turn higher than Model 2; Model 5 are higher than Model 4.

4.2 Sample Size Study

From Tables 7-11, we can roughly guess how many sample size are needed to achieve adequate statistical power 0.8. For example, in Model 1, regression coefficients $b_1=0.14$, empirical power is over 0.8 when sample size reached 500 except two conditions ($a_1=0, a_3=0.14$ and $a_1=0.14, a_3=0$). We could guess that the required sample size is 200 to 500 for adequate statistical power 0.8. In Model 2, Table 8, regression coefficients $b_1=0.39$, we guess that it require 50 to 100 sample size for the most conditions.

Because the simulation range of sample size is broad, for example, 500 to 1,000. We couldn't clearly point out how many sample sizes were necessary when conducted moderated mediation effect for adequate statistical power. Using second-order standard errors has the smallest power and the required sample sizes is largest than other methods. To simplify, we refer to second-order standard errors as the baseline.

Hence, we try to find a smallest sample sizes for power 0.8 in each model.

4.2.1 Results and Discussion

Results are summaries in Tables 12 and 13. We know that statistical power for detecting moderated mediation effects was slight larger for Model 1 than Model 3, which in turn higher than Model 2 and Model 5 is higher than Model 4. Therefore, the required sample size was slight larger for Model 2 than Model 3, which in turn slight larger than Model 1 and Model 4 is slight larger than Model 5. Because we use 10 samples as a unit, the results may be same for some conditions.

For example, Table 12 (Models 1-3), if first regression coefficient is 0.14, the required sample size is around 400-470; regression coefficient combinations are (0.14, 0, 0.14) and (0.14, 0.14, 0), the required sample size is around 1040-1070; and (0.14, 0.14, 0.14) required sample size around 530-540. When first regression coefficient is 0.59, the required sample size is smaller than when first regression coefficient is 0.39 which in turn smaller than when first coefficient is 0.14. And this trend is same as in Models 4-5.

We found some rules from the study. The form of point estimate for Models 1-3 is $a(b+c)$. If the product of this point estimate is the same and the required sample size is almost equal. For example, if regression coefficients are (0.14, 0.14, 0.39) and (0.14, 0.39, 0.14), they all need sample sizes around 440 to reach appropriate statistical power 0.8. This conclusion is also the same for Models 4-5. The form of point estimate for Models 4-5 is $(a+b)(c+d)$. Therefore, if regression coefficients are (0.14, 0.39, 0.14, 0.39), (0.14, 0.39, 0.39, 0.14), (0.39, 0.14, 0.14, 0.39) and (0.39, 0.14, 0.39, 0.14), these four combinations all need sample size around 100.

5 Conclusions

Fritz and MacKinnon (2007) discussed the required sample size to detect the mediated effect. In their simulation, they had an initial sample sizes which were estimated using results from MacKinnon et al. (2002, 2004). First, they calculated empirical power for initial sample size compared to 0.8, and if the empirical power was larger than 0.8, the sample size for the next simulation was decreased, and if the empirical power was smaller than 0.8, the sample size was increased. This iterative process was repeated until the empirical power was within 0.001 of 0.8.

In our sample size study, the regression coefficients combinations are too many and we don't know the initial sample size. We just could guess rough sample size for empirical power over 0.8 in empirical power estimation. Then we use 10 samples as a unit and try to find required sample size for empirical power of 0.8. Although the sample sizes we study are not very accurate but it could provide guidelines for researchers studying moderated mediation model in determining sample size.

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Appendix A

Table 2 Type I Error Rates for Model 1

Sample Size	Test				
	first	second	boot	bc	bca
50	0	0	0.007	0.013	0.013
100	0	0	0.002	0.005	0.005
200	0.001	0.001	0.003	0.006	0.006
500	0	0	0.003	0.013	0.013
1000	0	0	0.002	0.005	0.006

Note. *First* refers to tests using first-order standard errors, *second* refers to using second-order standard errors, *boot* refers to rejection rates using percentile bootstrap confidence intervals, *bc* refers to rejection rates using bias-corrected bootstrap confidence intervals and *bca* refers to rejection rates using bias-corrected and accelerated bootstrap confidence intervals.

Table 3 Type I Error Rates for Model 2

Sample Size	Test				
	first	second	boot	bc	bca
50	0	0	0.001	0.004	0.004
100	0	0	0	0.007	0.007
200	0	0	0.002	0.006	0.006
500	0	0	0.002	0.005	0.005
1000	0	0	0.001	0.004	0.004

Table 4 Type I Error Rates for Model 3

Sample Size	Test				
	first	second	boot	bc	bca
50	0	0	0.002	0.006	0.007
100	0	0	0.002	0.007	0.008
200	0	0	0.001	0.004	0.004
500	0	0	0.003	0.007	0.007
1000	0	0	0	0.004	0.004

Table 5 Type I Error Rates for Model 4

Sample Size	Test				
	first	second	boot	bc	bca
50	0.001	0.001	0.003	0.004	0.005
100	0	0	0	0.004	0.004
200	0	0	0.003	0.011	0.012
500	0	0	0	0.004	0.004
1000	0	0	0	0.001	0.001

Table 6 Type I Error Rates for Model 5

Sample Size	Test				
	first	second	boot	bc	bca
50	0	0	0.001	0.003	0.002
100	0	0	0.002	0.006	0.006
200	0	0	0.001	0.009	0.009
500	0	0	0.002	0.011	0.01
1000	0	0	0.002	0.004	0.004

Table 7 Empirical Power for Model 1

Regression Coefficients			Sample	Test				
b_1	a_1	a_3	Size	first	second	boot	bc	bca
0.14	0	0.14	1000	0.768	0.748	0.849	0.879	0.879
		0.39	500	0.851	0.848	0.866	0.885	0.885
		0.59	500	0.870	0.868	0.879	0.883	0.883
	0.14	0	1000	0.806	0.781	0.868	0.898	0.898
		0.14	500	0.822	0.803	0.889	0.928	0.928
		0.39	500	0.863	0.858	0.872	0.882	0.882
	0.59	0	500	0.876	0.875	0.876	0.885	0.885
		0.14	500	0.845	0.844	0.868	0.890	0.896
		0.39	500	0.871	0.865	0.876	0.890	0.890
	0.39	0	500	0.871	0.869	0.882	0.886	0.886
		0.14	500	0.852	0.851	0.855	0.859	0.859
		0.39	500	0.863	0.863	0.878	0.887	0.887
		0.59	500	0.862	0.862	0.871	0.874	0.874
		0.39	500	0.863	0.862	0.862	0.861	0.861
		0.59	500	0.850	0.850	0.849	0.856	0.856
0.39	0	0.14	1000	0.905	0.905	0.906	0.913	0.913
		0.39	200	0.962	0.955	0.966	0.970	0.970
		0.59	100	0.921	0.900	0.953	0.963	0.963
	0.14	0	1000	0.913	0.913	0.920	0.919	0.919
		0.14	200	0.787	0.773	0.822	0.850	0.850
		0.39	100	0.846	0.830	0.912	0.939	0.939
	0.39	0	200	0.952	0.945	0.966	0.978	0.978
		0.14	100	0.969	0.964	0.979	0.981	0.981
		0.39	100	0.847	0.831	0.902	0.943	0.943
	0.59	0	100	0.952	0.950	0.969	0.976	0.976
		0.14	100	0.959	0.958	0.963	0.964	0.964
		0.39	100	0.892	0.886	0.931	0.950	0.950
		0.59	100	0.942	0.934	0.955	0.969	0.969
		0.39	50	0.701	0.681	0.756	0.805	0.805
		0.59	50	0.724	0.711	0.775	0.804	0.804
0.59	0	0.14	1000	0.939	0.939	0.941	0.941	0.941
		0.39	100	0.798	0.794	0.811	0.846	0.846
		0.59	50	0.860	0.844	0.882	0.920	0.920
	0.14	0	1000	0.933	0.931	0.937	0.940	0.940

Table 7 Empirical Power for Model 1 (Continue)

Regression Coefficients			Sample	Test				
b_1	a_1	a_3	Size	first	second	boot	bc	bca
		0.14	200	0.857	0.853	0.864	0.873	0.873
		0.39	100	0.959	0.956	0.967	0.972	0.972
		0.59	50	0.860	0.844	0.882	0.920	0.920
	0.39	0	100	0.798	0.787	0.818	0.841	0.841
		0.14	100	0.964	0.961	0.963	0.969	0.969
		0.39	50	0.897	0.886	0.913	0.944	0.944
		0.59	50	0.938	0.935	0.950	0.967	0.967
	0.59	0	100	0.979	0.978	0.982	0.986	0.986
		0.14	50	0.855	0.835	0.884	0.925	0.925
		0.39	50	0.945	0.942	0.956	0.967	0.967
		0.59	50	0.960	0.956	0.964	0.972	0.972

Table 8 Empirical Power for Model 2

Regression Coefficients			Sample	Test				
b_1	a_1	a_3	Size	first	second	boot	bc	bca
0.14	0	0.14	1000	0.817	0.801	0.883	0.915	0.915
		0.39	500	0.869	0.864	0.895	0.909	0.909
		0.59	500	0.858	0.856	0.867	0.867	0.867
	0.14	0	1000	0.78	0.761	0.869	0.891	0.891
		0.14	500	0.778	0.759	0.856	0.898	0.898
		0.39	500	0.861	0.853	0.863	0.875	0.875
		0.59	500	0.880	0.878	0.881	0.884	0.884
	0.39	0	500	0.845	0.834	0.877	0.896	0.896
		0.14	500	0.869	0.864	0.878	0.892	0.892
		0.39	500	0.880	0.875	0.879	0.883	0.882
		0.59	500	0.868	0.865	0.867	0.871	0.871
	0.59	0	500	0.872	0.867	0.879	0.892	0.892
		0.14	500	0.881	0.879	0.880	0.885	0.885
		0.39	500	0.888	0.888	0.888	0.892	0.892
		0.59	500	0.872	0.872	0.873	0.881	0.881
0.39	0	0.14	1000	0.846	0.845	0.840	0.849	0.849
		0.39	200	0.929	0.923	0.943	0.953	0.953

Table 8 Empirical Power for Model 2 (Continue)

Regression Coefficients			Sample	Test				
b_1	a_1	a_3	Size	first	second	boot	bc	bca
		0.59	100	0.874	0.862	0.924	0.954	0.954
	0.14	0	1000	0.875	0.873	0.879	0.886	0.886
		0.14	500	0.984	0.984	0.988	0.988	0.988
		0.39	100	0.826	0.798	0.886	0.921	0.921
		0.59	100	0.934	0.925	0.954	0.971	0.971
	0.39	0	200	0.958	0.953	0.966	0.971	0.971
		0.14	100	0.819	0.796	0.885	0.941	0.941
		0.39	100	0.934	0.930	0.951	0.966	0.966
		0.59	100	0.944	0.943	0.959	0.959	0.959
	0.59	0	100	0.869	0.856	0.927	0.950	0.950
		0.14	100	0.941	0.932	0.955	0.967	0.967
		0.39	100	0.952	0.949	0.963	0.966	0.966
		0.59	100	0.962	0.960	0.957	0.964	0.964
0.59	0	0.14	1000	0.869	0.869	0.871	0.873	0.873
		0.39	200	0.958	0.957	0.959	0.966	0.966
		0.59	100	0.970	0.965	0.961	0.973	0.973
	0.14	0	1000	0.863	0.862	0.853	0.858	0.858
		0.14	500	0.992	0.992	0.989	0.990	0.990
		0.39	100	0.917	0.911	0.915	0.929	0.929
		0.59	50	0.787	0.769	0.824	0.870	0.870
	0.39	0	200	0.961	0.957	0.956	0.959	0.959
		0.14	100	0.992	0.918	0.923	0.943	0.943
		0.39	50	0.828	0.809	0.862	0.907	0.907
		0.59	50	0.925	0.919	0.939	0.953	0.953
	0.59	0	100	0.957	0.954	0.964	0.973	0.973
		0.14	50	0.785	0.768	0.815	0.882	0.882
		0.39	50	0.917	0.909	0.928	0.956	0.956
		0.59	50	0.943	0.937	0.940	0.961	0.961

Table 9 Empirical Power for Model 3

Regression Coefficients			Sample	Test				
b_1	a_1	a_3	Size	first	second	boot	bc	bca
0.14	0	0.14	1000	0.799	0.776	0.877	0.902	0.902
		0.39	500	0.841	0.828	0.860	0.876	0.876
		0.59	500	0.871	0.869	0.876	0.886	0.886
	0.14	0	1000	0.780	0.756	0.854	0.897	0.897
		0.14	500	0.789	0.778	0.861	0.901	0.901
		0.39	500	0.867	0.863	0.876	0.892	0.892
	0.39	0.59	500	0.876	0.873	0.877	0.886	0.886
		0	500	0.846	0.840	0.873	0.890	0.890
		0.14	500	0.884	0.881	0.889	0.900	0.900
	0.59	0.39	500	0.885	0.883	0.892	0.893	0.892
		0.59	500	0.845	0.845	0.839	0.844	0.844
		0	500	0.853	0.850	0.861	0.869	0.869
		0.14	500	0.883	0.881	0.881	0.890	0.890
	0.39	0	0.14	1000	0.881	0.878	0.887	0.888
0.39			200	0.955	0.952	0.963	0.972	0.972
0.59			100	0.894	0.882	0.938	0.966	0.966
0.14		0	1000	0.881	0.879	0.881	0.885	0.885
		0.14	200	0.745	0.730	0.793	0.826	0.826
		0.39	100	0.822	0.808	0.885	0.921	0.921
0.39		0.59	100	0.924	0.917	0.950	0.963	0.963
		0	200	0.949	0.946	0.962	0.972	0.972
		0.14	100	0.850	0.830	0.890	0.929	0.929
0.59		0.39	100	0.947	0.942	0.966	0.975	0.975
		0.59	100	0.959	0.958	0.964	0.973	0.973
		0	100	0.899	0.889	0.936	0.955	0.955
		0.14	100	0.936	0.931	0.951	0.965	0.965
0.59		0	0.39	100	0.959	0.955	0.960	0.964
	0.59		50	0.735	0.725	0.771	0.808	0.808
	0.14		1000	0.923	0.923	0.921	0.923	0.923
	0.14	0	1000	0.923	0.923	0.921	0.923	0.923
		0.39	100	0.774	0.764	0.784	0.815	0.815
		0.59	50	0.696	0.678	0.750	0.801	0.801

Table 9 Empirical Power for Model 3 (Continue)

Regression Coefficients			Sample	Test				
b_1	a_1	a_3	Size	first	second	boot	bc	bca
		0.14	200	0.835	0.828	0.831	0.850	0.850
		0.39	100	0.924	0.920	0.938	0.950	0.950
		0.59	50	0.835	0.819	0.860	0.907	0.907
	0.39	0	200	0.974	0.972	0.978	0.979	0.979
		0.14	100	0.934	0.931	0.931	0.947	0.947
		0.39	50	0.849	0.835	0.878	0.911	0.911
		0.59	50	0.941	0.935	0.959	0.974	0.974
	0.59	0	100	0.981	0.980	0.981	0.987	0.987
		0.14	50	0.847	0.833	0.880	0.911	0.911
		0.39	50	0.940	0.932	0.947	0.961	0.961
		0.59	50	0.956	0.955	0.968	0.981	0.981

Table 10 Empirical Power for Model 4

a ₁	a ₃	b ₁	b ₃	Sample		Test				
				Size	first	second	boot	bc	bca	
0	0.14	0	0.14	1000	0.666	9,642	0.816	0.847	0.847	
			0.39	1000	0.869	0.867	0.871	0.878	0.878	
			0.59	1000	0.854	0.853	0.856	0.853	0.853	
			0	1000	0.673	0.647	0.801	0.862	0.862	
			0.14	1000	0.834	0.827	0.848	0.862	0.862	
		0.39	1000	0.870	0.869	0.879	0.879	0.879		
		0.59	1000	0.882	0.882	0.880	0.883	0.883		
		0.39	0	1000	0.883	0.878	0.875	0.886	0.886	
			0.14	1000	0.869	0.869	0.862	0.866	0.866	
			0.39	1000	0.867	0.866	0.873	0.874	0.874	
	0.59		1000	0.883	0.882	0.871	0.870	0.870		
	0		1000	0.868	0.866	0.865	0.875	0.875		
	0.39	0	0.14	1000	0.864	0.862	0.856	0.860	0.860	
			0.39	1000	0.898	0.898	0.899	0.901	0.901	
			0.59	1000	0.887	0.887	0.884	0.885	0.885	
			0.14	1000	0.944	0.944	0.952	0.960	0.906	
			0.39	200	0.894	0.876	0.929	0.947	0.947	
		0.59	0	0.14	200	0.963	0.958	0.958	0.967	0.967
				0	1000	0.950	0.939	0.946	0.951	0.951
				0.14	200	0.720	0.701	0.837	0.882	0.882
0.39				200	0.958	0.955	0.963	0.972	0.972	
0.59				200	0.949	0.945	0.954	0.955	0.955	
0.39	0		200	0.936	0.928	0.951	0.969	0.969		
	0.14		200	0.950	0.959	0.963	0.969	0.969		
	0.39		200	0.965	0.965	0.962	0.965	0.965		
	0.59		200	0.955	0.955	0.957	0.961	0.961		
	0		200	0.953	0.951	0.954	0.958	0.958		
0.59	0.14		200	0.971	0.968	0.976	0.979	0.979		
	0.39		200	0.955	0.955	0.955	0.958	0.958		
	0.59		200	0.953	0.953	0.949	0.951	0.951		
	0.14		1000	0.947	0.947	0.950	0.954	0.954		
	0.39		100	0.729	0.705	0.807	0.858	0.858		
0.14	0	0.59	100	0.928	0.911	0.952	0.963	0.963		
		0	1000	0.947	0.947	0.947	0.947	0.947		

Table 10 Empirical Power for Model 4 (Continue)

a ₁	a ₃	b ₁	b ₃	Sample	Test				
				Size	first	second	boot	bc	bca
			0.14	200	0.866	0.858	0.887	0.910	0.910
			0.39	100	0.904	0.891	0.928	0.956	0.956
			0.59	100	0.962	0.960	0.966	0.979	0.979
		0.39	0	100	0.727	0.700	0.809	0.861	0.861
			0.14	100	0.921	0.912	0.956	0.964	0.964
			0.39	100	0.967	0.962	0.963	0.976	0.976
			0.59	100	0.972	0.971	0.967	0.977	0.977
		0.59	0	100	0.944	0.938	0.967	0.977	0.977
			0.14	100	0.946	0.942	0.951	0.967	0.967
			0.39	100	0.963	0.961	0.957	0.963	0.963
			0.59	100	0.964	0.964	0.959	0.962	0.962
0.14	0	0	0.14	1000	0.684	0.658	0.818	0.869	0.869
			0.39	1000	0.860	0.859	0.867	0.873	0.873
			0.59	1000	0.877	0.877	0.877	0.877	0.877
		0.14	0	1000	0.679	0.650	0.823	0.879	0.879
			0.14	1000	0.850	0.846	0.859	0.874	0.874
			0.39	1000	0.858	0.857	0.863	0.868	0.868
			0.59	1000	0.880	0.880	0.883	0.883	0.883
		0.39	0	1000	0.850	0.845	0.862	0.871	0.872
			0.14	1000	0.872	0.869	0.972	0.880	0.880
			0.39	1000	0.856	0.856	0.845	0.846	0.846
			0.59	1000	0.864	0.864	0.862	0.863	0.863
		0.59	0	1000	0.882	0.880	0.885	0.886	0.886
			0.14	1000	0.881	0.881	0.880	0.875	0.875
			0.39	1000	0.877	0.877	0.869	0.870	0.870
			0.59	1000	0.876	0.876	0.873	0.873	0.873
	0.14	0	0.14	1000	0.934	0.928	0.945	0.951	0.951
			0.39	200	0.677	0.651	0.756	0.809	0.809
			0.59	500	0.987	0.986	0.991	0.991	0.991
		0.14	0	1000	0.931	0.929	0.945	0.956	0.956
			0.14	500	0.981	0.978	0.985	0.992	0.992
			0.39	500	0.991	0.991	0.988	0.990	0.990
			0.59	500	0.993	0.993	0.991	0.991	0.991
		0.39	0	500	0.983	0.983	0.985	0.986	0.986

Table 10 Empirical Power for Model 4 (Continue)

a ₁	a ₃	b ₁	b ₃	Sample	Test				
				Size	first	second	boot	bc	bca
			0.14	500	0.992	0.992	0.989	0.991	0.991
			0.39	500	0.991	0.991	0.992	0.991	0.991
			0.59	500	0.993	0.993	0.990	0.989	0.989
		0.59	0	500	0.986	0.986	0.990	0.990	0.990
			0.14	500	0.992	0.992	0.993	0.992	0.992
			0.39	500	0.991	0.991	0.992	0.991	0.991
			0.59	500	0.993	0.993	0.990	0.989	0.989
	0.39	0	0.14	1000	0.959	0.958	0.956	0.961	0.961
			0.39	200	0.986	0.983	0.985	0.988	0.988
			0.59	100	0.889	0.876	0.920	0.941	0.941
		0.14	0	1000	0.951	0.950	0.952	0.954	0.954
			0.14	200	0.855	0.844	0.882	0.905	0.905
			0.39	100	0.846	0.830	0.894	0.926	0.926
			0.59	100	0.914	0.911	0.917	0.949	0.949
		0.39	0	200	0.980	0.978	0.985	0.985	0.985
			0.14	100	0.846	0.831	0.892	0.932	0.932
			0.39	100	0.919	0.912	0.917	0.932	0.932
			0.59	100	0.930	0.927	0.923	0.932	0.932
		0.59	0	100	0.878	0.865	0.914	0.939	0.939
			0.14	100	0.906	0.901	0.913	0.929	0.929
			0.39	100	0.938	0.936	0.937	0.947	0.947
			0.59	100	0.924	0.923	0.917	0.921	0.921
	0.59	0	0.14	1000	0.954	0.954	0.952	0.953	0.953
			0.39	100	0.792	0.779	0.833	0.855	0.855
			0.59	100	0.982	0.974	0.986	0.992	0.992
		0.14	0	1000	0.961	0.961	0.961	0.962	0.962
			0.14	200	0.898	0.891	0.917	0.924	0.924
			0.39	100	0.964	0.961	0.968	0.976	0.976
			0.59	50	0.771	0.749	0.788	0.848	0.848
		0.39	0	100	0.788	0.772	0.821	0.854	0.854
			0.14	100	0.968	0.964	0.974	0.980	0.908
			0.39	50	0.797	0.788	0.822	0.833	0.883
			0.59	50	0.850	0.839	0.857	0.884	0.884
		0.59	0	100	0.979	0.977	0.984	0.988	0.988

Table 10 Empirical Power for Model 4 (Continue)

a ₁	a ₃	b ₁	b ₃	Sample	Test				
				Size	first	second	boot	bc	bca
			0.14	50	0.767	0.749	0.791	0.853	0.853
			0.39	50	0.840	0.835	0.846	0.893	0.892
			0.59	50	0.865	0.861	0.849	0.871	0.871
0.39	0	0	0.14	1000	0.952	0.949	0.952	0.954	0.954
			0.39	200	0.906	0.895	0.945	0.965	0.965
			0.59	200	0.939	0.935	0.942	0.953	0.953
		0.14	0	1000	0.947	0.946	0.953	0.961	0.961
			0.14	200	0.735	0.709	0.834	0.878	0.878
			0.39	200	0.961	0.958	0.966	0.974	0.974
			0.59	200	0.951	0.960	0.954	0.956	0.956
		0.39	0	200	0.902	0.890	0.942	0.957	0.957
			0.14	200	0.944	0.944	0.949	0.962	0.962
			0.39	200	0.967	0.967	0.972	0.971	0.971
			0.59	200	0.959	0.956	0.957	0.957	0.957
		0.59	0	200	0.953	0.948	0.956	0.963	0.963
			0.14	200	0.953	0.951	0.958	0.963	0.963
			0.39	200	0.965	0.965	0.956	0.961	0.961
			0.59	200	0.962	0.961	0.961	0.963	0.963
	0.14	0	0.14	1000	0.950	0.948	0.953	0.956	0.956
			0.39	200	0.989	0.985	0.990	0.996	0.996
			0.59	100	0.900	0.889	0.925	0.949	0.949
		0.14	0	1000	0.956	0.956	0.958	0.960	0.960
			0.14	200	0.838	0.829	0.872	0.899	0.899
			0.39	100	0.844	0.821	0.894	0.925	0.925
			0.59	100	0.919	0.914	0.917	0.928	0.928
		0.39	0	200	0.978	0.973	0.984	0.987	0.987
			0.14	100	0.844	0.833	0.878	0.912	0.912
			0.39	100	0.920	0.917	0.926	0.936	0.936
			0.59	100	0.930	0.930	0.925	0.935	0.935
		0.59	0	100	0.894	0.883	0.925	0.941	0.941
			0.14	100	0.905	0.899	0.909	0.923	0.923
			0.39	100	0.930	0.929	0.916	0.937	0.937
			0.59	100	0.950	0.949	0.937	0.937	0.937
	0.39	0	0.14	1000	0.955	0.955	0.956	0.955	0.955

Table 10 Empirical Power for Model 4 (Continue)

Regression coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₃	Size	first	second	boot	bc	bca
			0.39	100	0.817	0.806	0.840	0.876	0.876
			0.59	100	0.981	0.977	0.983	0.993	0.993
		0.14	0	1000	0.962	0.962	0.959	0.960	0.960
			0.14	200	0.888	0.883	0.888	0.898	0.898
			0.39	100	0.973	0.967	0.979	0.979	0.979
			0.59	50	0.795	0.769	0.822	0.875	0.875
		0.39	0	100	0.824	0.811	0.840	0.877	0.877
			0.14	100	0.970	0.967	0.968	0.974	0.974
			0.39	50	0.820	0.798	0.849	0.889	0.889
			0.59	50	0.884	0.874	0.873	0.906	0.905
		0.59	0	100	0.981	0.978	0.984	0.988	0.988
			0.14	50	0.826	0.815	0.824	0.886	0.886
			0.39	50	0.893	0.883	0.875	0.910	0.910
			0.59	50	0.892	0.889	0.879	0.901	0.901
	0.59	0	0.14	1000	0.968	0.968	0.967	0.966	0.966
			0.39	100	0.845	0.835	0.846	0.864	0.864
			0.59	100	0.994	0.992	0.992	0.995	0.995
		0.14	0	1000	0.966	0.966	0.963	0.960	0.960
			0.14	200	0.887	0.886	0.887	0.939	0.939
			0.39	100	0.979	0.978	0.978	0.981	0.981
			0.59	50	0.903	0.896	0.900	0.928	0.928
		0.39	0	100	0.825	0.822	0.830	0.855	0.855
			0.14	100	0.980	0.980	0.975	0.975	0.975
			0.39	50	0.917	0.906	0.922	0.945	0.945
			0.59	50	0.966	0.960	0.956	0.971	0.971
		0.59	0	50	0.771	0.761	0.796	0.846	0.846
			0.14	50	0.888	0.876	0.894	0.925	0.925
			0.39	50	0.967	0.963	0.956	0.970	0.970
			0.59	50	0.974	0.974	0.970	0.978	0.978
0.59	0	0	0.14	1000	0.957	0.957	0.953	0.955	0.955
			0.39	200	0.988	0.985	0.992	0.993	0.993
			0.59	100	0.935	0.924	0.950	0.974	0.974
		0.14	0	1000	0.959	0.958	0.957	0.961	0.961
			0.14	200	0.875	0.863	0.897	0.915	0.915

Table 10 Empirical Power for Model 4 (Continue)

Regression Coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₃	Size	first	second	boot	bc	bca
			0.39	100	0.930	0.920	0.942	0.960	0.960
			0.59	100	0.952	0.950	0.958	0.965	0.965
		0.39	0	100	0.744	0.720	0.807	0.858	0.858
			0.14	100	0.912	0.893	0.934	0.959	0.959
			0.39	100	0.959	0.950	0.956	0.966	0.966
			0.59	100	0.960	0.957	0.960	0.966	0.966
		0.59	0	100	0.924	0.921	0.936	0.955	0.955
			0.14	100	0.956	0.956	0.969	0.976	0.976
			0.39	100	0.957	0.956	0.953	0.961	0.961
			0.59	100	0.960	0.960	0.953	0.957	0.957
	0.14	0	0.14	1000	0.955	0.955	0.957	0.955	0.955
			0.39	100	0.828	0.816	0.854	0.893	0.893
			0.59	100	0.975	0.972	0.981	0.985	0.985
		0.14	0	1000	0.958	0.958	0.957	0.959	0.959
			0.14	200	0.886	0.880	0.894	0.910	0.910
			0.39	100	0.957	0.954	0.973	0.972	0.972
			0.59	100	0.994	0.993	0.993	0.997	0.997
		0.39	0	100	0.790	0.770	0.829	0.875	0.875
			0.14	100	0.967	0.962	0.970	0.980	0.980
			0.39	50	0.787	0.779	0.815	0.867	0.867
			0.59	50	0.836	0.829	0.826	0.865	0.865
		0.59	0	100	0.976	0.976	0.979	0.984	0.984
			0.14	50	0.759	0.736	0.776	0.852	0.852
			0.39	50	0.842	0.831	0.840	0.871	0.871
			0.59	50	0.863	0.857	0.846	0.881	0.881
	0.39	0	0.14	1000	0.963	0.963	0.957	0.958	0.958
			0.39	100	0.845	0.837	0.857	0.871	0.871
			0.59	100	0.990	0.990	0.996	0.997	0.997
		0.14	0	1000	0.958	0.958	0.959	0.956	0.956
			0.14	200	0.885	0.885	0.882	0.891	0.891
			0.39	100	0.976	0.976	0.975	0.983	0.983
			0.59	50	0.893	0.885	0.888	0.923	0.923
		0.39	0	100	0.835	0.830	0.844	0.859	0.859
			0.14	100	0.979	0.977	0.984	0.986	0.986

Table 10 Empirical Power for Model 4 (Continue)

Regression Coefficients				Sample Size	Test				
a_1	a_3	b_1	b_3		first	second	boot	bc	bca
			0.39	50	0.936	0.931	0.929	0.956	0.956
			0.59	50	0.957	0.950	0.947	0.972	0.972
		0.59	0	100	0.992	0.992	0.992	0.994	0.994
			0.14	50	0.901	0.886	0.894	0.933	0.933
			0.39	50	0.952	0.947	0.946	0.966	0.965
			0.59	50	0.977	0.976	0.971	0.974	0.974
	0.59	0	0.14	1000	0.962	0.962	0.961	0.954	0.954
			0.39	100	0.870	0.866	0.863	0.882	0.882
			0.59	50	0.892	0.820	0.806	0.854	0.854
		0.14	0	1000	0.995	0.995	0.956	0.956	0.956
			0.14	200	0.914	0.914	0.905	0.910	0.910
			0.39	100	0.978	0.977	0.975	0.975	0.975
			0.59	50	0.892	0.820	0.806	0.854	0.854
		0.39	0	100	0.880	0.876	0.886	0.892	0.892
			0.14	100	0.973	0.972	0.972	0.978	0.978
			0.39	50	0.946	0.941	0.942	0.961	0.961
			0.59	50	0.988	0.988	0.984	0.987	0.987
		0.59	0	50	0.835	0.824	0.831	0.860	0.860
			0.14	50	0.927	0.921	0.923	0.937	0.937
			0.39	50	0.978	0.975	0.979	0.981	0.981
			0.59	50	0.998	0.997	0.995	0.996	0.996

Table 11 Empirical Power for Model 5

a ₁	a ₃	b ₁	b ₂	Sample	Test					
				Size	first	second	boot	bc	bca	
0	0.14	0	0.14	1000	0.713	0.685	0.826	0.864	0.864	
			0.39	1000	0.846	0.841	0.856	0.854	0.854	
			0.59	1000	0.858	0.858	0.854	0.860	0.860	
		0.14	0	1000	0.738	0.706	0.859	0.898	0.898	
			0.14	1000	0.855	0.840	0.872	0.886	0.886	
			0.39	1000	0.871	0.868	0.868	0.872	0.872	
	0.39	0	0.59	1000	0.886	0.886	0.879	0.874	0.874	
			0	1000	0.880	0.877	0.887	0.896	0.896	
			0.14	1000	0.875	0.873	0.878	0.880	0.880	
		0.59	0	1000	0.861	0.860	0.871	0.869	0.869	
			0.14	1000	0.871	0.870	0.862	0.867	0.867	
			0.39	1000	0.871	0.870	0.875	0.876	0.876	
	0.59	0	0.14	0.14	1000	0.857	0.857	0.858	0.862	0.862
				0.39	1000	0.870	0.870	0.863	0.866	0.866
				0.59	1000	0.886	0.886	0.880	0.882	0.882
			0.14	0	1000	0.969	0.967	0.969	0.972	0.972
				0.39	200	0.918	0.907	0.951	0.964	0.964
				0.59	200	0.944	0.941	0.954	0.959	0.959
0.39		0.14	0	1000	0.964	0.964	0.967	0.970	0.970	
			0.14	200	0.798	0.776	0.870	0.911	0.911	
			0.39	200	0.950	0.946	0.958	0.967	0.967	
		0.59	0	200	0.948	0.948	0.939	0.943	0.943	
			0	200	0.916	0.907	0.941	0.958	0.958	
			0.14	200	0.975	0.973	0.978	0.983	0.983	
0.59	0.39	0	200	0.956	0.955	0.954	0.956	0.956		
		0.39	200	0.951	0.951	0.945	0.949	0.949		
		0.59	200	0.951	0.951	0.945	0.949	0.949		
	0.59	0	200	0.957	0.956	0.956	0.966	0.966		
		0.14	200	0.954	0.953	0.949	0.951	0.951		
		0.39	200	0.951	0.950	0.940	0.956	0.956		
0.59	0	0.59	200	0.965	0.964	0.960	0.956	0.956		
		0.14	1000	0.964	0.964	0.968	0.970	0.970		
		0.39	100	0.789	0.763	0.860	0.904	0.904		
	0.14	0	1000	0.931	0.925	0.949	0.962	0.962		
		0	1000	0.976	0.976	0.970	0.975	0.975		

Table 11 Empirical Power for Model 5 (Continue)

Regression Coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₂	Size	first	second	boot	bc	bca
			0.14	200	0.908	0.903	0.919	0.934	0.934
			0.39	100	0.933	0.924	0.949	0.966	0.966
			0.59	100	0.952	0.949	0.958	0.973	0.973
		0.39	0	100	0.766	0.749	0.834	0.877	0.877
			0.14	100	0.923	0.913	0.947	0.970	0.970
			0.39	100	0.959	0.955	0.960	0.967	0.967
			0.59	100	0.953	0.952	0.955	0.960	0.960
		0.59	0	100	0.953	0.951	0.949	0.973	0.973
			0.14	100	0.956	0.955	0.961	0.969	0.969
			0.39	100	0.966	0.965	0.967	0.971	0.971
			0.59	100	0.960	0.959	0.957	0.963	0.963
0.14	0	0	0.14	1000	0.713	0.687	0.817	0.869	0.869
			0.39	1000	0.867	0.865	0.869	0.871	0.871
			0.59	1000	0.861	0.861	0.860	0.870	0.870
		0.14	0	1000	0.717	0.690	0.828	0.876	0.876
			0.14	1000	0.850	0.847	0.864	0.881	0.881
			0.39	1000	0.864	0.862	0.866	0.866	0.866
			0.59	1000	0.887	0.887	0.875	0.878	0.878
		0.39	0	1000	0.863	0.860	0.874	0.879	0.879
			0.14	1000	0.876	0.876	0.874	0.879	0.879
			0.39	1000	0.874	0.873	0.970	0.968	0.968
			0.59	1000	0.889	0.889	0.884	0.884	0.884
		0.59	0	1000	0.865	0.863	0.866	0.867	0.867
			0.14	1000	0.859	0.858	0.858	0.858	0.858
			0.39	1000	0.884	0.884	0.880	0.879	0.879
			0.59	1000	0.870	0.870	0.869	0.872	0.872
	0.14	0	0.14	1000	0.969	0.968	0.976	0.979	0.979
			0.39	500	0.993	0.993	0.989	0.991	0.991
			0.59	500	0.989	0.989	0.989	0.990	0.990
		0.14	0	500	0.654	0.622	0.753	0.800	0.800
			0.14	500	0.986	0.986	0.987	0.992	0.992
			0.39	500	0.986	0.985	0.987	0.987	0.987
			0.59	500	0.84	0.984	0.983	0.984	0.984
		0.39	0	500	0.990	0.989	0.993	0.993	0.993

Table 11 Empirical Power for Model 5 (Continue)

Regression Coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₂	Size	first	second	boot	bc	bca
			0.14	500	0.991	0.991	0.991	0.992	0.992
			0.39	500	0.990	0.990	0.989	0.986	0.986
			0.59	500	0.994	0.994	0.994	0.993	0.993
		0.59	0	500	0.998	0.998	0.997	0.998	0.998
			0.14	500	0.990	0.990	0.986	0.988	0.988
			0.39	500	0.991	0.991	0.991	0.991	0.991
			0.59	500	0.995	0.995	0.996	0.994	0.994
	0.39	0	0.14	1000	0.972	0.972	0.970	0.975	0.975
			0.39	100	0.703	0.684	0.807	0.866	0.866
			0.59	100	0.893	0.882	0.906	0.942	0.942
		0.14	0	1000	0.977	0.977	0.981	0.979	0.979
			0.14	200	0.905	0.897	0.926	0.946	0.946
			0.39	100	0.882	0.868	0.914	0.943	0.943
			0.59	100	0.888	0.886	0.899	0.924	0.924
		0.39	0	200	0.994	0.992	0.998	0.998	0.998
			0.14	100	0.886	0.869	0.919	0.944	0.944
			0.39	100	0.917	0.914	0.916	0.944	0.944
			0.59	100	0.917	0.915	0.921	0.933	0.933
		0.59	0	100	0.888	0.877	0.910	0.939	0.939
			0.14	100	0.928	0.926	0.927	0.946	0.946
			0.39	100	0.922	0.921	0.925	0.935	0.935
			0.59	100	0.938	0.936	0.936	0.937	0.937
	0.59	0	0.14	1000	0.967	0.967	0.964	0.966	0.966
			0.39	100	0.842	0.826	0.874	0.903	0.903
			0.59	100	0.986	0.984	0.989	0.994	0.994
		0.14	0	1000	0.974	0.974	0.970	0.969	0.969
			0.14	200	0.912	0.909	0.915	0.926	0.926
			0.39	100	0.977	0.976	0.974	0.986	0.986
			0.59	100	0.997	0.997	0.996	0.998	0.998
		0.39	0	100	0.841	0.823	0.873	0.904	0.904
			0.14	100	0.900	0.968	0.974	0.977	0.977
			0.39	50	0.992	0.991	0.993	0.995	0.995
			0.59	50	0.993	0.993	0.990	0.990	0.990
		0.59	0	100	0.988	0.984	0.991	0.993	0.993

Table 11 Empirical Power for Model 5 (Continue)

Regression Coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₂	Size	first	second	boot	bc	bca
			0.14	50	0.991	0.990	0.990	0.993	0.993
			0.39	50	0.991	0.989	0.993	0.992	0.992
			0.59	50	0.993	0.993	0.990	0.990	0.990
0.39	0	0	0.14	1000	0.975	0.972	0.973	0.977	0.977
			0.39	200	0.919	0.913	0.948	0.959	0.959
			0.59	200	0.963	0.961	0.966	0.972	0.972
		0.14	0	1000	0.968	0.967	0.970	0.976	0.976
			0.14	200	0.779	0.746	0.858	0.916	0.916
			0.39	200	0.963	0.960	0.967	0.971	0.971
			0.59	200	0.956	0.964	0.951	0.957	0.957
		0.39	0	200	0.911	0.898	0.943	0.961	0.961
			0.14	200	0.938	0.934	0.952	0.959	0.959
			0.39	200	0.959	0.958	0.956	0.961	0.961
			0.59	200	0.960	0.959	0.953	0.960	0.960
		0.59	0	200	0.951	0.948	0.947	0.954	0.954
			0.14	200	0.963	0.962	0.959	0.964	0.964
			0.39	200	0.961	0.961	0.960	0.960	0.960
			0.59	200	0.953	0.952	0.947	0.947	0.947
	0.14	0	0.14	1000	0.970	0.970	0.969	0.972	0.972
			0.39	200	0.985	0.984	0.989	0.994	0.994
			0.59	100	0.895	0.890	0.917	0.933	0.933
		0.14	0	1000	0.969	0.969	0.967	0.969	0.969
			0.14	200	0.877	0.868	0.904	0.924	0.924
			0.39	100	0.868	0.847	0.893	0.924	0.924
			0.59	100	0.927	0.921	0.924	0.947	0.947
		0.39	0	200	0.987	0.986	0.993	0.994	0.994
			0.14	100	0.876	0.856	0.912	0.935	0.935
			0.39	100	0.930	0.925	0.927	0.939	0.939
			0.59	100	0.915	0.913	0.908	0.917	0.917
		0.59	0	100	0.891	0.884	0.913	0.933	0.933
			0.14	100	0.918	0.914	0.927	0.939	0.939
			0.39	100	0.928	0.924	0.919	0.936	0.936
			0.59	100	0.927	0.926	0.914	0.919	0.919
	0.39	0	0.14	1000	0.973	0.973	0.976	0.976	0.976

Table 11 Empirical Power for Model 5 (Continue)

Regression Coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₂	Size	first	second	boot	ba	bca
			0.39	100	0.845	0.832	0.865	0.899	0.899
			0.59	100	0.985	0.984	0.985	0.991	0.991
		0.14	0	1000	0.966	0.964	0.966	0.966	0.966
			0.14	200	0.904	0.902	0.912	0.925	0.925
			0.39	100	0.972	0.969	0.979	0.981	0.981
			0.59	50	0.825	0.807	0.834	0.886	0.886
		0.39	0	100	0.860	0.854	0.871	0.904	0.904
			0.14	100	0.966	0.965	0.971	0.978	0.978
			0.39	50	0.843	0.830	0.838	0.893	0.893
			0.59	50	0.893	0.883	0.884	0.912	0.912
		0.59	0	100	0.992	0.992	0.988	0.994	0.994
			0.14	50	0.817	0.808	0.832	0.881	0.881
			0.39	50	0.906	0.897	0.872	0.908	0.908
			0.59	50	0.898	0.888	0.874	0.902	0.902
	0.59	0	0.14	1000	0.976	0.976	0.974	0.974	0.974
			0.39	100	0.860	0.857	0.861	0.880	0.880
			0.59	50	0.995	0.994	0.995	0.996	0.996
		0.14	0	1000	0.978	0.978	0.977	0.975	0.975
			0.14	200	0.893	0.893	0.901	0.909	0.909
			0.39	100	0.983	0.983	0.985	0.986	0.986
			0.59	50	0.915	0.908	0.925	0.952	0.952
		0.39	0	100	0.861	0.856	0.864	0.883	0.883
			0.14	100	0.984	0.982	0.985	0.984	0.984
			0.39	50	0.932	0.927	0.927	0.952	0.952
			0.59	50	0.971	0.969	0.966	0.979	0.979
		0.59	0	50	0.820	0.806	0.813	0.862	0.862
			0.14	50	0.912	0.906	0.904	0.933	0.933
			0.39	50	0.960	0.955	0.957	0.970	0.970
			0.59	50	0.975	0.975	0.968	0.978	0.978
0.59	0	0	0.14	1000	0.978	0.976	0.974	0.978	0.978
			0.39	100	0.754	0.738	0.825	0.869	0.869
			0.59	100	0.939	0.931	0.958	0.968	0.968
		0.14	0	1000	0.972	0.972	0.970	0.973	0.973
			0.14	200	0.883	0.877	0.909	0.921	0.921

Table 11 Empirical Power for Model 5 (Continue)

Regression Coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₂	Size	first	second	boot	bc	bca
			0.39	100	0.907	0.901	0.942	0.957	0.957
			0.59	100	0.966	0.964	0.961	0.969	0.969
		0.39	0	200	0.991	0.989	0.990	0.994	0.994
			0.14	100	0.915	0.899	0.930	0.953	0.953
			0.39	100	0.964	0.962	0.964	0.968	0.968
			0.59	100	0.964	0.964	0.967	0.972	0.972
		0.59	0	100	0.950	0.942	0.948	0.960	0.960
			0.14	100	0.946	0.943	0.948	0.960	0.960
			0.39	100	0.965	0.92	0.959	0.965	0.965
			0.59	100	0.965	0.963	0.966	0.965	0.965
	0.14	0	0.14	1000	0.970	0.970	0.969	0.968	0.968
			0.39	100	0.834	0.823	0.869	0.903	0.903
			0.59	100	0.975	0.974	0.984	0.989	0.989
		0.14	0	1000	0.971	0.971	0.973	0.973	0.973
			0.14	200	0.895	0.891	0.899	0.912	0.912
			0.39	100	0.961	0.957	0.971	0.980	0.980
			0.59	100	0.995	0.994	0.994	0.995	0.995
		0.39	0	100	0.849	0.836	0.874	0.895	0.895
			0.14	100	0.975	0.972	0.980	0.988	0.988
			0.39	50	0.803	0.792	0.817	0.867	0.867
			0.59	50	0.861	0.854	0.867	0.885	0.885
		0.59	0	100	0.984	0.982	0.982	0.987	0.987
			0.14	100	0.992	0.992	0.992	0.996	0.996
			0.39	50	0.851	0.840	0.856	0.880	0.880
			0.59	50	0.883	0.878	0.860	0.890	0.890
	0.39	0	0.14	1000	0.966	0.966	0.968	0.965	0.965
			0.39	100	0.885	0.877	0.878	0.903	0.903
			0.59	100	0.996	0.995	0.995	0.996	0.996
		0.14	0	1000	0.972	0.971	0.971	0.972	0.972
			0.14	200	0.909	0.908	0.914	0.927	0.927
			0.39	100	0.980	0.979	0.976	0.983	0.983
			0.59	50	0.918	0.907	0.989	0.942	0.942
		0.39	0	100	0.879	0.875	0.881	0.899	0.899
			0.14	100	0.979	0.979	0.980	0.985	0.985

Table 11 Empirical Power for Model 5 (Continue)

Regression Coefficients				Sample	Test				
a ₁	a ₃	b ₁	b ₂	Size	first	second	boot	bc	bca
			0.39	50	0.920	0.914	0.921	0.947	0.947
			0.59	50	0.966	0.965	0.963	0.976	0.976
		0.59	0	100	0.993	0.993	0.992	0.994	0.994
			0.14	50	0.924	0.915	0.919	0.942	0.941
			0.39	50	0.962	0.956	0.953	0.967	0.967
			0.59	50	0.969	0.967	0.968	0.977	0.977
	0.59	0	0.14	1000	0.969	0.969	0.969	0.971	0.971
			0.39	100	0.872	0.870	0.859	0.880	0.880
			0.59	50	0.854	0.848	0.844	0.871	0.871
		0.14	0	1000	0.977	0.977	0.975	0.971	0.971
			0.14	200	0.90	0.905	0.905	0.915	0.915
			0.39	100	0.980	0.9778	0.980	0.978	0.978
			0.59	50	0.947	0.942	0.937	0.954	0.954
		0.39	0	100	0.891	0.889	0.887	0.901	0.901
			0.14	100	0.982	0.981	0.983	0.985	0.985
			0.39	50	0.967	0.963	0.965	0.972	0.972
			0.59	50	0.990	0.989	0.979	0.988	0.988
		0.59	0	50	0.841	0.831	0.826	0.879	0.879
			0.14	50	0.936	0.931	0.936	0.955	0.955
			0.39	50	0.984	0.981	0.981	0.988	0.988
			0.59	50	0.994	0.994	0.987	0.992	0.992

Table 12 Required Sample Sizes with 0.8 Empirical Power for Models 1-3

Regression Coefficients			Model 1	Model 2	Model 3	
			Sample Size			
0.14	0	0.14	1040	1070	1040	
		0.39	460	460	460	
		0.59	440	430	440	
	0.14	0	1040	1060	1050	
		0.14	530	540	530	
		0.39	430	440	440	
	0.39	0	460	470	460	
		0.14	440	430	440	
		0.39	420	420	420	
	0.59	0	0.59	420	420	420
			0	410	440	430
			0.14	410	420	430
0.39		0	410	420	410	
		0.39	400	420	410	
		0.59	400	420	410	
0.39	0	0.14	750	850	800	
		0.39	130	150	140	
		0.59	90	100	90	
	0.14	0	740	840	780	
		0.14	220	240	230	
		0.39	100	110	100	
	0.39	0	80	80	80	
		0.39	130	150	140	
		0.14	100	110	100	
	0.59	0	0.39	80	80	80
			0.59	70	70	70
			0	90	100	90
0.14		0	80	80	80	
		0.39	70	70	70	
		0.59	60	70	70	
0.59	0	0.14	660	820	740	
		0.39	110	130	110	
		0.59	60	70	70	
	0.14	0	660	830	730	

Table 12 Required Sample Sizes with 0.8 Empirical Power for Models 1-3
(Continue)

Regression Coefficients		Model 1	Model 2	Model 3
		Sample Size		
	0.14	180	230	200
	0.39	70	80	80
	0.59	50	60	50
0.39	0	110	130	120
	0.14	70	80	80
	0.39	50	60	50
	0.59	40	50	40
0.59	0	70	70	70
	0.14	50	60	50
	0.39	40	50	40
	0.59	40	40	40

Note. First column of regression coefficients is a_1 in Model 1, b_1 in Model 2 and a_1 in Model 3; second column: b_1 in Model 1, a_1 in Model 2 and b_1 in Model 3; third column: b_2 in Model 1, a_3 in Model 2 and b_3 in Model 3.

Table 13 Required Sample Sizes with 0.8 Statistical Power for Models 4-5

Regression Coefficients				Model 4	Model 5	Regression Coefficients				Model 4	Model 5
0	0.14	0	0.14	1220	1180	0.39	0	0.39	0	170	160
			0.39	850	850				0.14	140	140
			0.59	840	840				0.39	120	130
	0.14	0	0	1240	1170				0.59	120	130
			0.14	900	880		0.59	0	0	130	130
			0.39	850	860				0.14	130	130
			0.59	830	830				0.39	130	130
	0.39	0	0	880	850				0.59	120	120
			0.14	860	830	0.14	0	0.14	0.14	650	570
			0.39	840	820				0.39	130	120
			0.59	820	820				0.59	90	90
	0.59	0	0	850	830		0.14	0	0	610	580
			0.14	840	820				0.14	190	180
			0.39	830	830				0.39	100	100
			0.59	830	810				0.59	80	80
0.39	0	0.14	0.14	670	600		0.39	0	0	130	120
			0.39	170	160				0.14	100	100
			0.59	140	130				0.39	80	80
	0.14	0	0	640	590				0.59	70	70
			0.14	230	220		0.59	0	0	90	90
			0.39	140	140				0.14	80	80
			0.59	130	130				0.39	80	70
	0.39	0	0	170	170				0.59	70	70
			0.14	140	140	0.39	0	0.14	0.14	620	550
			0.39	130	130				0.39	100	100
			0.59	120	120				0.59	70	60
	0.59	0	0	130	130		0.14	0	0	590	560
			0.14	130	130				0.14	170	160
			0.39	120	120				0.39	70	70
			0.59	120	120				0.59	60	50
0.59	0	0.14	0.14	560	560		0.39	0	0	100	100
			0.39	110	110				0.14	70	70
			0.59	80	80				0.39	50	50
	0.14	0	0	550	550				0.59	50	50

Table 13 Required Sample Sizes with 0.8 Statistical Power for Models 4-5

(Continue)

Regression Coefficients				Model 4	Model 5	Regression Coefficients				Model 4	Model 5
			0.14	170	170			0.59	0	70	60
			0.39	90	90			0.14	0	60	50
			0.59	70	70			0.39	0	50	50
		0.39	0	120	110			0.59	0	50	40
			0.14	90	90		0.59	0	0.14	580	530
			0.39	70	70			0.39	0	100	90
			0.59	60	60			0.59	0	60	50
		0.59	0	80	80			0.14	0	580	520
			0.14	70	70			0.14	0	160	150
			0.39	70	60			0.39	0	70	60
			0.59	60	60			0.59	0	50	50
0.14	0	0	0.14	1220	1180			0.39	0	100	90
			0.39	870	860			0.14	0	70	60
			0.59	840	830			0.39	0	50	40
		0.14	0	1230	1170			0.59	0	40	40
			0.14	910	880			0.59	0	60	60
			0.39	850	840			0.14	0	50	50
			0.59	820	810			0.39	0	40	40
		0.39	0	860	830			0.59	0	40	40
			0.14	830	820		0	0	0.14	620	580
			0.39	820	820			0.39	0	120	110
			0.59	810	810			0.59	0	80	80
		0.59	0	840	830			0.14	0	610	580
			0.14	830	810			0.14	0	180	170
			0.39	830	810			0.39	0	90	90
			0.59	830	810			0.59	0	70	70
	0.14	0	0.14	720	670	0.59		0.39	0	120	120
			0.39	260	250			0.14	0	90	90
			0.59	240	230			0.39	0	70	70
		0.14	0	720	670			0.59	0	60	70
			0.14	320	300			0.59	0	80	80
			0.39	230	230			0.14	0	70	70
			0.59	230	220			0.39	0	60	70
		0.39	0	260	250			0.59	0	60	60

Table 13 Required Sample Sizes with 0.8 Statistical Power for Models 4-5

(Continue)

Regression Coefficients		Model 4	Model 5	Regression Coefficients		Model 4	Model 5	
	0.14	230	230	0.14	0	610	570	
	0.39	220	220		0.39	110	100	
	0.59	210	210		0.59	70	70	
0.59	0	230	230	0.14	0	600	560	
	0.14	230	220		0.14	170	160	
	0.39	220	210		0.39	80	70	
	0.59	220	210		0.59	60	60	
0.39	0	0.14	630	560	0.39	0	110	100
	0.39	130	120		0.14	80	70	
	0.59	90	90		0.39	60	60	
0.14	0	630	560		0.59	50	50	
	0.14	200	180	0.59	0	70	70	
	0.39	100	90		0.14	60	60	
	0.59	80	80		0.39	50	50	
0.39	0	130	120		0.59	50	50	
	0.14	100	100	0.39	0	0.14	590	540
	0.39	80	80		0.39	100	90	
	0.59	80	80		0.59	60	60	
0.59	0	90	90	0.14	0	600	550	
	0.14	80	80		0.14	160	150	
	0.39	70	70		0.39	0	60	
	0.59	70	70		0.59	50	50	
0.59	0	0.14	600	540	0.39	0	100	90
	0.39	110	100		0.14	60	60	
	0.59	70	70		0.39	50	40	
0.14	0	610	540		0.59	40	40	
	0.14	170	160	0.59	0	60	60	
	0.39	80	70		0.14	50	50	
	0.59	60	60		0.39	40	40	
0.39	0	110	100		0.59	40	40	
	0.14	80	70	0.59	0	0.14	580	550
	0.39	60	60		0.39	90	90	
	0.59	50	50		0.59	50	50	
0.59	0	70	70	0.14	0	570	540	

Table 13 Required Sample Sizes with 0.8 Statistical Power for Models 4-5
(Continue)

Regression Coefficients			Model 4	Model 5	Regression Coefficients			Model 4	Model 5			
0.39	0	0	0.14	60	60	0.39	0	0.14	160	150		
			0.39	50	50			0.39	60	60		
			0.59	50	50			0.59	40	40		
			0.14	650	600			0.14	60	60		
			0.39	170	160			0.39	40	40		
			0.59	130	130			0.59	30	30		
			0.14	0	650			600	0.59	0	50	50
			0.14	240	220			0.14	40	40		
			0.39	140	140			0.39	40	30		
			0.59	130	130			0.59	30	50		

Note. First and 7th columns is coefficient a_1 , second and 8th is coefficient a_3 , third and 9th is coefficient b_1 in Models 4-5; 4th and 10th is coefficient b_3 and b_2 for Model 4 and 5, respectively.

Appendix B

R code of Model 1

```
sample.size=c(50,100,200,500,1000);
[2] c=0;
[3] a1=0.14;
[4] b1=0.14;
[5] b2=0.14;
[6] x=1;
alpha=0.05;
nrep=1000;
nboot=1000;
p1= rep(0,nrep);
p2=rep(0,nrep);
Z1= rep(0,nrep);
Z2=rep(0,nrep);
percent= rep(0,nboot);
bc= rep(0,nboot);
bca=rep(0,nboot);
output=array(rep(0),c(5,5));
for(t in 1:5)
{
n=sample.size[t];
for(i in 1:nrep)
{
[23] X=rnorm(n,0,1);
r1=rnorm(n,0,1);
r2=rnorm(n,0,1);
[26] M=a1*X+r1;
[27] Y=b1*M+b2*X*M+c*X+r2;
[28] MVmodel=lm(M~X);
[29] fit.a1=summary(MVmodel)$coefficients[2,1];
[30] SEa1=summary(MVmodel)$coefficients[2,2];
[31] DVmodel=lm(Y~M+X+X*M);
[32] fit.b1=summary(DVmodel)$coefficients[2,1];
[33] SEb1=summary(DVmodel)$coefficients[2,2];
```

```

[34] fit.b2=summary(DVmodel)$coefficients[4,1];
[35] SEb2=summary(DVmodel)$coefficients[4,2];
[36] co=vcov(DVmodel);
[37] SEb12=co[2,4];
[38] PE=fit.a1*(fit.b1+fit.b2*x); # Point Estaimtor
[39] se1=sqrt(((fit.b1+fit.b2*x)^2)*SEa1^2+(fit.a1^2)*
      (SEb1^2+2*SEb12*x+(SEb2^2)*(x^2))); # first-order delta method
[40] se2=sqrt(((fit.b1+fit.b2*x)^2)*SEa1^2+(fit.a1^2+SEa1^2)*
      (SEb1^2+2*SEb12*x+(SEb2^2)*(x^2))); # second-order delta method

z1=PE/se1;
z2=PE/se2;
Z1[i]=z1;
Z2[i]=z2;

# first
if(Z1[i]<qnorm(alpha/2)|Z1[i]>qnorm(1-(alpha/2)))
{
p1[i]=1;
}
else
{
p1[i]=0;
}

# second
if(Z2[i]<qnorm(alpha/2)|Z2[i]>qnorm(1-(alpha/2)))
{
p2[i]=1;
}
else
{
p2[i]=0;
}

# bootstrap
boot.PE=rep(0,nrep);
for(j in 1:nboot)

```

```

{
r=sample(1:n,n,replace=T);
Xnew=X[r];
Mnew=M[r];
Ynew=Y[r];
boot.MVmodel=lm(Mnew~Xnew);
boot.a1=summary(boot.MVmodel)$coefficients[2,1];
boot.DVmodel=lm(Ynew~Mnew+Xnew+Xnew*Mnew);
boot.b1=summary(boot.DVmodel)$coefficients[2,1];
boot.b2=summary(boot.DVmodel)$coefficients[4,1];
boot.PE[j]=boot.a1*(boot.b1+boot.b2*x);
}

```

```

# Percentile Bootstrap
sorting=sort(boot.PE);
lower=round((alpha/2)*nboot);
upper=round((1-(alpha/2))*nboot);
if(lower==0)
{
lower=1;
}
else
{
lower=lower;
}
boot.lower=sorting[lower];
boot.upper=sorting[upper];
if(boot.lower<=0 && boot.upper>=0)
{
percent[i]=0;
}
else
{
percent[i]=1;
}

```

```

# Bias Corrected

```

```

bias=rep(0,nboot);
for(k in 1:nboot)
{
if(boot.PE[k]<=PE)
{
bias[k]=1;
}
else
{
bias[k]=0;
}
}
z0=mean(bias);
lower.limit=round((pnorm(2*qnorm(z0)+qnorm(alpha/2)))*nboot);
upper.limit=round((pnorm(2*qnorm(z0)+qnorm(1-(alpha/2)))*nboot);
if(lower.limit==0)
{
lower.limit=1;
}
else
{
lower.limit=lower.limit;
}
bias.lower=sorting[lower.limit];
bias.upper=sorting[upper.limit];
if(bias.lower<=0 && bias.upper>=0)
{
bc[i]=0;
}
else
{
bc[i]=1;
}
# Bias-Corrected Acceleration
bca.PE=rep(0,n);
for(q in 1:n)
{
bca.X=X[-q];

```

```

bca.M=M[-q];
bca.Y=Y[-q];
bca.MVmodel=lm(bca.M~bca.X);
bca.a1=summary(bca.MVmodel)$coefficients[2,1];
bca.DVmodel=lm(bca.Y~bca.M+bca.X+bca.X*bca.M);
bca.b1=summary(bca.DVmodel)$coefficients[2,1];
bca.b2=summary(bca.DVmodel)$coefficients[4,1];
bca.PE[q]=bca.a1*(bca.b1+bca.b2*x);
}
mean.bca.PE=mean(bca.PE);
a=mean.bca.PE-bca.PE;
acce=sum(a^3)/(6*((sum(a^2))^(3/2)));
bca.lower.limit=round((pnorm(qnorm(z0)+(qnorm(z0)+qnorm(alpha/2))/(1-acce*(qnorm(z0)+qnorm(alpha/2)))))*nboot);
bca.upper.limit=round((pnorm(qnorm(z0)+(qnorm(z0)+qnorm(1-alpha/2))/(1-acce*(qnorm(z0)+qnorm(1-alpha/2)))))*nboot);
if(bca.lower.limit==0)
{
bca.lower.limit=1;
}
else
{
bca.lower.limit=lower.limit;
}
bca.lower=sorting[bca.lower.limit];
bca.upper=sorting[bca.upper.limit];
if(bca.lower<=0 && bca.upper>=0)
{
bca[i]=0;
}
else
{
bca[i]=1;
}
}
output[1,t]=mean(p1);
output[2,t]=mean(p2);
output[3,t]=mean(percent);

```

```

output[4,t]=mean(bc);
output[5,t]=mean(bca);
}
output;

```

Note. The values for relevant coefficients were varied and those relevant coefficients values included 0, 0.14, 0.39, and 0.59. Other models we just change some parts that we mark.

Model 2

```

[2] a2=1;
    c1=0;
    c2=0;
    c3=0;
[3] b1=0.14;
[4] a1=0.14;
[5] a3=0.14;
[6] w=1;
[23] X=rnorm(n,0,1);
      W=rnorm(n,0,1);
[26] M=a1*X+a2*W+a3*X*W+r1;
[27] Y=b1*M+c1*X+c2*W+c3*X*W+r2;
[28] MVmodel=lm(M~X+W+X*W);
[29] fit.a1=summary(MVmodel)$coefficients[2,1];
[30] SEa1=summary(MVmodel)$coefficients[2,2];
[31] fit.a3=summary(MVmodel)$coefficients[4,1];
[32] SEa3=summary(MVmodel)$coefficients[4,2];
[33] co=vcov(MVmodel);
[34] SEa13=co[2,4];
[35] DVmodel=lm(Y~M+X+W+X*W);
[36] fit.b1=summary(DVmodel)$coefficients[2,1];
[37] SEb1=summary(DVmodel)$coefficients[2,2];
[38] PE=(fit.a1+fit.a3*w)*fit.b1; # Point Estimate
[39] se1=sqrt((((fit.a1+fit.a3*w)^2)*SEb1^2+(fit.b1^2)*(SEa1^2+2*SEa13*w
      +(SEa3^2)*(w^2)));
[40] se2=sqrt((((fit.a1+fit.a3*w)^2)*SEb1^2+(fit.b1^2+SEb1^2)*
      (SEa1^2+2*SEa13*w+(SEa3^2)*(w^2)));

```

Model 3

```
[2] b2=1;
     c =0;
[3] a1=0.14;
[4] b1=0.14;
[5] b3=0.14;
[6] w=1;
[23] X=rnorm(n,0,1);
      W=rnorm(n,0,1);
[26] M=a1*X+r1;
[27] Y=b1*M+b2*W+b3*M*W+c*X+r2;
[28] MVmodel=lm(M~X);
[29] fit.a1=summary(MVmodel)$coefficients[2,1];
[30] SEa1=summary(MVmodel)$coefficients[2,2];
[31] DVmodel=lm(Y~M+W+X+M*W);
[32] fit.b1=summary(DVmodel)$coefficients[2,1];
[33] SEb1=summary(DVmodel)$coefficients[2,2];
[34] fit.b3=summary(DVmodel)$coefficients[5,1];;
[35] SEb3=summary(DVmodel)$coefficients[5,2];
[36] co=vcov(DVmodel);
[37] SEb13=co[2,5];
[38] PE=fit.a1*(fit.b1+fit.b3*w);
[39] se1=sqrt((((fit.b1+fit.b3*w)^2)*(SEa1^2)+(fit.a1^2)*
             (SEb1^2+2*SEb13*w+(SEb3^2)*(w^2))));
[40] se2=sqrt((((fit.b1+fit.b3*w)^2)*(SEa1^2)
             +(fit.a1^2+SEa1^2)*(SEb1^2+2*SEb13*w+(SEb3^2)*(w^2))));
```


Model 4

```
[2] a2=1;
     b2=1;
     c1=0;
     c2=0;
     c3=0;
[3] a1=0.14;
[4] a3=0.14;
[5] b1=0.14;
     b3=0.14
[6] w=1;
     z=1;
[23] X=rnorm(n,0,1);
      W=rnorm(n,0,1);
      Z=rnorm(n,0,1);
[26] M=a1*X+a2*W+a3*X*W+r1;
[27] Y=b1*M+b2*Z+b3*M*Z+c1*X+c2*W+c3*X*W+r2;
[28] MVmodel=lm(M~X+W+X*W);
[29] fit.a1=summary(MVmodel)$coefficients[2,1];
[30] SEa1=summary(MVmodel)$coefficients[2,2];
[31] fit.a3=summary(MVmodel)$coefficients[4,1];
[32] SEa3=summary(MVmodel)$coefficients[4,2];
[33] a13=vcov(MVmodel);
[34] SEa13=a13[2,4];
[35] DVmodel=lm(Y~M+Z+X+W+M*Z+X*W);
[36] fit.b1=summary(DVmodel)$coefficients[2,1];
[37] SEb1=summary(DVmodel)$coefficients[2,2];
     fit.b3=summary(DVmodel)$coefficients[6,1];
     SEb3=summary(DVmodel)$coefficients[6,2];
     b13=vcov(DVmodel);
     SEb13=b13[2,6];
[38] PE=(fit.a1+fit.a3*w)*(fit.b1+fit.b3*z);
[39] se1=sqrt((((fit.a1+fit.a3*w)^2)*(SEb1^2+2*SEb13*z+(SEb3^2)*(z^2))
             +((fit.b1+fit.b3*z)^2)*(SEa1^2+2*SEa13*w+(SEa3^2)*(w^2))));
[40] se2=sqrt((((fit.a1+fit.a3*w)^2)*(SEb1^2+2*SEb13*z+(SEb3^2)*(z^2))
             +((fit.b1+fit.b3*z)^2)*(SEa1^2+2*SEa13*w+(SEa3^2)*(w^2))
             +(SEb1^2+2*SEb13*z+(SEb3^2)*(z^2))*(SEa1^2+2*SEa13*w+(SEa3^2)*(w^2))));
```

Model 5

```
[2] a2 =1;
      c1=0;
      c2=0;
      c3=0;
[3] a1=0.14;
[4] a3=0.14;
[5] b1=0.14;
      b2=0.14
[6] w=1;
[23] X=rnorm(n,0,1);
      W=rnorm(n,0,1);
[26] M=a1*X+a2*W+a3*X*W+r1;
[27] Y=b1*M+b2*M*W+c1*X+c2*W+c3*X*W+r2;
[28] MVmodel=lm(M~X+W+X*W);
[29] fit.a1=summary(MVmodel)$coefficients[2,1];
[30] SEa1=summary(MVmodel)$coefficients[2,2];
[31] fit.a3=summary(MVmodel)$coefficients[4,1];
[32] SEa3=summary(MVmodel)$coefficients[4,2];
[33] a13=vcov(MVmodel);
[34] SEa13=a13[2,4];
[35] DVmodel=lm(Y~M+X+W+M*W+X*W);
[36] fit.b1=summary(DVmodel)$coefficients[2,1];
[37] SEb1=summary(DVmodel)$coefficients[2,2];
      fit.b2=summary(DVmodel)$coefficients[5,1];
      SEb2=summary(DVmodel)$coefficients[5,2];
      b12=vcov(DVmodel);
      SEb12=b13[2,5];
[38] PE=(fit.a1+fit.a3*w)*(fit.b1+fit.b2*w);
[39] se1=sqrt(((fit.b1+fit.b2*w)^2*(SEa1^2+2*SEa13*w+(SEa3^2)*(w^2))
      +((fit.a1+fit.a3*w)^2)*(SEb1^2+2*SEb12*w+(SEb2^2)*(w^2))));
[40] se2=sqrt(((fit.b1+fit.b2*w)^2*(SEa1^2+2*SEa13*w+(SEa3^2)*(w^2))
      +((fit.a1+fit.a3*w)^2)*(SEb1^2+2*SEb12*w+(SEb2^2)*(w^2))
      +(SEa1^2+2*SEa13*w+(SEa3^2)*(w^2))*(SEb1^2+2*SEb12*w+(SEb2^2)*(w^2)))));
```

Appendix C

Point Estimates of Moderated Mediation Effects

The indirect effect of X on Y in Models 1-5 can be easily derived using a matrix algebra approach described by Sobel (1986) and Bollen (1987, 1989). Let \mathbf{B} is a matrix which the rows and columns are correspond to all the variables in the regression equations. This \mathbf{B} matrix may be collapsed by noting that interaction term is an exact function of variables already represented, \mathbf{B}^* . The indirect effects of X on Y can be obtained using the formula:

$$\mathbf{F} = (\mathbf{I} - \mathbf{B}^*)^{-1} - \mathbf{I} - \mathbf{B}^*$$

The indirect effects may also be obtained by the chain rule from elementary calculus (Stolzenberg, 1980; Preacher et al., 2007).

Model 1

(a) Using a matrix algebra approach,

$$\mathbf{B} = \begin{matrix} X \\ XM \\ M \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ c' & b_2 & b_1 & 0 \end{bmatrix}$$

$$\mathbf{B}^* = \begin{matrix} X \\ M \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ c' + b_2M & b_1 + b_2X & 0 \end{bmatrix}$$

$$(\mathbf{I} - \mathbf{B}^*)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ a_1(b_1 + b_2X) - (c' + b_2M) & b_1 + b_2X & 0 \end{bmatrix}$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{B}^*)^{-1} - \mathbf{I} - \mathbf{B}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1(b_1 + b_2X) & 0 & 0 \end{bmatrix}$$

The point estimate of the indirect effect of X on Y is $f(\hat{\theta}|X) = \hat{a}_1(\hat{b}_1 + \hat{b}_2X)$.

(b) Using chain rule,

$$\frac{\partial Y}{\partial X} = \frac{\partial Y}{\partial M} \cdot \frac{\partial M}{\partial X} = (b_1 + b_2X) \cdot a_1$$

Model 2

(a) Using a matrix algebra approach,

$$\mathbf{B} = \begin{matrix} X \\ W \\ M \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 \\ c'_1 & c'_2 & c'_3 & b_1 & 0 \end{bmatrix}$$

$$\mathbf{B}^* = \begin{matrix} X \\ W \\ M \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 + a_3W & a_2 + a_3X & 0 & 0 \\ c'_1 + c'_3W & c'_2 + c'_3X & b_1 & 0 \end{bmatrix}$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{B}^*)^{-1} - \mathbf{I} - \mathbf{B}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_1(a_1 + a_3W) & b_1(a_2 + a_3X) & 0 & 0 \end{bmatrix}$$

The point estimate of Model 2 is $f(\hat{\theta}|W) = b_1(a_1 + a_3W)$

(b) Using chain rule,

$$\frac{\partial Y}{\partial X} = \frac{\partial Y}{\partial M} \cdot \frac{\partial M}{\partial X} = (a_1 + a_3W) \cdot b_1$$

Model 3

(a) Using a matrix algebra approach,

$$\mathbf{B} = \begin{matrix} X \\ M \\ W \\ MW \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c' & b_1 & b_2 & b_3 & 0 \end{bmatrix}$$

$$\mathbf{B}^* = \begin{matrix} X \\ M \\ W \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c' & b_1 + b_3W & b_2 + b_3M & 0 \end{bmatrix}$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{B}^*)^{-1} - \mathbf{I} - \mathbf{B}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1(b_1 + b_3W) & 0 & 0 & 0 \end{bmatrix}$$

The point estimate of Model 2 is $f(\hat{\theta}|W) = a_1(b_1 + b_3W)$

(b) Using chain rule,

$$\frac{\partial Y}{\partial X} = \frac{\partial Y}{\partial M} \cdot \frac{\partial M}{\partial X} = (b_1 + b_3W) \cdot a_1$$

Model 4

(a) Using a matrix algebra approach,

$$\mathbf{B} = \begin{matrix} X \\ W \\ XW \\ M \\ Z \\ MZ \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c'_1 & c'_2 & c'_3 & b_1 & b_2 & b_3 & 0 \end{bmatrix}$$

$$\mathbf{B}^* = \begin{matrix} X \\ W \\ M \\ Z \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 + a_3W & a_2 + a_3X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c'_1 + c'_3W & c'_2 + c'_3X & b_1 + b_3Z & b_2 + b_3M & 0 \end{bmatrix}$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{B}^*)^{-1} - \mathbf{I} - \mathbf{B}^*$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (a_1 + a_3W)(b_1 + b_3Z) & (a_2 + a_3X)(b_1 + b_3Z) & 0 & 0 & 0 \end{bmatrix}$$

The point estimate of Model 2 is $f(\hat{\boldsymbol{\theta}}|W, Z) = (a_1 + a_3W)(b_1 + b_3Z)$

(b) Using chain rule,

$$\frac{\partial Y}{\partial X} = \frac{\partial Y}{\partial M} \cdot \frac{\partial M}{\partial X} = (b_1 + b_3Z) \cdot (a_1 + a_3W)$$

Model 5

(a) Using a matrix algebra approach,

$$\mathbf{B} = \begin{matrix} X \\ W \\ XW \\ M \\ MW \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ c'_1 & c'_2 & c'_3 & b_1 & b_2 & 0 \end{bmatrix}$$

$$\mathbf{B}^* = \begin{matrix} X \\ W \\ M \\ Y \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 + a_3W & a_2 + a_3X & 0 & 0 \\ c'_1 + c'_3W & c'_2 + c'_3X + b_2M & b_1 + b_2W & 0 \end{bmatrix}$$

$$\mathbf{F} = (\mathbf{I} - \mathbf{B}^*)^{-1} - \mathbf{I} - \mathbf{B}^*$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ (a_1 + a_3W)(b_1 + b_2W) & (a_2 + a_3X)(b_1 + b_2W) & 0 & 0 & 0 & 0 \end{bmatrix}$$

The point estimate of Model 2 is $f(\hat{\boldsymbol{\theta}}|W) = (a_1 + a_3W)(b_1 + b_3W)$

(b) Using chain rule,

$$\frac{\partial Y}{\partial X} = \frac{\partial Y}{\partial M} \cdot \frac{\partial M}{\partial X} = (b_1 + b_2W) \cdot (a_1 + a_3W)$$

Appendix D

First- and Second-Order Variance Approximations

Let $\hat{\boldsymbol{\theta}}$ is a column vector of the sample coefficients involved in an indirect effect.

Let $\boldsymbol{\mu}$ is a column vector of the means of the regression coefficients, i.e., $\boldsymbol{\mu} = E[\hat{\boldsymbol{\theta}}]$.

Let $f(\hat{\boldsymbol{\theta}})$ is the effect of interest, a differentiable function of the coefficients in $\hat{\boldsymbol{\theta}}$.

Let $\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})$ is the covariance matrix of $\hat{\boldsymbol{\theta}}$.

Let $\mathbf{D} = \partial_{\boldsymbol{\theta}} f(\hat{\boldsymbol{\theta}})$ is the gradient of $f(\hat{\boldsymbol{\theta}})$ evaluated at $\boldsymbol{\mu}$.

Let $\mathbf{H} = \partial^2_{\boldsymbol{\theta}} f(\hat{\boldsymbol{\theta}})$ is the Hessian of $f(\hat{\boldsymbol{\theta}})$ evaluated at $\boldsymbol{\mu}$.

The gradient and Hessian are, respectively, matrices of first and second partial derivatives of $f(\hat{\boldsymbol{\theta}})$ with respect to all relevant free parameters.

Because $\text{var}[f(\hat{\boldsymbol{\theta}})] = E[f^2(\hat{\boldsymbol{\theta}})] - E^2[f(\hat{\boldsymbol{\theta}})]$,

$$f(\hat{\boldsymbol{\theta}}) \approx f(\boldsymbol{\mu}) + \mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) + \frac{1}{2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) \quad (\text{by Taylor's theorem})$$

$$\begin{aligned} E[f(\hat{\boldsymbol{\theta}})] &\approx E\left[f(\boldsymbol{\mu}) + \mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) + \frac{1}{2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})\right] \\ &= f(\boldsymbol{\mu}) + \frac{1}{2}E\left[(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})\right] \end{aligned}$$

$$= f(\boldsymbol{\mu}) + \frac{1}{2} \text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\}$$

$$\begin{aligned} f^2(\hat{\boldsymbol{\theta}}) &\approx \left(f(\boldsymbol{\mu}) + \mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) + \frac{1}{2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})\right)^2 \\ &= f^2(\boldsymbol{\mu}) + \mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{D} + \frac{1}{4}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) \\ &\quad + 2f(\boldsymbol{\mu})\mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) + f(\boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) \\ &\quad + \mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) \end{aligned}$$

$$\begin{aligned}
\mathbb{E}[f^2(\hat{\boldsymbol{\theta}})] &\approx \mathbb{E}[f^2(\boldsymbol{\mu}) + \mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{D} \\
&\quad + \frac{1}{4}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) + 2f(\boldsymbol{\mu})\mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) \\
&\quad + f(\boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu}) + \mathbf{D}'(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})] \\
&= f^2(\boldsymbol{\mu}) + \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{4}\mathbb{E}[(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})' \mathbf{H}(\hat{\boldsymbol{\theta}} - \boldsymbol{\mu})] \\
&\quad + f(\boldsymbol{\mu})\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\} \\
&= f^2(\boldsymbol{\mu}) + \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{4}(\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\})^2 + \frac{1}{2}\text{tr}\{(\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\} \\
&\quad + f(\boldsymbol{\mu})\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\}
\end{aligned}$$

$$\begin{aligned}
\text{var}[f(\hat{\boldsymbol{\theta}})] &= \mathbb{E}[f^2(\hat{\boldsymbol{\theta}})] - \mathbb{E}^2[f(\hat{\boldsymbol{\theta}})] \\
&= f^2(\boldsymbol{\mu}) + \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{4}(\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\})^2 + \frac{1}{2}\text{tr}\{(\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\} + f(\boldsymbol{\mu})\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\} \\
&\quad - \left(f(\boldsymbol{\mu}) + \frac{1}{2}\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\}\right)^2 \\
&= f^2(\boldsymbol{\mu}) + \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{4}(\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\})^2 + \frac{1}{2}\text{tr}\{(\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\} + f(\boldsymbol{\mu})\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\} \\
&\quad - f^2(\boldsymbol{\mu}) - f(\boldsymbol{\mu})\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\} - \frac{1}{4}(\text{tr}\{\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\})^2 \\
&= \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{2}\text{tr}\{(\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}
\end{aligned}$$

$\mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D}$ is the first-order part and $\frac{1}{2}\text{tr}\{(\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2\}$ is the second-order part.

Simple Mediation

$$\hat{\boldsymbol{\theta}} = [a \quad b]' \quad f(\hat{\boldsymbol{\theta}}) = \hat{a}\hat{b} \quad \mathbf{D}' = [\hat{b} \quad \hat{a}]$$

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s_{\hat{a}}^2 & 0 \\ 0 & s_{\hat{b}}^2 \end{bmatrix}$$

$$\begin{aligned} \text{var}[f(\hat{\boldsymbol{\theta}})] &\approx \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{2} \text{tr} \left\{ (\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2 \right\} \\ &= [\hat{b} \quad \hat{a}] \begin{bmatrix} s_{\hat{a}}^2 & 0 \\ 0 & s_{\hat{b}}^2 \end{bmatrix} \begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} + \frac{1}{2} (2s_{\hat{a}}^2 s_{\hat{b}}^2) \\ &= \hat{b}^2 s_{\hat{a}}^2 + \hat{a} s_{\hat{b}}^2 + s_{\hat{a}}^2 s_{\hat{b}}^2 \end{aligned}$$

Model 1

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1 \quad \hat{b}_1 \quad \hat{b}_2]' \quad f(\hat{\boldsymbol{\theta}}|X) = \hat{a}_1(\hat{b}_1 + \hat{b}_2 X) \quad \mathbf{D}' = [\hat{b}_1 + \hat{b}_2 X \quad \hat{a}_1 \quad \hat{a}_1 X]$$

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & X \\ 1 & 0 & 0 \\ X & 0 & 0 \end{bmatrix} \quad \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s_{\hat{a}_1}^2 & 0 & 0 \\ 0 & s_{\hat{b}_1}^2 & s_{\hat{b}_1, \hat{b}_2} \\ 0 & s_{\hat{b}_1, \hat{b}_2} & s_{\hat{b}_2}^2 \end{bmatrix}$$

$$\begin{aligned} \text{var}[f(\hat{\boldsymbol{\theta}}|X)] &\approx \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{2} \text{tr} \left\{ (\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2 \right\} \\ &= [(\hat{b}_1 + \hat{b}_2 X) s_{\hat{a}_1}^2 \quad \hat{a}_1 s_{\hat{b}_1}^2 + \hat{a}_1 s_{\hat{b}_1, \hat{b}_2} X \quad \hat{a}_1 s_{\hat{b}_1, \hat{b}_2} + \hat{a}_1 s_{\hat{b}_2}^2] \begin{bmatrix} \hat{b}_1 + \hat{b}_2 X \\ \hat{a}_1 \\ \hat{a}_1 X \end{bmatrix} \\ &\quad + s_{\hat{a}_1}^2 s_{\hat{b}_1}^2 + 2s_{\hat{a}_1}^2 s_{\hat{b}_1, \hat{b}_2} X + s_{\hat{a}_1}^2 s_{\hat{b}_2}^2 X^2 \\ &= (\hat{b}_1 + \hat{b}_2 X)^2 s_{\hat{a}_1}^2 + \hat{a}_1^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} X + s_{\hat{b}_2}^2 X^2) \\ &\quad + s_{\hat{a}_1}^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} X + s_{\hat{b}_2}^2 X^2) \\ &= (\hat{b}_1 + \hat{b}_2 X)^2 s_{\hat{a}_1}^2 + (\hat{a}_1^2 + s_{\hat{a}_1}^2) (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} X + s_{\hat{b}_2}^2 X^2) \end{aligned}$$

Model 2

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1 \quad \hat{a}_3 \quad \hat{b}_1]' \quad f(\hat{\boldsymbol{\theta}}|W) = \hat{b}_1(\hat{a}_1 + \hat{a}_3W) \quad \mathbf{D}' = [\hat{b}_1 \quad \hat{b}_1W \quad \hat{a}_1 + \hat{a}_3W]$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & W \\ 1 & W & 0 \end{bmatrix} \quad \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s_{\hat{a}_1}^2 & s_{\hat{a}_1, \hat{a}_3} & 0 \\ s_{\hat{a}_1, \hat{a}_3} & s_{\hat{a}_3}^2 & 0 \\ 0 & 0 & s_{\hat{b}_1}^2 \end{bmatrix}$$

$$\begin{aligned} \text{var}[f(\hat{\boldsymbol{\theta}}|X)] &\approx \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{2} \text{tr} \left\{ (\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2 \right\} \\ &= [\hat{b}_1 s_{\hat{a}_1}^2 + \hat{b}_1 s_{\hat{a}_1, \hat{a}_3} W \quad \hat{b}_1 s_{\hat{a}_1, \hat{a}_3} + \hat{b}_1 s_{\hat{a}_3}^2 W \quad s_{\hat{b}_1}^2 (\hat{a}_1 + \hat{a}_3W)] \begin{bmatrix} \hat{b}_1 \\ \hat{b}_1W \\ \hat{a}_1 + \hat{a}_3W \end{bmatrix} \\ &\quad + s_{\hat{b}_1}^2 (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3}W + s_{\hat{a}_3}^2 W^2) \\ &= (\hat{a}_1 + \hat{a}_3W)^2 s_{\hat{b}_1}^2 + \hat{b}_1^2 (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3}W + s_{\hat{a}_3}^2 W^2) \\ &\quad + s_{\hat{b}_1}^2 (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3}W + s_{\hat{a}_3}^2 W^2) \\ &= (\hat{a}_1 + \hat{a}_3W)^2 s_{\hat{b}_1}^2 + (\hat{b}_1^2 + s_{\hat{b}_1}^2)(s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3}W + s_{\hat{a}_3}^2 W^2) \end{aligned}$$

Model 3

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1 \quad \hat{b}_1 \quad \hat{b}_3]' \quad f(\hat{\boldsymbol{\theta}}|W) = \hat{a}_1(\hat{b}_1 + \hat{b}_3W) \quad \mathbf{D}' = [\hat{b}_1 + \hat{b}_3W \quad \hat{a}_1 \quad \hat{a}_1W]$$

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & W \\ 1 & 0 & 0 \\ W & 0 & 0 \end{bmatrix} \quad \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s_{\hat{a}_1}^2 & 0 & 0 \\ 0 & s_{\hat{b}_1}^2 & s_{\hat{b}_1, \hat{b}_3} \\ 0 & s_{\hat{b}_1, \hat{b}_3} & s_{\hat{b}_3}^2 \end{bmatrix}$$

$$\begin{aligned} \text{var}[f(\hat{\boldsymbol{\theta}}|X)] &\approx \mathbf{D}'\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}})\mathbf{D} + \frac{1}{2} \text{tr} \left\{ (\mathbf{H}\hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2 \right\} \\ &= [s_{\hat{a}_1}^2 (\hat{b}_1 + \hat{b}_3W) \quad \hat{a}_1 s_{\hat{b}_1}^2 + \hat{a}_1 s_{\hat{b}_1, \hat{b}_3} W \quad \hat{a}_1 s_{\hat{b}_1, \hat{b}_3} + \hat{a}_1 s_{\hat{b}_3}^2 W] \\ &\quad + s_{\hat{a}_1}^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3}W + s_{\hat{b}_3}^2 W^2) \end{aligned}$$

$$\begin{aligned}
&= (\hat{b}_1 + \hat{b}_3 W)^2 s_{\hat{a}_1}^2 + \hat{a}_1^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} W + s_{\hat{b}_3}^2 W^2) \\
&\quad + s_{\hat{a}_1}^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} W + s_{\hat{b}_3}^2 W^2) \\
&= (\hat{b}_1 + \hat{b}_3 W)^2 s_{\hat{a}_1}^2 + (\hat{a}_1^2 + s_{\hat{a}_1}^2) (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} W + s_{\hat{b}_3}^2 W^2)
\end{aligned}$$

Model 4

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1 \quad \hat{a}_3 \quad \hat{b}_1 \quad \hat{b}_3]' \quad f(\hat{\boldsymbol{\theta}}|W, Z) = (\hat{a}_1 + \hat{a}_3 W)(\hat{b}_1 + \hat{b}_3 Z)$$

$$\mathbf{D}' = [\hat{b}_1 + \hat{b}_3 Z \quad \hat{b}_1 W + \hat{b}_3 WZ \quad \hat{a}_1 + \hat{a}_3 W \quad \hat{a}_1 Z + \hat{a}_3 WZ]$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & Z \\ 0 & 0 & W & WZ \\ 1 & W & 0 & 0 \\ Z & WZ & 0 & 0 \end{bmatrix} \quad \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s_{\hat{a}_1}^2 & s_{\hat{a}_1, \hat{a}_3} & 0 & 0 \\ s_{\hat{a}_1, \hat{a}_3} & s_{\hat{a}_3}^2 & 0 & 0 \\ 0 & 0 & s_{\hat{b}_1}^2 & s_{\hat{b}_1, \hat{b}_3} \\ 0 & 0 & s_{\hat{b}_1, \hat{b}_3} & s_{\hat{b}_3}^2 \end{bmatrix}$$

$$\text{var}[f(\hat{\boldsymbol{\theta}}|X)] \approx \mathbf{D}' \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) \mathbf{D} + \frac{1}{2} \text{tr} \left\{ (\mathbf{H} \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2 \right\}$$

$$= (\hat{a}_1 + \hat{a}_3 W)^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} Z + s_{\hat{b}_3}^2 Z^2)$$

$$+ (\hat{b}_1 + \hat{b}_3 Z)^2 (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2)$$

$$+ (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_3} Z + s_{\hat{b}_3}^2 Z^2) (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2)$$

Model 5

$$\hat{\boldsymbol{\theta}} = [\hat{a}_1 \quad \hat{a}_3 \quad \hat{b}_1 \quad \hat{b}_2]' \quad f(\hat{\boldsymbol{\theta}}|W) = (\hat{a}_1 + \hat{a}_3 W)(\hat{b}_1 + \hat{b}_2 W)$$

$$\mathbf{D}' = [\hat{b}_1 + \hat{b}_2 W \quad \hat{b}_1 W + \hat{b}_2 W^2 \quad \hat{a}_1 + \hat{a}_3 W \quad \hat{a}_1 W + \hat{a}_3 W^2]$$

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & W \\ 0 & 0 & W & W^2 \\ 1 & W & 0 & 0 \\ W & W^2 & 0 & 0 \end{bmatrix} \quad \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} s_{\hat{a}_1}^2 & s_{\hat{a}_1, \hat{a}_3} & 0 & 0 \\ s_{\hat{a}_1, \hat{a}_3} & s_{\hat{a}_3}^2 & 0 & 0 \\ 0 & 0 & s_{\hat{b}_1}^2 & s_{\hat{b}_1, \hat{b}_2} \\ 0 & 0 & s_{\hat{b}_1, \hat{b}_2} & s_{\hat{b}_2}^2 \end{bmatrix}$$

$$\begin{aligned} \text{var}[f(\hat{\boldsymbol{\theta}}|X)] &\approx \mathbf{D}' \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}) \mathbf{D} + \frac{1}{2} \text{tr} \left\{ (\mathbf{H} \hat{\boldsymbol{\Sigma}}(\hat{\boldsymbol{\theta}}))^2 \right\} \\ &= (\hat{b}_1 + \hat{b}_2 W)^2 (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2) \\ &\quad + (\hat{a}_1 + \hat{a}_3 W)^2 (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} W + s_{\hat{b}_2}^2 W^2) \\ &\quad + (s_{\hat{a}_1}^2 + 2s_{\hat{a}_1, \hat{a}_3} W + s_{\hat{a}_3}^2 W^2) (s_{\hat{b}_1}^2 + 2s_{\hat{b}_1, \hat{b}_2} W + s_{\hat{b}_2}^2 W^2) \end{aligned}$$