國立高雄大學統計學研究所

碩士論文

A study of some different concepts of symmetry on the real line

實數上有關對稱性概念的探討

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摘要

近年來，陸續有對正實數上，對稱性概念的研究。例如對數對稱，倒數對稱，及同時具備兩種對稱性，所謂雙重對稱等。在本論文中，將把這些對稱性，推廣至實數上。我們將利用偏斜分佈表示法，及先前的一些研究結果，刻劃在實數上的雙重對稱。除此之外，我們也呈現一些關於反對稱(即在實數上與對數對稱相似的對稱)的有趣例子。

關鍵字: 倒數對稱，對數對稱，反對稱，雙重對稱，偏斜表示法，刻劃。

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ABSTRACT

Recently, different concepts of symmetry on R^+ such as R-symmetry, log-symmetry, and doubly symmetry are studied. Analogue concept and their properties of these symmetries on R will be studied in this work. Based on skewing representation and previous studies, characterizations of doubly symmetry on R will be given. Among others, some interesting examples of the so-called I-symmetry, that is the analogue of log-symmetry on R, will also be presented.

Key words and phrases: R-symmetry; log-symmetry; I-symmetry; doubly symmetry; skewing representation; characterization.

1 Introduction

A random variable (r.v.) X is said to be symmetric about a constant μ, if $X-\mu$ and $\mu-X$ have the same distribution, denote it by $X-\mu \stackrel{d}{=} \mu-X$. If $\mu=0$, we simply say X is symmetric. Recently, different concepts of symmetry on R^+ are introduced and investigated. Mudholkar and Wang (2007) gave the definition of R-symmetric distribution on R^+. According to their definition, a positive r.v. X with probability density function (p.d.f.) f_X is said to be R-symmetric about the R-center θ, where $\theta>0$, if $f_X(\theta x)=f_X\left(\theta/x\right),\ x>0$. Earlier in 1965, Seshadri studied another non-ordinary symmetry. He characterized those nonnegative r.v.'s X's on R^+ such that $X \stackrel{d}{=} 1/X$. Jones (2008) refered to this as log-symmetry since $\log X \stackrel{d}{=} -\log X$. When X is defined on R^+, Jones (2008) also studied R-symmetry and log-symmetry about δ, $\delta>0$. The latter is, $X/\delta \stackrel{d}{=} \delta/X$, which is equivalent to ordinary symmetry about $\log\delta$ of the r.v. $\log X$. Among others, Jones (2008) pointed out that when X has a p.d.f. f, $X/\delta \stackrel{d}{=} \delta/X$ is equivalent to $x^2 f_X(\delta x)=f_X\left(\delta/x\right),\ x>0$.

Jones and Arnold (2008) studied the r.v.'s defined on R^+ which are both R-symmetry and log-symmetry, the so-called doubly symmetry. An example of doubly symmetric distribution is lognormal. They also characterized the class of absolutely continuous r.v.'s defined on R^+ that are doubly symmetric, which turns out to be a proper subset of absolutely continuous and moment-equivalent to the lognormal distribution.

In this work, we will investigate natural analogue of the concepts of R-symmetry, log-symmetry, and doubly symmetry on R. More precisely, we call the analogue of log-symmetry on R as I-symmetry. Here 'I' stands for 'inverse'. Throughout this work, unless it is stated, every r.v. is assumed to follow an absolutely continuous distribution. Also for an r.v., say X, let f_X denote the p.d.f. of X.

First, we give the definitions of those symmetries mentioned above.

Definition 1. An r.v. X defined on R is said to be R-symmetric about the R-center θ, where $\theta>0$, if

$$f_X(\theta x)=f_X\left(\frac{\theta}{x}\right),\ x\in R\backslash\{0\}, \tag{1}$$

or equivalently, if

$$f_X(x)=f_X\left(\frac{\theta^2}{x}\right),\ x\in R\backslash\{0\}.$$

It can be shown easily from (1), if X is R-symmetric on R, then $f_X(0)=0$.

Definition 2. An r.v. X defined on R is said to be I-symmetric about δ, where $\delta>0$, if

$$\frac{X}{\delta} \stackrel{d}{=} \frac{\delta}{X},$$

or equivalently, if

$$x^2 f_X(\delta) = f_X\left(\frac{\delta}{x}\right),\ x \in R\backslash\{0\}. \tag{2}$$

Definition 3. An r.v. X defined on R is said to be doubly symmetric about (θ, δ), where $\theta, \delta > 0$, if X is both R-symmetric about θ and I-symmetric about δ.

In Section 2, based on mixture distribution, we investigate the relationship between doubly symmetry on R^+ and doubly symmetry on R. In Sections 3, 4, and 5, we give some propositions of R-symmetry, I-symmetry, and doubly symmetry, respectively. Next, in Section 6, we discuss the connection between mixture distribution representation and skewing representation. By the skewing representation, we investigate and characterize doubly symmetry. Finally, in Section 7, we give some interesting examples of I-symmetry.

2 Preliminary Results

Let X be an r.v. defined on R. Obviously f_X can have the following mixture representation:

$$f_X(x) = \begin{cases} a f_1(x), & x > 0, \\ (1-a) f_2(-x), & x \leq 0, \end{cases} \tag{3}$$

where

$$a = P(X > 0) = \int_0^\infty f_X(x)dx, \tag{4}$$

and $a \in [0, 1]$. Then both $f_1(x) = f_X(x)/a$ and $f_2(x) = f_X(-x)/(1-a)$ are p.d.f.'s on R^+. Note that f_1 is defined to be 0 if $a = 0$, and f_2 is defined to be 0 if $a = 1$. It can be seen if $a = 0$, then X is defined on R^-; if $a = 1$, then X is defined on R^+. Based on the above representation, we have the following simple lemma.

Lemma 1. Let X be an r.v. defined on R with $0 < a < 1$, where a is defined in (4). Then f_X is doubly symmetric about (θ, δ) if and only if both f_1 and f_2 are doubly symmetric about (θ, δ), where f_1 and f_2 are given in (3).

By Lemma 1, we have the following two immediate consequences.

Corollary 1: Let X be an r.v. defined on R with $0 < a < 1$, where a is defined in (4). Then f_X is R-symmetric about θ if and only if both the f_1 and f_2 given in (3) are R-symmetric about θ.

Corollary 2: Let X be an r.v. defined on R with $0 < a < 1$, where a is defined in (4). Then f_X is I-symmetric about δ if and only if both f_1 and f_2 given in (3) are log-symmetric about δ.

Remark 1: Suppose X is symmetric about 0. Clearly, the constant a in (3) is equal to 0.5 and $f_1(x) = f_2(x) = f_{|X|}(x)$, $x > 0$. Then according to Corollary 2, X is I-symmetric about δ if and only if $|X|$ is log-symmetric about δ. Also according to Lemma 1, X is doubly symmetric about (θ, δ) if and only if $|X|$ is doubly symmetric about (θ, δ). Furthermore, according to Corollary 1, X is R-symmetric about θ if and only if $|X|$ is R-symmetric about θ.

3 R-Symmetry on *R*

In this section, we give some simple properties of R-symmetry. The first proposition gives the mode of an R-symmetric distribution.

Proposition 1. Let the r.v. X defined on R be R-symmetric about θ. Then $\max_{x\in R} f_X(x) = \max\{f_X(\theta), f_X(-\theta)\}$.

Proof: Obviously we only need to prove the case $0 < a < 1$, where a is defined in (4). According to Mudholkar and Wang (2007), for an r.v. which is R-symmetric about θ on R^+, then θ is the mode. From Corollary 1, as X is R-symmetric about θ on R, $f_X(x) = af_1(x)$, $a \in [0, 1]$, $x > 0$, where f_1 is R-symmetric about θ on R^+, then $\max_{x\in R^+} f_X(x) = \max_{x\in R^+} af_1(x) = af_1(\theta) = f_X(\theta)$. Note that as mentioned it before, being R-symmetric, $f_X(0) = 0$. Similarly, we have $\max_{x\in R^-} f_X(x) = \max_{x\in R^-}(1-a)f_2(x) = (1-a)f_2(-\theta) = f_X(-\theta)$, where f_2 is R-symmetric about θ on R^+. Hence $\max_{x\in R} f_X(x) = \max\{f_X(\theta), f_X(-\theta)\}$. This completes the proof.

Proposition 2. Let the r.v. X defined on R be R-symmetric about θ. Then for every constant $a > 0$, aX is R-symmetric about $a\theta$.

Proof: Upon changing of variable, it yields

$$f_{aX}(x) = f_X\left(\frac{x}{a}\right)\frac{1}{a},\ x \in R. \tag{5}$$

Therefore,

$$f_{aX}(a\theta x) = f_X\left(\frac{a\theta x}{a}\right)\frac{1}{a} = f_X\left(\frac{a\theta}{ax}\right)\frac{1}{a} = f_{aX}\left(\frac{a\theta}{x}\right),\ x \in R\backslash\{0\},$$

where the first and last equalities are by (5), and the second equality is by (1). This completes the proof.

For independent r.v.'s X and Y defined on R^+ which are R-symmetric about θ_1 and θ_2, respectively, Mudholkar and Wang (2007) proved that XY is R-symmetric about $\theta_1\theta_2$. The next proposition shows that this property also holds for R-symmetry on R.

Proposition 3. Let the independent r.v.'s X and Y defined on R be R-symmetric about θ_1 and θ_2, respectively. Then XY is R-symmetric about $\theta_1\theta_2$.

Proof: Let $X_1 = X/\theta_1$, $Y_1 = Y/\theta_2$, and $W = X_1 Y_1$. Then both X_1 and Y_1 are R-symmetric about 1 and

$$f_W(w) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X_1}\left(\frac{w}{y}\right) f_{Y_1}(y) dy = \int_{-\infty}^{\infty} \frac{1}{|t|} f_{X_1}(wt) f_{Y_1}\left(\frac{1}{t}\right) dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{|t|} f_{X_1}(wt) f_{Y_1}(t) dt, \ w \in R, \tag{6}$$

where the change of variable $y = 1/t$, and $f_{Y_1}(t) = f_{Y_1}(1/t)$ are used. On the other hand, since $f_{X_1}(wt) = f_{X_1}(1/(wt))$, from (6) we obtain

$$f_W\left(\frac{1}{w}\right) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X_1}\left(\frac{1}{wy}\right) f_{Y_1}(y) dy = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X_1}(wy) f_{Y_1}(y) dy, \ w \in R\backslash\{0\}.$$

This proves W is R-symmetric about 1. Consequently, $XY = \theta_1\theta_2 W$ is R-symmetric about $\theta_1\theta_2$. This completes the proof.

Obviously, Proposition 3 can be easily extended to the situation of n r.v.'s. However, if X is R-symmetric about θ, $1/X$ may not be R-symmetric for any center c. Consequently, under the conditions of Proposition 3, X/Y may not be R-symmetric about any center c. We give an example in the following.

Example 1. Let X and Y be i.i.d. with the distribution of the root-reciprocal of $IG(1, \lambda)$ (IG stands for inverse Gaussian). That is

$$f_X(x) = f_Y(x) = \sqrt{\frac{2\lambda}{\pi}} \exp\left(-\frac{\lambda}{2}\left(\frac{1}{x} - x\right)^2\right), \ x > 0.$$

Then as pointed out by Mudholkar and Wang (2007), both X and Y are R-symmetric in R about 1. It can be found that the p.d.f.'s of $U = XY$, $V = 1/X$, and $W = X/Y$ are given by

$$f_U(u) = \int_0^{\infty} \frac{2\lambda}{\pi x} \exp\left(-\frac{\lambda}{2}\left(\frac{1}{x} - x\right)^2\right) \exp\left(-\frac{\lambda}{2}\left(\frac{u}{x} - \frac{x}{u}\right)^2\right) dx, \ u > 0,$$

$$f_V(v) = \frac{\sqrt{2\lambda}}{\sqrt{\pi} v^2} \exp\left(-\frac{\lambda}{2}\left(\frac{1}{v} - v\right)^2\right), \ v > 0,$$

and

$$f_W(w) = \frac{2\lambda}{\pi} \int_0^{\infty} x \exp\left(-\frac{\lambda}{2}\left(\frac{1}{x} - x\right)^2\right) \exp\left(-\frac{\lambda}{2}\left(\frac{1}{xw} - xw\right)^2\right) dx$$
$$= \frac{\lambda e^{2\lambda}}{\pi w} \int_0^{\infty} \exp\left(-\frac{\lambda}{2}\left(\frac{1}{w} + w\right)\left(\frac{1}{y} + y\right)\right) dy, \ w > 0,$$

respectively.

Now

$$f_U\left(\frac{1}{u}\right) = \int_0^\infty \frac{2\lambda}{\pi x} \exp\left(-\frac{\lambda}{2}\left(\frac{1}{x} - x\right)^2\right) \exp\left(-\frac{\lambda}{2}\left(\frac{1}{ux} - ux\right)^2\right) dx$$

$$= \int_0^\infty \frac{2\lambda}{\pi t} \exp\left(-\frac{\lambda}{2}\left(\frac{u}{t} - \frac{t}{u}\right)^2\right) \exp\left(-\frac{\lambda}{2}\left(\frac{1}{t} - t\right)^2\right) dt = f_U(u),\ u \neq 0.$$

This proves U is R-symmetric about 1.

Next for $v \neq 0$,

$$\frac{f_V(v)}{f_V(\theta^2/v)} = \frac{\frac{2\lambda}{\sqrt{\pi}} \exp(-\frac{2\lambda}{\pi}(v - \frac{1}{v})^2)\frac{1}{v^2}}{\frac{2\lambda}{\sqrt{\pi}} \exp(-\frac{2\lambda}{\pi}(\frac{\theta^2}{v} - \frac{v}{\theta^2})^2)\frac{v^2}{\theta^4}} = \frac{\theta^4}{v^4} \exp\left(-\frac{2\lambda}{\pi}\left(\theta^2 - \frac{1}{\theta^2}\right)\left(\frac{v^2}{\theta^2} - \frac{\theta^2}{v^2}\right)\right), \tag{7}$$

which can be shown easily is a strictly monotone decreasing function of v. Also $f_V(v)/f_V(\theta^2/v) = 1$ when $v = \theta$. Hence for every $\theta > 0$, $f_V(v)/f_V(\theta^2/v) \neq 1$, if $v \neq \theta$. This proves V is not R-symmetric about any center θ.

Finally,

$$\frac{f_W(w)}{f_W(\theta^2/w)} = \frac{\theta^2}{w^2} \frac{\int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{1}{w} + w\right)\left(\frac{1}{y} + y\right)\right) dy}{\int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{\theta^2}{w} + \frac{w}{\theta^2}\right)\left(\frac{1}{y} + y\right)\right) dy},\ w \neq 0. \tag{8}$$

Clearly when $\theta^2 = 1$, $f_W(w)/f_W(c/w) \neq 1$ for $w \neq 1$. For $\theta^2 \neq 1$, since $(\theta^2 + 1/\theta^2)/2 > 1$, and $g(z) = e^{-z}$, $z > 0$, is a strictly decreasing function of z, we have

$$\frac{\int_0^\infty \exp\left(-\lambda\left(\frac{1}{y} + y\right)\right) dy}{\int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\theta^2 + \frac{1}{\theta^2}\right)\left(\frac{1}{y} + y\right)\right) dy} > 1.$$

Hence if $\theta^2 > 1$, then $\theta^2/w^2 > 1$, and it yields $f_W(1)/f_W(\theta^2/1) > 1$. Now consider the case $\theta^2 < 1$. Assume there exists a θ, $0 < \theta < 1$, such that $f_W(w) = f_W(\theta^2/w)$, for $w \in R^+$. Then from (8) we obtain

$$\int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{1}{w} + w\right)\left(\frac{1}{y} + y\right)\right) dy = \frac{w^2}{\theta^2} \int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{\theta^2}{w} + \frac{w}{\theta^2}\right)\left(\frac{1}{y} + y\right)\right) dy.$$

By replacing w by w/θ^2, it yields

$$\int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{1}{w} + w\right)\left(\frac{1}{y} + y\right)\right) dy = \frac{w^4}{\theta^8} \int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{\theta^4}{w} + \frac{w}{\theta^2}\right)\left(\frac{1}{y} + y\right)\right) dy.$$

Repeating this procedure, in the kth time replacing w by w/θ^{2k}, we obtain

$$\int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{1}{w} + w\right)\left(\frac{1}{y} + y\right)\right) dy = w^{2n} \int_0^\infty \frac{1}{\theta^{2n^2}} \exp\left(-\frac{\lambda}{2}\left(\frac{\theta^{2n}}{w} + \frac{w}{\theta^{2n}}\right)\left(\frac{1}{y} + y\right)\right) dy$$

holds for every $n \geq 1$. Let

$$g_n(y) = \frac{1}{\theta^{2n^2}} \exp\left(-\frac{\lambda}{2}\left(\frac{\theta^{2n}}{w} + \frac{w}{\theta^{2n}}\right)\left(\frac{1}{y} + y\right)\right),\ n \geq 1.$$

Since for every $y \geq 0$,

$$\lim_{n\to\infty} g_n(y) = 0,$$

also it can be shown easily that for every $y \geq 0$, $g_n(y)$ is decreasing in n for n large enough, by Lebesgue's monotone convergence theorem, we arrive at the following contradiction

$$\int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{1}{w}+w\right)\left(\frac{1}{y}+y\right)\right)dy = \lim_{n\to\infty} w^{2n}\int_0^\infty g_n(y)dy = 0,\ 0 < w < 1.$$

This completes the proof that W is not R-symmetric about any center θ.

4 I-Symmetry

We now give some simple properties of I-symmetry.

Proposition 4. Let the r.v. X defined on R be I-symmetric about δ. Then $P(-\delta < X \leq \delta) = 1/2$.

Proof: From Corollary 2, for X being I-symmetric about δ, then f_1 and f_2 are log-symmetric about δ, where f_1 and f_2 are given in (3). According to Jones (2008), for an r.v. defined on R^+ which is log-symmetric about δ, then δ is the median. Hence

$$P(0 < X \leq \delta | X > 0) = \int_0^\delta f_1(x)dx = 1/2,$$

and

$$P(0 < -X \leq \delta | X \leq 0) = \int_0^\delta f_2(x)dx = 1/2.$$

This completes the proof.

The proofs of the following three propositions are similar to those of the situation of log-symmetry on R^+, hence are omitted.

Proposition 5. Let the r.v. X defined on R be I-symmetric about δ. Then for every constant $a > 0$, aX is I-symmetric about $a\delta$.

Proposition 6. Let the independent r.v.'s X and Y defined on R be I-symmetric about δ_1 and δ_2, respectively. Then XY is I-symmetric about $\delta_1\delta_2$.

Again the above proposition also holds for n r.v.'s as the situation of log-symmetric on R^+ (see Jones (2008)). On the other hand, although X is R-symmetric may not imply $1/X$ is R-symmetric, it can be seen easily that this is true for I-symmetric. We state this as follows.

Proposition 7. Let the independent r.v.'s X and Y defined on R be I-symmetric about δ_1 and δ_2, respectively. Then X/Y is I-symmetric about δ_1/δ_2. In particular, $1/X$ is I-symmetric about $1/\delta_1$.

5 Doubly-Symmetry in $\boldsymbol{R}$

In this section we give some simple properties of doubly symmetry.

Proposition 8. Let the r.v. X defined on R be doubly symmetric about (θ, δ). Then for any constant $a > 0$, aX is doubly symmetric about $(a\theta, a\delta)$.

Proposition 9. Let the independent r.v.'s X and Y defined on R be doubly symmetric about (θ_2, δ_2) and (θ_2, δ_2), respectively. Then XY is doubly symmetric about $(\theta_1\theta_2, \delta_1\delta_2)$.

The proofs of the above two propositions are trivial hence are omitted.

Although the ratio of two independent R-symmetric r.v.'s may not be R-symmetric about any center, yet as pointed out in the following result, this is true if both X and Y are doubly symmetric.

Proposition 10. Let the independent r.v.'s X and Y defined on R be doubly symmetric about (θ_1, δ_1) and (θ_2, δ_2), respectively. Then X/Y is doubly symmetric about $(\theta_1\theta_2/\delta_2^2, \delta_1/\delta_2)$. In particular, $1/X$ is doubly symmetric about $(\theta_1/\delta_1^2, 1/\delta_1)$.

Proof: Based on Proposition 7, we only need to prove the "R-symmetric part". Since $Y/\delta_2 \stackrel{d}{=} \delta_2/Y$ if and only if $1/Y \stackrel{d}{=} Y/\delta_2^2$, also by Proposition 2, Y/δ_2^2 is R-symmetric about θ_2/δ_2^2, we obtain $1/Y$ is also R-symmetric about θ_2/δ_2^2. The proof now follows by Proposition 3.

6 Main Results

For a p.d.f. f_X, except (3), it can also be represented as

$$f_X(x) = 2f(x)G(x), \ x \in R, \tag{9}$$

where

$$f(x) = \frac{f_X(x) + f_X(-x)}{2} = \begin{cases} \frac{1}{2}(af_1(x) + (1-a)f_2(x)), & x > 0, \\ \frac{1}{2}(af_1(-x) + (1-a)f_2(-x)), & x \leq 0, \end{cases} \tag{10}$$

is a symmetric p.d.f., and

$$G(x) = \frac{f_X(x)}{f_X(x) + f_X(-x)} = \begin{cases} af_1(x)/(af_1(x) + (1-a)f_2(x)), & x > 0, \\ (1-a)f_2(-x)/(af_1(-x) + (1-a)f_2(-x)), & x \leq 0, \end{cases} \tag{11}$$

is a skewing function. In this section, we will characterize doubly symmetry through skewing representation.

When f_X is represented as in (9), we are interested in knowing that is it possible that f is not doubly symmetric, yet f_X is doubly symmetric? The next lemma will answer this question.

Lemma 2. Let the r.v. X defined on R be doubly symmetric about (θ, δ). Then f is also doubly symmetric about (θ, δ), where f is the symmetric p.d.f. given in (9). Consequently, $|X|$ is doubly symmetric about (θ, δ) on R^+.

Proof: That X is doubly symmetric about (θ, δ) implies

$$f(\theta x) = \frac{f_X(\theta x) + f_X(-\theta x)}{2} = \frac{f_X\left(\theta/x\right) + f_X\left(-\theta/x\right)}{2} = f\left(\frac{\theta}{x}\right),\ x \in R\backslash\{0\},$$

and

$$x^2 f(\delta x) = \frac{x^2 f_X(\delta x) + x^2 f_X(-\delta x)}{2} = \frac{f_X\left(\delta/x\right) + f_X\left(-\delta/x\right)}{2} = f\left(\frac{\delta}{x}\right),\ x \in R\backslash\{0\}.$$

This proves the first assertion. The second assertion follows immediately by noting $|X|$ has the p.d.f. $2f(x),\ x > 0$. This completes the proof.

Jones and Arnold (2008) characterized the doubly symmetry on R^+. By using their characterization and the skewing representation of a distribution as in (9), we can characterize the doubly symmetry on R.

Theorem 1. Let the r.v. X defined on R be doubly symmetric about (θ, δ). Let $k = \delta/\theta$. Also let f_X be represented as in (9). Then f has the form

$$f(x) \propto \sum_{i=-\infty}^{\infty} \theta^{-2i} k^{2i(i+1)} x^{2i} |x| \omega(\theta^{-2} k^{4(i-1)} x^2) I(\theta k^{-2i} < |x| \leq \theta k^{2-2i}),\ x \in R\backslash\{0\}, \tag{12}$$

where ω is a nonnegative function on $(k^{-4}, 1]$ and chosen to satisfy

$$\psi(u) = \psi\left(\frac{1}{k^4 u}\right),\ k^{-4} < u \leq 1, \tag{13}$$

where

$$\psi(u) \equiv u\omega(u), \tag{14}$$

and G is chosen to satisfy

$$G(\theta x) = G\left(\frac{\theta}{x}\right),\ \text{and}\ G(\delta x) = G\left(\frac{\delta}{x}\right)\ x \in R\backslash\{0\}. \tag{15}$$

Proof: First from Lemma 2, we obtain $X_1 = |X|$ is doubly symmetric. Now by Jones and Arnold (2008),

$$f_{X_1}(x) \propto \sum_{i=-\infty}^{\infty} \theta^{-2i} k^{2i(i+1)} x^{2i+1} \omega(\theta^{-2} k^{4(i-1)} x^2) I(\theta k^{-2i} < x \leq \theta k^{2-2i}),\ x > 0,$$

where the nonnegative function ω defined on $(k^{-4}, 1]$ satisfying (13) and (14). Note that $f(x) = f_{X_1}(|x|)/2$, $x \in R$, hence (12) is obtained immediately.

Next due to the doubly symmetric property of X, we have (1) and (2). Then by the representation of (9), (1) and (2) in turn imply

$$2f(\theta x)G(\theta x) = 2f\left(\frac{\theta}{x}\right)G\left(\frac{\theta}{x}\right), \tag{16}$$

and

$$2x^2 f(\delta x)G(\delta x) = 2f\left(\frac{\delta}{x}\right)G\left(\frac{\delta}{x}\right), \tag{17}$$

respectively. Again, from Lemma 2, we have $f(\theta x) = f(\theta/x)$ and $x^2 f(\delta x) = f(\delta/x)$, these together with (16) and (17), imply (15) immediately. This completes the proof.

Theorem 2. Let the p.d.f. of the r.v. X be written as in (9). Then X is doubly symmetric about (θ, δ) if and only if

(i) f is doubly symmetric about (θ, δ),

(ii) G satisfies (15).

Proof: The "if" part is obvious. By Lemma 2 and Theorem 1, the "only if" follows immediately. This completes the proof.

We give an illustration of Theorem 2.

Example 2. Let the p.d.f. of the r.v. X be written as in (8), where

$$f(x) = \frac{1}{2\sqrt{2\pi}\sigma|x|}\exp\left(-\frac{(\log|x| - \mu)^2}{2\sigma^2}\right),$$

and

$$G(x) = \frac{1}{2} + \frac{\varepsilon}{2}\operatorname{sgn}(x)\cos\left(\frac{2\pi(\log|x| - \mu)}{\sigma^2}\right), \ |\varepsilon| \le 1.$$

Note that $X_1 = |X|$ has $\mathcal{L}og\mathcal{N}ormal(\mu, \sigma^2)$ distribution, which is doubly symmetric about $(e^{\mu-\sigma^2}, e^{\mu})$. From Remark 1, f is doubly symmetric about $(\theta, \delta) = (e^{\mu-\sigma^2}, e^{\mu})$. Also,

$$\begin{aligned}
G(e^{\mu-\sigma^2}x) &= \frac{1}{2} + \frac{\varepsilon}{2}\operatorname{sgn}(x)\cos\left(\frac{2\pi(\log|x| - \sigma^2)}{\sigma^2}\right) \\
&= \frac{1}{2} + \frac{\varepsilon}{2}\operatorname{sgn}(x)\cos\left(\frac{2\pi(\log|x|)}{\sigma^2} - 2\pi\right) \\
&= \frac{1}{2} + \frac{\varepsilon}{2}\operatorname{sgn}(x)\cos\left(\frac{2\pi(-\log|x|)}{\sigma^2} + 2\pi\right) \\
&= \frac{1}{2} + \frac{\varepsilon}{2}\operatorname{sgn}(x)\cos\left(\frac{2\pi(-\log|x|)}{\sigma^2} - 2\pi\right) \\
&= \frac{1}{2} + \frac{\varepsilon}{2}\operatorname{sgn}(x)\cos\left(\frac{2\pi(-\log|x| - \sigma^2)}{\sigma^2}\right) = G\left(\frac{e^{\mu-\sigma^2}}{x}\right),
\end{aligned}$$

and

$$
\begin{aligned}
G(e^{\mu}x) &= \frac{1}{2} + \frac{\varepsilon}{2}\,\mathrm{sgn}(x)\cos\left(\frac{2\pi(\log|x|)}{\sigma^2}\right) \\
&= \frac{1}{2} + \frac{\varepsilon}{2}\,\mathrm{sgn}(x)\cos\left(\frac{2\pi(-\log|x|)}{\sigma^2}\right) = G\left(\frac{e^{\mu}}{x}\right).
\end{aligned}
$$

That is the conditions for G in (15) are satisfied. Therefore, X is doubly symmetric about $(e^{\mu-\sigma^2}, e^{\mu})$.

7 Some Interesting Examples of I-Symmetry

7.1 Characterization of I-Symmetry

First we give a characterization by Seshadri (1965) of the p.d.f. of an r.v. defined on R^+ which is log-symmetric about 1.

Lemma 3. Let X be an r.v. defined on R^+. Then $X \overset{d}{=} 1/X$ if and only if

$$
f_X(x) = \frac{1}{x} g(\log x),\ x > 0, \tag{18}
$$

where g is a symmetric p.d.f.

The next lemma is an extension of the above lemma, which concerns r.v.'s defined on R.

Lemma 4. Let X be an r.v. defined on R. Then $X \overset{d}{=} 1/X$ if and only if

$$
f_X(x) = \frac{1}{|x|} g(\log|x|) G(x),\ x \in R\backslash\{0\}, \tag{19}
$$

where g is a symmetric p.d.f. and G is a skewing function which satisfies

$$
G(x) = G(1/x),\ x \in R\backslash\{0\}. \tag{20}
$$

Proof: First we prove the "if" part. Suppose (19) holds. Let $Z = 1/X$. Then

$$
f_Z(z) = f_X\left(\frac{1}{z}\right)\frac{1}{z^2} = \frac{|z|}{z^2} g\left(\log\left|\frac{1}{z}\right|\right) G\left(\frac{1}{z}\right) = \frac{1}{|z|} g(\log|z|) G(z) = f_X(z),\ z \in R\backslash\{0\},
$$

where the third equality is by the symmetry of g and (20). This proves the "if" part.

Next, assume $X \overset{d}{=} 1/X$. According to Corollary 2, both f_1 and f_2 are log-symmetric about 1, where f_1 and f_2 are given in (3). From Lemma 3, $f_1(x) = g_1(\log x)/x$ and $f_2(x) = g_2(\log x)/x$, where g_1 and g_2 are symmetric p.d.f.'s. By (9), (10), and (11),

$$
\begin{aligned}
f(x) &= \begin{cases} \frac{1}{2}(a g_1(\log x)/x + (1-a) g_2(\log x)/x), & x > 0, \\ \frac{1}{2}(a g_1(\log(-x))/(-x) + (1-a) g_2(\log(-x)))/(-x), & x < 0, \end{cases} \\
&= \frac{1}{2|x|}(a g_1(\log|x|) + (1-a) g_2(\log|x|)) \\
&= \frac{1}{2|x|} g(\log|x|),\ x \in R\backslash\{0\},
\end{aligned} \tag{21}
$$

and

$$G(x) = \begin{cases} ag_1(\log x)/(ag_1(\log x) + (1-a)g_2(\log x)), & x > 0, \\ (1-a)g_2(\log(-x))/(ag_1(\log(-x)) + (1-a)g_2(\log(-x))), & x < 0, \end{cases} \tag{22}$$

where $g(x) = ag_1(x) + (1-a)g_2(x)$, $x \in R$, which is a mixture p.d.f. of g_1 and g_2. Obviously g is also symmetric since both g_1 and g_2 are symmetric. Substituting (21) into (9), (19) follows immediately. The rest to be proved is G satisfies (20). Now from (22),

$$\begin{aligned} G\left(\frac{1}{x}\right) &= \begin{cases} ag_1(\log 1/x)/(ag_1(\log 1/x) + (1-a)g_2(\log 1/x)), & x > 0, \\ (1-a)g_2(\log(-x))/(ag_1(\log(-1/x)) + (1-a)g_2(\log(-1/x))), & x < 0. \end{cases} \\ &= \begin{cases} ag_1(\log x)/(ag_1(\log x) + (1-a)g_2(\log x)), & x > 0, \\ (1-a)g_2(\log(-x))/(ag_1(\log(-x)) + (1-a)g_2(\log(-x))), & x < 0. \end{cases} \\ &= G(x),\ x \in R\backslash\{0\}, \end{aligned}$$

where the second equality follows by the symmetry of g_1 and g_2. This completes the proof.

Remark 2. If the G in (20) is $G(x) = 0$, $x \le 0$ and $G(x) = 1$, $x > 0$, then $X > 0$ and f_X is given as in (18).

Consider an r.v. X which is I-symmetric about δ. Then by Proposition 5 and Lemma 4, the consequence given below follows immediately.

Theorem 3. Let X be an r.v. defined on R. Then $X/\delta \overset{d}{=} \delta/X$, $\delta > 0$, if and only if

$$f_X(x) = \frac{1}{|x|} g\left(\log \frac{|x|}{\delta}\right) G\left(\frac{x}{\delta}\right),\ x \in R\backslash\{0\}, \tag{23}$$

where g is a symmetric p.d.f. and G is a skewing function which satisfies (20).

We now give some examples.

Example 3. The function defined below is a skewing function satisfying (20),

$$G(x) = \begin{cases} \frac{1}{2}(1 + h(x)), & 0 \le |x| \le 1, \\ \frac{1}{2}(1 + h(\frac{1}{x})), & |x| > 1, \end{cases} \tag{24}$$

where $|h(x)| \le 1$, $|x| \le 1$, is an odd function. In particular when $h(x) = cx^n$, where $|c| \le 1$ and n is 0 or odd number, then

$$G(x) = \begin{cases} \frac{1}{2}(1 + cx^n), & 0 \le |x| \le 1, \\ \frac{1}{2}(1 + cx^{-n}), & |x| > 1. \end{cases} \tag{25}$$

Moreover $G(x) = 1/2$, $x \in R$, is a skewing function satisfying (20).

The following are examples to illustrate Theorem 3.

Example 4. Let X be $\mathcal{C}(0,1)$ distributed with p.d.f.

$$f_X(x) = \frac{1}{\pi(1+x^2)},\ x \in R.$$

Obviously $X \stackrel{d}{=} 1/X$. By choosing

$$g(x) = \frac{2e^x}{\pi(1+e^{2x})},\ x \in R, \tag{26}$$

and $G(x) = 1/2$, $x \in R$, then

$$f_X(x) = \frac{1}{\pi(1+x^2)} = \frac{1}{|x|} g(\log|x|)G(x),\ x \in R \backslash \{0\}.$$

On the other hand, if

$$f_X(x) = \frac{2}{\pi(1+x^2)} G(x),\ x \in R,$$

where G is given in (24), then it is still true that $X \stackrel{d}{=} 1/X$.

Example 5. Let $g(x) = \frac{1}{2}e^{-|x|}$, $x \in R$, the p.d.f. of a Laplace distribution, and

$$G(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases}$$

a skewing function which satisfies (20). Then

$$f_X(x) = \frac{1}{|x|} g(\log x)G(x) = \begin{cases} \frac{1}{2}, & 0 < x < 1, \\ \frac{1}{2x^2}, & x \geq 1. \end{cases}$$

This is the p.d.f. of U/V, where U and V are i.i.d. $\mathcal{U}(0,1)$ r.v.'s. Note that if $X \stackrel{d}{=} U/V$, where U and V are i.i.d. r.v.'s, then X is I-symmetric about 1.

Although the ratio of two i.i.d. r.v.'s has a distribution of I-symmetric about 1, conversely, for an r.v. Z which is log-symmetric about 1, we will show below that there may not exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$. As $\log Z$ is symmetric, if there is a symmetric r.v. which is not distributed as the difference of two i.i.d. r.v.'s, then this offers an example that a log-symmetric r.v. cannot be distributed as the ratio of two i.i.d. r.v.'s.

Throughout the rest of this section, for an r.v. Z, let $\psi_Z(t)$, $t \in R$, denote the characteristic function (ch.f.) of Z. First we give a lemma.

Lemma 5. Let the r.v. Z defined on R^+ be log-symmetric about 1. Also let $Z_1 = \log Z$. If there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$, then $\psi_{Z_1}(t) \geq 0$ for $t \in R$.

Proof: That $Z \stackrel{d}{=} X/Y$ implies $Z_1 \stackrel{d}{=} \log X - \log Y$. Obviously, $\log X$ and $\log Y$ are also i.i.d. Consequently, $\psi_{Z_1}(t) = \psi_{\log X}(t)\psi_{\log Y}(-t) = \psi_{\log X}(t)\psi_{\log X}(-t) = |\psi_{\log X}(t)|^2 \geq 0$. This completes the proof.

For a symmetric r.v. Z_1, let $Z = e^{Z_1}$. Then Z is log-symmetric about 1. According to Lemma 5, if $\psi_{Z_1}(t) \le 0$ for some $t \in R$, then there do not exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$. The following example was given by Seshadri (1965).

Example 6. Let the p.d.f. of the r.v. Z_1 mentioned above be

$$f_{Z_1}(z) = \frac{1}{\sqrt{2\pi}} z^2 e^{-z^2/2},\ z \in R.$$

Then

$$\psi_{Z_1}(t) = \sqrt{\frac{2}{\pi}}(1-t^2)e^{-t^2/2},\ t \in R.$$

That there do not exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$ follows by noting $\psi_{Z_1}(t) < 0$ when $|t| > 1$.

Next we give a sufficient condition for the distribution of Z defined on R^+ which can be represented as X/Y, where X and Y are i.i.d. r.v.'s.

Theorem 4. Let the r.v. Z defined on R^+ be log-symmetric about 1, and let $Z_1 = \log Z$. If $\sqrt{\psi_{Z_1}}$ is also a ch.f., then there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$.
Proof: That Z is log symmetric about 1 implies ψ_{Z_1} is real and even. Hence $\sqrt{\psi_{Z_1}}$ is also even. Let i.i.d. r.v.'s X_1 and Y_1 have ch.f. $\sqrt{\psi_{Z_1}}$. Then

$$\psi_{X_1-Y_1}(t) = \psi_{X_1}\psi_{Y_1}(-t) = \sqrt{\psi_{Z_1}(t)}\sqrt{\psi_{Z_1}(-t)} = (\sqrt{\psi_{Z_1}(t)})^2 = \psi_{Z_1}(t),\ t \in R.$$

Consequently, $Z_1 \stackrel{d}{=} X_1 - Y_1$, which in turn implies $Z \stackrel{d}{=} X/Y$, where $X = e^{X_1}$, and $Y = e^{Y_1}$. This completes the proof.

Theorem 4 has an immediate consequence.

Corollary 3. Let the r.v. Z defined on R^+ be log-symmetric about 1. Also let $Z_1 = \log Z$. If ψ_{Z_1} is infinitely divisible, then there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$.
Proof: That ψ_{Z_1} is infinitely divisible implies $\sqrt{\psi_{Z_1}}$ is also a ch.f. By Theorem 1, there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$. This completes the proof.

The following is the Pólya type criterion for ch.f.'s, which can be found in Chung (2001), p.191.

Theorem 5. Let the function ψ on R satisfy

$$\psi(0) = 1,\ \psi(t) \ge 0,\ \psi(t) = \psi(-t),\ t \in R, \tag{27}$$

and ψ is decreasing and continuous convex on R^+. Then ψ is a ch.f.

If a ch.f. ψ satisfies the sufficient condition of Theorem 5, then it is said to be a Pólya type ch.f. Theorems 4 and 5 yield the following consequence.

Corollary 4. Let the r.v. Z defined on R^+ be log-symmetric about 1. Also let $Z_1 = \log Z$. If ψ_{Z_1} is a Pólya type ch.f. satisfying

$$\psi_{Z_1}(t) > 0,\ \psi''_{Z_1}(t)\psi_{Z_1}(t) - \frac{1}{2}(\psi'_{Z_1}(t))^2 \geq 0,\ t > 0, \tag{28}$$

then there exist two i.i.d. r.v.'s X and Y such that $Z \overset{d}{=} X/Y$.

Proof: Firstly, we show that $\sqrt{\psi_{Z_1}}$ is also a Pólya type ch.f. Obviously, $\sqrt{\psi_{Z_1}}$ satisfies (27). Also, since ψ_{Z_1} is decreasing and continuous on R^+, so is $\sqrt{\psi_{Z_1}}$. Now

$$(\sqrt{\psi_{Z_1}(t)})'' = \frac{\psi''_{Z_1}(t)\psi_{Z_1}(t) - (\psi'_{Z_1}(t))^2/2}{2\psi_{Z_1}(t)\sqrt{\psi_{Z_1}(t)}} \geq 0 \tag{29}$$

by (28). Consequently, $\sqrt{\psi_{Z_1}}$ is convex on R^+. Therefore, $\sqrt{\psi_{Z_1}}$ is also a Pólya type ch.f. By Theorem 4, there exist two i.i.d. r.v.'s X and Y such that $Z \overset{d}{=} X/Y$. This completes the proof.

It is known that both $\mathcal{C}(0,1)$ and $\mathcal{N}(0,1)$ distributions are infinitely divisible. We now present some examples to illustrate Corollary 3.

Example 7. Let the p.d.f. of the r.v. Z be

$$f_Z(z) = \frac{1}{\pi z(1 + (\log z)^2)},\ z > 0.$$

Then Z is log symmetric about 1, and Z_1 is $\mathcal{C}(0,1)$ distributed, where $Z_1 = \log Z$. Since $\mathcal{C}(0,1)$ distribution is infinitely divisible, according to Corollary 3, there exist two i.i.d. r.v.'s X and Y such that $Z \overset{d}{=} X/Y$. As can be seen below, if the common p.d.f. of X and Y is

$$f_X(x) = \frac{1/2}{\pi x(1/4 + (\log x)^2)},\ x > 0,$$

then this can be served as an example. Let $W = X/Y$. Then

$$\begin{aligned} f_W(w) &= \int_0^\infty y f_X(wy) f_X(y) dy \\ &= \int_0^\infty y \frac{1/2}{\pi wy(1/4 + (\log wy)^2)} \frac{1/2}{\pi y(1/4 + (\log y)^2)} dy \\ &= \frac{1}{\pi w(1 + (\log w)^2)} = f_Z(w),\ w > 0. \end{aligned}$$

Example 8. Let the r.v. Z have $\mathcal{L}og\mathcal{N}ormal(0,1)$ distribution. Also let $Z_1 = \log Z$. Then Z is log-symmetric about 1, and Z_1 is $\mathcal{N}(0,1)$ distributed, which is infinitely divisible. Hence there exist two i.i.d. r.v.'s X and Y such that $Z \overset{d}{=} X/Y$. The r.v.'s X and Y with $\mathcal{L}og\mathcal{N}ormal(0,1/2)$ being their common distribution is an example.

Example 9. Let the r.v. Z_1 have the following Pólya type ch.f.

$$\psi_{Z_1}(t) = \frac{1}{1+|t|},\ t \in R.$$

Also let $Z = e^{Z_1}$. Then Z is log-symmetric about 1. Since

$$\psi''_{Z_1}(t)\psi_{Z_1}(t) - \frac{1}{2}(\psi'_{Z_1}(t))^2 = \frac{3}{2(1+t)^4} \geq 0,\ t \in R^+,$$

according to Corollary 4, there exist two i.i.d. r.v.'s X and Y such that $Z \overset{d}{=} X/Y$.

7.2 I-Symmetry Arising From Trigonometric Formulas

Let $Z = X/Y$. Although Z is I-symmetry about 1 if X and Y are i.i.d., the converse is not true. That the joint p.d.f. of X, Y satisfies

$$f_{X,Y}(x,y) = f_{X,Y}(y,x),\ x,y \in R, \tag{30}$$

is sufficient to imply Z is I-symmetric about 1. See also the following example by Jones (1999).

Example 10. Let (X,Y) have the polar representation

$$X = R\cos\Theta \text{ and } Y = R\sin\Theta, \tag{31}$$

where Θ is $\mathcal{U}(0,2\pi)$ distributed, and R is a positive r.v. independent with Θ. Then

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{x^2+y^2}} f_R(\sqrt{x^2+y^2}),\ x,y \in R, \tag{32}$$

which satisfies (30). Hence $\tan\Theta$ $(= Y/X)$ is I-symmetric about 1. In fact, $\tan\Theta$ is $\mathcal{C}(0,1)$ distributed.

Example 9 shows that there exists an I-symmetric distribution about 1 arising from trigonometric functions. Jones (1999) also pointed out if the Θ given in (31) is $\mathcal{U}(a,b)$ distributed, where $b-a = m\pi$, m is a positive integer, then $\tan\Theta$ has a $\mathcal{C}(0,1)$ distribution. It follows immediately that for S being an r.v. independent of Θ, where Θ is $\mathcal{U}(-\pi/2,\pi/2)$ distributed, then $\tan(n\Theta+S)$ is also $\mathcal{C}(0,1)$ distributed, where n is a positive integer. Furthermore Jones (1999) gave some multiple angle and angle sum formulas for tangent functions, which remain $\mathcal{C}(0,1)$ distributed. For example, the double angle formula for tangent function yields $(\tan\Theta - 1/\tan\Theta)/2$ is $\mathcal{C}(0,1)$ distributed. Also the multiple angle and angle sum formulas for sine and cosine functions yield some functions of X and Y have the same distribution as X and some functions of X and Y have a $\mathcal{C}(0,1)$ distribution. For example, $2XY/\sqrt{X^2+Y^2} \overset{d}{=} X$ and $2XY/(Y^2-X^2)$ is $\mathcal{C}(0,1)$ distributed (see Jones (1999)).

Inspired by Jones (1999), we present some related results in the following. Let

$$f_U(u) = \frac{2}{\pi}G(\tan u),\ u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{33}$$

where G is a skewing function satisfying (20). Let $T = \tan U$. Then it can be shown easily that T is I-symmetric about 1 with p.d.f.

$$f_T(t) = \frac{2}{\pi(1+t^2)} G(t), \ t \in R. \tag{34}$$

The following theorem points out that some of the results presented by Jones (1999) still hold for the r.v.'s U and T given above.

Theorem 6. Let T have the p.d.f. given in (34). Then

(i) $T \stackrel{d}{=} 1/T$;

(ii) $\tan(2U)$ is $\mathcal{C}(0,1)$ distributed;

(iii) $(T - 1/T)/2$ is $\mathcal{C}(0,1)$ distributed;

(iv) $\tan(2U + S)$ is $\mathcal{C}(0,1)$ distributed, where S is an r.v. independent with U;

(v) $2XY/(Y^2 - X^2)$ is $\mathcal{C}(0,1)$ distributed, where $X = R\cos U$, $Y = R\sin U$, and R is a positive r.v. independent with U;

(vi) let $V = \sin(4U)$, then

$$f_V(v) = \frac{1}{\pi\sqrt{1-v^2}}, \ |v| < 1.$$

The proof of the above theorem is standard hence is omitted.

Remark 3. If $G(x) = 1/2$, $x \in R$, that is U is $\mathcal{U}(-\pi/2, \pi/2)$ distributed, then $2U$ in (ii) and (iv) can be replaced by U, and $4U$ in (vi) can be replaced by U.

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