國立高雄大學統計學研究所

碩士論文

Some study of symmetric and skew distributions

對稱及偏斜分佈之一些探討

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中華民國九十八年七月

謝辭

一晃眼過去, 我終於要畢業了 。不敢說在學業上有何成就, 但是值得慶幸的是, 我遇到了一個好老師 。不僅在論文上的指導, 黃文璋老師讓我見識到, 待人處事該有的嚴謹及精確度 。心在南方 、投影片及論文, 一字一句的精雕細琢 、力求簡潔 。對粗心隨便的我來說, 眞的是很幸運的機緣 。雖然永遠沒辦法如老師那樣精細, 至少我見識到一個榜樣, 兼具身心的強健, 總是如標竿般引領學生前進, 謝謝老師!

另外謝謝蘭屏姊的陪伴, 每當我生活上有疑惑, 您總是耐心的提醒, 給予最好的建議, 有您在的所辦, 就是統研所溫暖的所在 。

最後感謝小竹 、小暉 、怡如 、聲杰等同學的扶持, 讓我終於脫離被孤立的魔咒 。你們幫我慶生的那天, 是研究所最快樂的事之一 。

張恩豪
於高雄大學統計學研究所
民國98年7月

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July 2009

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摘要

自從Azzalini (1985,1986) 發表偏斜常態分佈後, 一些基於常見對稱分佈的偏斜分佈之研究, 便如雨後春筍般出現 。這些偏斜分佈, 不僅包含原本的對稱分佈性質, 此外還有偏斜的特性, 所以可以用來處理更廣泛的問題 。

本論文分三部分, 探討三個關於偏斜-對稱分佈的主題 。在第一章, 先針對兩個獨立隨機變數的乘積, 給出只要乘積為對稱, 則兩個隨機變數中，至少有一個也會是對稱的條件 。接著, 對某些常見的二元隨機變數, 以邊際分佈的相同與否, 分別加以討論: 不僅給出兩個隨機變數中, 極大值與極小值的線性組合之機率密度函數, 並且提供線性組合之一些偏斜的性質 。

在第二章, 對於兩個獨立的廣義偏斜常態隨機變數, 我們給出其比值的機率密度函數 。而何時比值會服從偏斜柯西分佈? 我們將給出充分必要條件 。

在第三章, 對多元廣義偏斜常態隨機變數, 我們給一些關於二次形式及二次形式比值的動差之公式 。二次形式的討論中, 將用到正定矩陣和反矩陣 。

關鍵字: 兩元隨機變數, 橢圓分佈, 可交換性, 廣義偏斜常態分佈, 反矩陣, 線性組合, 動差, 常態分佈, 正定矩陣, 乘積, 二次形式, 比值, 偏斜, 偏斜柯西分佈, 偏斜常態分佈, 偏斜t分佈, 對稱性

Some study of symmetric and skew distributions

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ABSTRACT

Since Azzalini (1985,1986) introduced the univariate skew-normal distribution, there are many investigations about the skew distributions based on certain symmetric probability density functions. Because these classes of the skew distributions include the original symmetric distribution and have some properties like the original one and yet is skew, hence it is more useful to handle related problems.

In this thesis, we consider three topics of the symmetric and skew distributions. In Chapter 1, we will discuss the case $Z = UV$ first, where U and V are assumed to be independent. Under some conditions, we will show that if Z is symmetric, then at least one of U and V is symmetrically distributed. Next for certain bivariate symmetric random variables X and Y, we will find the distributions of $M = aU + bV$, where a and b are constants, $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$. When X and Y are assumed or not assumed to be identically distributed, we will present the distributions and skew properties of M, respectively.

In Chapter 2, we will present the probability density function of the ratio of two generalized skew-normal distributed random variables. We also give necessary and sufficient conditions when the ratio is skew-Cauchy distributed.

In Chapter 3, some formulas for the central inverse moments of a quadratic form and of the ratio of two quadratic forms are established for multivariate skew normal random variables. They relate the quadratic forms which are determined by positive definite matrices to that defined by the inverse matrices.

Key words and phrases: Bivariate random variables, elliptical distribution, exchangeable, generalized skew-normal, inverse matrix, linear combination, moments, normal distribution, positive definite matrix, product, quadratic forms, ratio, skew, skew-Cauchy, skew-normal, skew-t, symmetric.

Chapter 1

Symmetric and Skew Distributions

1.1 Introduction

It is known that symmetric distributions are not suitable for modeling all types of data. Azzalini (1985, 1986) introduced the univariate skew-normal distribution having the probability density function (p.d.f.) of the form

$$2\phi(x)\Phi(\alpha x), \ x, \alpha \in \mathcal{R}, \tag{1}$$

where ϕ and Φ are the p.d.f. and cumulative distribution function (c.d.f.) of the standard normal distribution, respectively. For a random variable X, we write $X \sim \mathcal{SN}(\alpha)$, if X has the p.d.f. as given in (1). This class of distributions includes the $\mathcal{N}(0,1)$ distribution and has some properties like the normal and yet is skew. Since then there are many investigating about skew distributions, also more general definitions of skew distributions are given. Azzalini and Dalla Valle (1996) extended the results to the multivariate setting with the p.d.f. of the form

$$2\phi_p(\mathbf{x}, \Omega)\Phi(\alpha'\mathbf{x}), \quad \mathbf{x}, \alpha \in \mathcal{R}^p, \ \Omega > 0,$$

where $\phi_p(\mathbf{x}, \Omega)$ is the p-dimensional normal p.d.f. with zero mean vector and correlation matrix Ω.

In some sense, skew distributions and symmetric distributions are closely related. For example, it is an interesting problem to determine the distribution of V, by giving the distribution of Z and U, where $Z = UV$, U and V are assumed to be independent. It turns out (see Hunag and Su (2008)) that when U is symmetrically distributed, the distribution of V can be determined. Furthermore, unless there is no solution, otherwise all distributions of V which satisfy $Z = UV$, form a so-called skew class (see the definition in Section 3).

On the other hand, as mentioned by Viana and Olkin (2000), "Observations between related measurements, such as with eyes, ears, siblings, etc., possess intrinsic symmetries that may be relevant for assessing an underlying physiological process," bivariate exchangeable random variables play an important role in modeling observation taken from both sides of the same individual. Nagarajah (1982) obtained the distribution of a linear combination of order statistics from a bivariate, exchangeable and normal random variables. Huang and Chen (2007) pointed out that both the maximum and the minimum of two independent and identically distributed (i.i.d.) symmetric distributed random variables X and Y, belong to the skew class of X. The connections between order statistics and skew distributions of bivariate random variables, were also investigated by authors such as Loperfido (2002), Azzalini and Captanio (2003), and Loperfido (2008), etc.

In Section 2, let $Z = UV$, where U and V are assumed to be independent, under some condition, we will show that if Z is symmetric, then at least one of U and V must be symmetric. In Section 3, for certain bivariate exchangeable random variables X and Y, we find the distributions of $M = aU + bV$, linear combinations of order statistics $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$. It is interesting to know whether the sum of two random variables, which belong to the same skew class, will also belong to the same skew class. Let T and S be independent, such that $T \sim \mathcal{N}(0, 1)$ and $S \sim \mathcal{SN}(\alpha)$, then

$$Z = \frac{aT + bS}{\sqrt{a^2 + b^2}} \sim \mathcal{SN}\left(\frac{b\alpha}{\sqrt{a^2(1 + \alpha^2) + b^2}}\right),$$

where $a^2 + b^2 \neq 0$. Also if T and S are i.i.d. $\mathcal{N}(0, 1)$ distributed, then for $|\delta| < 1$,

$$Z = \delta|T| + \sqrt{1 - \delta^2}S \sim \mathcal{SN}\left(\frac{\delta}{\sqrt{1 - \delta^2}}\right).$$

See, e.g., Azzalini (2005). Linear combinations of two non-independent skew normal distributed random variables can also be skew normal distributed. An example can be found in Gupta and Brown (2001). But usually for two random variables belonging to the same skew class, their sum may not belong to the same skew class. Also it will be shown when (X, Y) are exchangeable elliptical and symmetric random variables, M has a skew distribution of ξX, for some suitable constant ξ. For certain bivariate exchangeable distributions of (X, Y), necessary and sufficient conditions that M has skew distribution of X, will be given for each case. Finally, in Section 4, we investigate the situation that X and Y are not identically distributed. It turns out that when the p.d.f. of (X, Y) has an elliptical form, the distribution of M is the mixture of two skew distributions of αX and βY for suitable constants α and β. Note that the p.d.f. of linear combinations of order statistics for continuous random variable (X, Y) with a general p.d.f. was given by Gupta and Gupta (2009).

1.2 Symmetric property for product of independent random variables

Let $Z = UV$, where U and V are assumed to be independent random variables. As $-(UV) = (-U)V = U(-V)$, Z is symmetric if one of U and V is symmetric. Conversely, it is interesting to know if Z is symmetric, whether at least one of U and V is symmetric? The following theorem provides some partial answer.

Theorem 1.1. Let $Z = UV$, where U and V are independent random variables, with continuous p.d.f.'s f_U and f_V, respectively. Let $a = \inf\{u | f_U(u) \neq f_U(-u), u > 0\}$, and $b = \inf\{v | f_V(v) \neq f_V(-v), v > 0\}$. Assume $a, b > 0$. If Z is symmetric, then at least one of U and V is symmetrically distributed.

Proof. Suppose the contrary that neither f_U nor f_V is symmetric. Then $0 < a, b < \infty$. Without loss of generality, assume $a \geq b$.

The continuous assumption of f_U and f_V yield

$$
\begin{aligned}
f_U(u) &= f_U(-u), \quad \forall u \in [0, a], & (2)\\
f_V(v) &= f_V(-v), \quad \forall v \in [0, b], & (3)
\end{aligned}
$$

also there exists an $\epsilon > 0$, such that

$$f_U(u) \neq f_U(-u), \quad \forall u \in (a, a+\epsilon], \tag{4}$$

and

$$f_V(v) \neq f_V(-v), \quad \forall v \in (b, b+\epsilon]. \tag{5}$$

By changing of variables, it yields

$$
\begin{aligned}
f_Z((a+\epsilon)b) &= \int_{-\infty}^{\infty} \left|\frac{1}{u}\right| f_U(u) f_V\left(\frac{(a+\epsilon)b}{u}\right) du \\
&= \int_0^{\infty} \frac{1}{u}\left(f_U(u) f_V\left(\frac{(a+\epsilon)b}{u}\right) + f_U(-u) f_V\left(-\frac{(a+\epsilon)b}{u}\right)\right) du \\
&= \int_0^{\infty} (A(u) + B(u)) du. & (6)
\end{aligned}
$$

Similarly

$$f_Z(-(a+\epsilon)b) = \int_0^{\infty} (C(u) + D(u)) du, \tag{7}$$

where for $u > 0$,

$$
\begin{aligned}
A(u) &= \frac{1}{u} f_U(u) f_V\left(\frac{(a+\epsilon)b}{u}\right), \\
B(u) &= \frac{1}{u} f_U(-u) f_V\left(-\frac{(a+\epsilon)b}{u}\right), \\
C(u) &= \frac{1}{u} f_U(u) f_V\left(-\frac{(a+\epsilon)b}{u}\right),
\end{aligned}
$$

and

$$D(u) = \frac{1}{u} f_U(-u) f_V\left(\frac{(a+\epsilon)b}{u}\right).$$

(6) and (7) can be rewritten as

$$f_Z((a+\epsilon)b) = \int_0^a (A(u)+B(u))du + \int_a^{a+\epsilon} (A(u)+B(u))du + \int_{a+\epsilon}^{\infty} (A(u)+B(u))du, \tag{8}$$

and

$$f_Z(-(a+\epsilon)b) = \int_0^a (C(u)+D(u))du + \int_a^{a+\epsilon} (C(u)+D(u))du + \int_{a+\epsilon}^{\infty} (C(u)+D(u))du. \tag{9}$$

From (2), we have

$$\int_0^a A(u)du = \int_0^a D(u)du, \ and \ \int_0^a B(u)du = \int_0^a C(u)du.$$

Hence

$$\int_0^a (A(u)+B(u))du = \int_0^a (C(u)+D(u))du.$$

As $0 < (a+\epsilon)b/u < b$, $\forall\ u \in [a+\epsilon, \infty)$, (3) implies

$$\int_{a+\epsilon}^{\infty} A(u)du = \int_{a+\epsilon}^{\infty} C(u)du, \ and \ \int_{a+\epsilon}^{\infty} B(u)du = \int_{a+\epsilon}^{\infty} D(u)du.$$

Hence

$$\int_{a+\epsilon}^{\infty} (A(u)+B(u))du = \int_{a+\epsilon}^{\infty} (C(u)+D(u))du.$$

As Z is assumed to be symmetric, $f_Z((a+\epsilon)b) = f_Z(-(a+\epsilon)b)$, this in turn implies

$$\int_a^{a+\epsilon} (A(u)+B(u))du = \int_a^{a+\epsilon} (C(u)+D(u))du,$$

which can be rewritten as

$$\int_a^{a+\epsilon} \frac{1}{u}(f_U(u)-f_U(-u))\left(f_V\left(\frac{(a+\epsilon)b}{u}\right) - f_V\left(-\frac{(a+\epsilon)b}{u}\right)\right) du = 0. \tag{10}$$

In view of the mean-value theorem for integrals, (10) yields

$$\frac{\epsilon}{c}(f_U(c)-f_U(-c))\left(f_V\left(\frac{(a+\epsilon)b}{c}\right) - f_V\left(-\frac{(a+\epsilon)b}{c}\right)\right) = 0, \tag{11}$$

for some $c \in (a, a+\epsilon)$. Furthermore, the assumption $a \geq b$ implies $(a+\epsilon)b/c \in (b, b+\epsilon]$, $\forall c \in (a, a+\epsilon)$. Hence

$$f_U(c) \neq f_U(-c) \text{ and } f_V\left(\frac{(a+\epsilon)b}{c}\right) \neq f_V\left(-\frac{(a+\epsilon)b}{c}\right), \quad \forall c \in (a, a+\epsilon).$$

Therefore (11) cannot hold for any $c \in (a, a+\epsilon)$, which contradicts to the assumption that Z is symmetric. Hence at least one of a and b is infinity, or equivalently to say that at least one of f_U and f_V is symmetric. This completes the proof.

1.3 Linear combinations of order statistics for bivariate exchangeable and symmetric random variables

Throughout this section, for bivariate random variables(X, Y), let U =max$\{X, Y\}$, V =min $\{X, Y\}$ and $M = aU + bV$, where $a, b \in \mathcal{R}$, such that $a^2 + b^2 \neq 0$. First we give a definition, which can be found in Huang and Chen (2007).

Definition 1.1. Let f be a symmetric p.d.f. A random variable X, with p.d.f. f_X, is said to have a skew distribution of f, denote this by $X \sim S(f)$, and $f_X \in S(f)$, if

$$f_X(x) = 2f(x)G(x), \quad x \in \mathcal{R}, \tag{12}$$

where G is a skew function, that is

$$0 \leq G(x) \leq 1 \;\; and \;\; G(x) + G(-x) = 1, x \in \mathcal{R}, \tag{13}$$

and

$$S(f) = \{h | h(x) = 2f(x)G(x), \; x \in \mathcal{R}, \; for\ some\ skew\ function\ G\}.$$

$S(f)$ is said to be the skew class generated by f. When a p.d.f. $g \in S(f)$, g is said to have a skew p.d.f. of f.

Similarly we can define $X \sim S(Y)$, and $X \sim S(F)$, where Y and F are symmetric random variable and symmetric distribution, respectively. Also if F is a common distribution, such as $\mathcal{C}(0,1)$ distribution, then X is said to have a skew-$\mathcal{C}(0,1)$ distribution, denote this by $X \sim$ skew-$\mathcal{C}(0,1)$. Note that for every $\alpha \in \mathcal{R}$, $S\mathcal{N}(\alpha)$ is also a skew-$\mathcal{N}(0,1)$ distribution with $G(x) = \Phi(\alpha x)$. Also the random variable $|T|$ in the Introduction is skew-$\mathcal{N}(0,1)$ distributed with $G(x) = 1, \; x \geq 0, G(x) = 0, \; x < 0$.

Theorem 1.2. Let the continuous function $f(x, y)$ be the joint p.d.f. of (X, Y). Also assume

$$f(x, y) = f(y, x), \; (x, y) \in \mathcal{R}^2, \tag{14}$$

and

$$f(x, y) = f(-x, -y), \;\; (x, y) \in \mathcal{R}^2. \tag{15}$$

Then we have
(i) X and Y are identically distributed ;

(ii) X and Y are symmetrically distributed ;

(iii) both U and $V \sim S(f_X)$.

Proof. (i)The assertion follows from

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} f(y,x)dy = f_Y(x), \quad x \in \mathcal{R}.$$

(ii)By changing of variable and using (15), we have

$$f_X(-x) = \int_{-\infty}^{\infty} f(-x,y)dy = \int_{-\infty}^{\infty} f(-x,-w)dw = \int_{-\infty}^{\infty} f(x,w)dw = f_X(x), \quad x \in \mathcal{R},$$

and the assertion follows.
(iii)First (14) yields $f_{U,V}(u,v) = 2f(u,v),\ v < u$. Hence $f_U(u) = \int_{-\infty}^{u} 2f(u,v)dv = 2f_X(u)G(u)$, $u \in \mathcal{R}$, where

$$G(u) = \begin{cases} \frac{\int_{-\infty}^{u} f(u,v)dv}{f_X(u)}, & f_X(u) \neq 0, \\ \frac{1}{2}, & f_X(u) = 0. \end{cases}$$

Obviously, $0 \leq G(u) \leq 1,\ u \in \mathcal{R}$, and if $f_X(u) \neq 0$, then

$$G(-u) \quad = \quad \frac{1}{f_X(-u)} \int_{-\infty}^{-u} f(-u,v)dv = \frac{1}{f_X(u)} \int_{u}^{\infty} f(-u,-v)dv = \frac{1}{f_X(u)} \int_{u}^{\infty} f(u,v)dv.$$

Hence $G(u) + G(-u) = 1,\ u \in \mathcal{R}$, and G is a skew function follows. Consequently, $U \sim S(f_X)$. Similarly, we have $f_V(v) = 2f_X(v)(1 - G(v)),\ v \in \mathcal{R}$. This completes the proof.

The next theorem gives the distribution of the linear combinations of the order statistics of bivariate random variables which satisfies the conditions in Theorem 1.2. The proof is easy hence is omitted.

Theorem 1.3. Let (X,Y) be defined as in Theorem 2. Then the p.d.f of M is

$$f_M(m) = \begin{cases} \frac{2}{|a|} \int_{-\infty}^{m/a} f(m/a,n)dn, & b = 0, m \in \mathcal{R}, \\ \frac{2}{|b|} \int_{m/b}^{\infty} f(n,m/b)dn, & a = 0, m \in \mathcal{R}, \\ \frac{2}{|ab|} \int_{-\infty}^{bm/(a+b)} f((m-n)/a,n/b)dn, & a^{-1} + b^{-1} > 0, m \in \mathcal{R}, \\ \frac{2}{|ab|} \int_{bm/(a+b)}^{\infty} f((m-n)/a,n/b)dn, & a^{-1} + b^{-1} < 0, m \in \mathcal{R}, \\ \frac{2}{a^2} \int_{-\infty}^{\infty} f((m+n)/a,n/a)dn, & a^{-1} + b^{-1} = 0, m/a > 0. \end{cases}$$

Obviously, if X and Y are i.i.d. with the common marginal distribution being symmetric about zero, then the joint p.d.f. f of (X,Y) satisfies the conditions in Theorem 1.2. The following corollary indicates that when the joint p.d.f. of (X,Y) has an exchangeable elliptical form, then the conditions in Theorem 1.2 are also satisfied. Throughout this work, (X, Y) is said to be elliptical distributed, if the p.d.f. of (X, Y) has the following form:

$$f(x, y) = |A|^{-1/2} g(z' A^{-1} z), \tag{16}$$

where $z = (x, y)'$, A is a positive definite 2×2 matrix, and g is a function from $\mathcal{R}^+$ to $\mathcal{R}^+$ satisfying $\int_0^\infty g(y)dy = 1/\pi$.

Corollary 1.1. Let (X, Y) be exchangeable elliptical random variables with

$$\mathbf{A} = c^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \tag{17}$$

where $c > 0$, and $|\rho| < 1$, that is the joint p.d.f. of (X, Y) has the form

$$f(x, y) = \frac{1}{c^2\sqrt{1-\rho^2}} g\left(\frac{x^2 + y^2 - 2\rho xy}{c^2(1-\rho^2)}\right), \quad x, y \in \mathcal{R}. \tag{18}$$

We have

(i) $f(x, y)$ satisfies (14) and (15) ;

(ii) if $a + b \neq 0$, then the p.d.f. of M is

$$f_M(m) = \frac{2}{c\xi} \int_{-\infty}^{\alpha m} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr, \; m \in \mathcal{R}, \tag{19}$$

where

$$\xi = \sqrt{a^2 + b^2 + 2ab\rho}, \; \alpha = \frac{1}{c\xi} \frac{a - b}{|a + b|} \sqrt{\frac{1-\rho}{1+\rho}} \; ; \tag{20}$$

(iii) if $a + b = 0$, then the p.d.f. of M is

$$f_M(m) = \frac{2}{c\xi} \int_{-\infty}^{\infty} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr, \; m/a > 0 \; ;$$

(iv) $M \sim S(\xi X)$.

Proof. That (i) holds is obvious. We only prove (ii) and (iv), as (iii) can be obtained similarly as (ii). First for the case $b = 0$, by Theorem 1.3, we have

$$\begin{aligned} f_M(m) &= \frac{2}{c^2|a|\sqrt{1-\rho^2}} \int_{-\infty}^{m/a} g\left(\frac{m^2 + a^2n^2 - 2a\rho mn}{c^2a^2(1-\rho^2)}\right) dn \\ &= \frac{2}{c^2|a|\sqrt{1-\rho^2}} \int_{-\infty}^{m/a} g\left(\left(\frac{an - \rho m}{ca\sqrt{1-\rho^2}}\right)^2 + \frac{m^2}{c^2a^2}\right) dn \\ &= \frac{2}{c|a|} \int_{-\infty}^{\frac{\sqrt{1-\rho}m}{ca\sqrt{1+\rho}}} g\left(r^2 + \frac{m^2}{c^2a^2}\right) dr, \quad m \in \mathcal{R}, \end{aligned} \tag{21}$$

where we have used the change of variable $r = (an - \rho m)(ca\sqrt{1-\rho^2})^{-1}$. Obviously, (21) coincides with $b = 0$ in (19). The proof of the case $a = 0$ is similar.

Next assume $a \neq 0$, $b \neq 0$. First consider $a^{-1} + b^{-1} > 0$. Again by Theorem 1.3, the result follows by noting

$$\begin{aligned}
f_M(m) &= \frac{2}{c^2|ab|\sqrt{1-\rho^2}} \int_{-\infty}^{\frac{bm}{a+b}} g\left(\frac{(m-n)^2/a^2 + n^2/b^2 - 2\rho(m-n)n/(ab)}{c^2(1-\rho^2)}\right) dn \\
&= \frac{2}{c^2|ab|\sqrt{1-\rho^2}} \int_{-\infty}^{\frac{bm}{a+b}} g\left(\left(\frac{\xi n - (b^2+ab)\xi m}{cab\sqrt{1-\rho^2}}\right)^2 + \frac{m^2}{c^2\xi^2}\right) dn \\
&= \frac{2}{c\xi}\int_{-\infty}^{\alpha m} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr, \quad m \in \mathcal{R}, \qquad (22)
\end{aligned}$$

where a change of variable is used again in the last step. Finally along the lines of the above proof, the case $a^{-1} + b^{-1} < 0$ can be obtained easily. This completes the proof of (ii).

We now prove (iv). From (18), we obtain the following marginal p.d.f. of X

$$f_X(m) = \frac{1}{c}\int_{-\infty}^{\infty} g\left(r^2 + \frac{m^2}{c^2}\right) dr, \; m \in \mathcal{R}.$$

Consequently, the p.d.f. of $X_1 = \xi X$ is

$$f_{X_1}(m) = \frac{1}{c\xi}\int_{-\infty}^{\infty} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr, \; m \in \mathcal{R}.$$

Now for the case $a + b \neq 0$, $f_M(m)$ can be rewritten as

$$\begin{aligned}
f_M(m) &= 2\frac{1}{c\xi}\int_{-\infty}^{\infty} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr \frac{\frac{1}{c\xi}\int_{-\infty}^{\alpha m} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr}{\frac{1}{c\xi}\int_{-\infty}^{\infty} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr} \\
&= 2 f_{X_1}(m) G_1(m),
\end{aligned}$$

where

$$G_1(m) = \frac{\frac{1}{c\xi}\int_{-\infty}^{\alpha m} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr}{\frac{1}{c\xi}\int_{-\infty}^{\infty} g\left(r^2 + \frac{m^2}{c^2\xi^2}\right) dr}, m \in \mathcal{R},$$

is a skew function; and for the case $a + b = 0$, we get $f_M(m) = 2 f_{X_1}(m) I_{\{m/a>0\}}$, where the indicator function $I_{\{m/a>0\}}$ is a skew function. This completes the proof of (iv).

Result (ii) of Corollary 1.1 can also be found in Loperfido (2008) with a different proof. Following the notation of Loperfido (2008), here M has a $SE_1[0, \mathbf{A}, \alpha, g]$ distribution, where $\mathbf{A}$ and α are given in (17) and (20), respectively. Yet the α given in (7) of Loperfido (2008) is in error. Note that in general when X and Y are i.i.d. symmetrically distributed, $f_M(m)$ may not have a simple form and M may not be $S(\xi X)$ distributed, as the case when X and Y are exchangeable elliptical distributed.

We give some corollaries in the following. Part (i) of the first corollary is due to Nagaraja (1982), and part (ii) is obvious.

Corollary 1.2. Let (X, Y) be bivariate normally distributed with the joint p.d.f.

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}, \quad (x,y) \in \mathcal{R}^2, |\rho| < 1. \tag{23}$$

(i) The p.d.f. of M is given by

$$f_M(m) = \begin{cases} \frac{2}{|a|}\phi(m/a)\Phi(\sqrt{\frac{1-\rho}{1+\rho}}m/a), & b = 0, m \in \mathcal{R}, \\ \frac{2}{|b|}\phi(m/b)\Phi(-\sqrt{\frac{1-\rho}{1+\rho}}m/b), & a = 0, m \in \mathcal{R}, \\ \frac{2}{\xi}\phi\left(\frac{m}{\xi}\right)\Phi(-\eta m), & a^{-1}+b^{-1} > 0, m \in \mathcal{R}, \\ \frac{2}{\xi}\phi\left(\frac{m}{\xi}\right)\Phi(\eta m), & a^{-1}+b^{-1} < 0, m \in \mathcal{R}, \\ \frac{2}{|a|\sqrt{2(1-\rho)}}\phi\left(\frac{m}{|a|\sqrt{2(1-\rho)}}\right), & a^{-1}+b^{-1} = 0, m/a > 0, \end{cases} \tag{24}$$

where ξ is defined in (20) and

$$\eta = \sqrt{\frac{1-\rho}{1+\rho}}\frac{b-a}{\xi(b+a)} \ ;$$

(ii) $M \sim$ skew-$\mathcal{N}(0,1)$ if and only if $\xi = 1$. In particular, if $\rho = 0$, that is X and Y are i.i.d. $\mathcal{N}(0,1)$ distributed, then $M \sim$ *skew*-$\mathcal{N}(0,1)$ if and only if $a^2 + b^2 = 1$.

The above corollary indicates that when (X, Y) has a bivariate normal distribution with p.d.f. given in (23), and $\mathcal{N}(0,1)$ being the common marginal distribution of X and Y, then $M \sim$skew-$\mathcal{N}(0,\xi^2)$ in each case. The next corollary consider bivariate Cauchy distributed random variables where the common marginal distribution of X and Y is $\mathcal{C}(0,1)$.

Corollary 1.3. Let (X, Y) be bivariate Cauchy distributed with the joint p.d.f.

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}\left(1+\frac{x^2+y^2-2\rho xy}{1-\rho^2}\right)^{3/2}}, \quad (x,y) \in \mathcal{R}^2, |\rho| < 1.$$

Then

(i) the p.d.f. of M is

$$f_M(m) = \begin{cases} \frac{|a|}{\pi(m^2+a^2)}\left(1+\frac{\sqrt{1-\rho}m}{\sqrt{2m^2+a^2(1+\rho)}}\right), & b = 0, m \in \mathcal{R}, \\ \frac{|b|}{\pi(m^2+b^2)}\left(1-\frac{\sqrt{1-\rho}m}{\sqrt{2m^2+b^2(1+\rho)}}\right), & a = 0, m \in \mathcal{R}, \\ \frac{\xi}{\pi(m^2+\xi^2)}(1+A), & a^{-1}+b^{-1} > 0, m \in \mathcal{R}, \\ \frac{\xi}{\pi(m^2+\xi^2)}(1-A), & a^{-1}+b^{-1} < 0, m \in \mathcal{R}, \\ \frac{2|a|\sqrt{2(1-\rho)}}{\pi(m^2+2a^2(1-\rho))}, & a^{-1}+b^{-1} = 0, m/a > 0, \end{cases} \tag{25}$$

where

$$A = \frac{(b-a)\sqrt{1-\rho}m}{\xi\sqrt{2m^2+(a+b)^2(1+\rho)}}, \tag{26}$$

and ξ is defined in (20) ;

(ii) $M \sim$ skew-$C(0,1)$ if and only if $\xi = 1$.

Proof. We only prove (i). Let $b = 0$, then $M = aU$. Following Theorem 1.3, we have

$$f_M(m) \quad = \quad \frac{2}{|a|}\int_{-\infty}^{m/a} \frac{1}{2\pi\sqrt{1-\rho^2}\left(1+\frac{m^2/a^2+n^2-2\rho mn/a}{1-\rho^2}\right)^{3/2}}dn, \quad m \in \mathcal{R},$$

and the result follows immediately. The case $a = 0$ can be obtained similarly.

Next consider the case $a, b \neq 0$. First assume $a^{-1} + b^{-1} > 0$. Then Theorem 1.3 implies

$$\begin{aligned} f_M(m) \quad &= \quad \frac{2}{|ab|}\int_{-\infty}^{bm/(a+b)} \frac{1}{2\pi\sqrt{1-\rho^2}\left(1+\frac{(m-n)^2/a^2+n^2/b^2-2\rho(m-n)n/ab}{1-\rho^2}\right)^{3/2}}dn \\ &= \quad \frac{\xi^2 n - b^2 m - ab\rho m}{\pi(m^2+\xi^2)\sqrt{a^2b^2(1-\rho^2)+b^2(m-n)^2+a^2n^2-2ab\rho(m-n)n}}\Bigg|_{-\infty}^{bm/(a+b)} \\ &= \quad \frac{\xi}{\pi(m^2+\xi^2)}(1+A), \quad m \in \mathcal{R}, \end{aligned}$$

where A is defined in (26), and the result follows. The case $a^{-1}+b^{-1} < 0$ can be proved similarly.

For the last case $a^{-1} + b^{-1} = 0$, again Theorem 1.3 yields

$$f_M(m) \quad = \quad \frac{2}{a^2}\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}\left(1+\frac{(m+n)^2+n^2-2\rho(m+n)n}{a^2(1-\rho^2)}\right)^{3/2}}dn, \quad m/a > 0,$$

and the conclusion follows by finishing the integration.

According to Kotz and Nadarajah (2004), the general form of the joint p.d.f. of bivariate t distribution has the following elliptical form

$$f(x,y) = \frac{|\mathbf{A}|^{-1/2}}{2\pi\left(1+\boldsymbol{z}'\mathbf{A}^{-1}\boldsymbol{z}/n\right)^{(n+2)/2}}, \ \boldsymbol{z}' = (x,y) \in \mathcal{R}^2, \ n \in \mathcal{N},$$

where $\mathbf{A}$ is given in (17) with $c^2 = 1$. If $n = 1$, then this becomes bivariate Cauchy distribution and has been discussed in Corollary 1.3. The next corollary considers the case $n = 2$, where both X and Y have $\mathcal{T}_2$ as the marginal distribution. That is the p.d.f. of X is

$$f_X(x) = \frac{1}{(x^2+2)^{3/2}} \quad , x \in \mathcal{R}.$$

Again the proof is omitted.

Corollary 1.4. Let (X, Y) be bivariate $\mathcal{T}_2$ distributed with the joint p.d.f.

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}\left(1+\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right)^2}, \quad (x,y) \in \mathcal{R}^2, |\rho| < 1.$$

(i) Then the p.d.f. of M is

$$f_M(m) = \begin{cases} \frac{a^2}{(m^2+2a^2)^{3/2}}(1+A), & b=0, m \in \mathcal{R}, \\ \frac{b^2}{(m^2+2b^2)^{3/2}}(1-B), & a=0, m \in \mathcal{R}, \\ \frac{\xi^2}{(m^2+2\xi^2)^{3/2}}(1+C), & a^{-1}+b^{-1} > 0, m \in \mathcal{R}, \\ \frac{\xi^2}{(m^2+2\xi^2)^{3/2}}(1-C), & a^{-1}+b^{-1} < 0, m \in \mathcal{R}, \\ \frac{4a^2(1-\rho)}{(m^2+4a^2(1-\rho))^{3/2}}, & a^{-1}+b^{-1} = 0, m/a > 0, \end{cases} \tag{27}$$

where

$$\begin{aligned} A &= \frac{m(m^2+2a^2)^{1/2}\sqrt{1-\rho^2}}{\pi(m^2+(1+\rho)a^2)} + \frac{2\arctan\left(\frac{m}{\sqrt{(m^2+2a^2)(1+\rho)}}\right)}{\pi}, \\ B &= \frac{m(m^2+2b^2)^{1/2}\sqrt{1-\rho^2}}{\pi(m^2+(1+\rho)b^2)} + \frac{2\arctan\left(\frac{m}{\sqrt{(m^2+2b^2)(1+\rho)}}\right)}{\pi}, \\ C &= \frac{m(b^2-a^2)(m+2\xi^2)^{1/2}\sqrt{1-\rho^2}}{\pi\xi^2(m^2+(a+b)^2(1+\rho))} + \frac{2\arctan\left(\frac{(a-b)(1-\rho)m}{(a+b)\sqrt{(m^2+2\xi^2)(1-\rho^2)}}\right)}{\pi}, \end{aligned}$$

and ξ is defined in (20) ;

(ii) $M \sim$ skew-$\mathcal{T}_2$ if and only if $\xi = 1$.

The following corollary shows when X and Y are i.i.d. symmetrically distributed, M may not be $S(\xi X)$ distributed. Note that when X and Y are i.i.d. $\mathcal{N}(0,1)$ distributed, they are still exchangeable elliptical distributed.

Corollary 1.5. Let X, Y be i.i.d. $\mathcal{C}(0,1)$ distributed. Then

(i) the p.d.f. of M is

$$f_M(m) = \begin{cases} \frac{|a|}{\pi(m^2+a^2)}\left(1+\frac{2\arctan(m/a)}{\pi}\right), & b=0, m \in \mathcal{R}, \\ \frac{|b|}{\pi(m^2+b^2)}\left(1-\frac{2\arctan(m/b)}{\pi}\right), & a=0, m \in \mathcal{R}, \\ \epsilon(m), & a^{-1}+b^{-1} > 0, m \in \mathcal{R}, \\ \delta(m), & a^{-1}+b^{-1} < 0, m \in \mathcal{R}, \\ \frac{4|a|}{\pi(m^2+4a^2)}, & a^{-1}+b^{-1} = 0, m/a > 0, \end{cases} \tag{28}$$

where

$$\delta(m) = \frac{\pi ab(m^2+a^2-b^2)\mathrm{sgn}(b)+\pi b(m^2+b^2-a^2)\mathrm{sgn}(a)+2a^3bm(\log(a^2/b^2))}{\pi^2(m^4+2(a^2+b^2)m^2+(a^2-b^2)^2)} - S,$$

$$\epsilon(m) = \frac{\pi ab(m^2+a^2-b^2)\mathrm{sgn}(b)+\pi b(m^2+b^2-a^2)\mathrm{sgn}(a)-2a^3bm(\log(a^2/b^2))}{\pi^2(m^4+2(a^2+b^2)m^2+(a^2-b^2)^2)} + S,$$

$$S = \frac{2((ab-b)m^2+a^2b-ab^3-b^3+a^3b)\arctan(m/(a+b))}{\pi^2(m^4+2(a^2+b^2)m^2+(a^2-b^2)^2)},$$

and for $x \neq 0$, $\mathrm{sgn}(x) = 1,\ x>0,\ \ =-1,\ x<0$;

(ii) $M \sim$ skew-$\mathcal{C}(0,1)$ if and only if $a=0$ and $|b|=1$, $b=0$ and $|a|=1$, or $|a|=1/2$ and $b=-a$.

For X and Y being i.i.d. $\mathcal{U}(-1,1)$, or i.i.d. Pearson Type II distributed, that is X has the p.d.f. $f_X(x) = (3/4)(1-x^2),\ \ x\in(-1,1)$, we have also obtained the p.d.f.'s of M in both cases. As they are rather cumbersome, hence are omitted. Still $M \sim$ skew-$\mathcal{U}(-1,1)$ and skew-Pearson Type II, respectively, if and only if $a=0$ and $|b|=1$, or $|a|=1$ and $b=0$ for both of the two distributions.

Finally, we give an corollary where although (14) and (15) for joint p.d.f. of (X,Y) are satisfied, yet not as in the above corollaries, neither (X,Y) is exchangeable elliptical distributed nor X and Y are i.i.d. Again the p.d.f. of M is very cumbersome hence is omitted.

Corollary 1.6. Let (X,Y) have the following joint p.d.f.

$$f(x,y) = \frac{21}{56\rho^2+24}(x^6+y^6+2\rho x^3y+2\rho xy^3+\rho^2x^2+\rho^2y^2), \quad -1\leq x,y\leq 1, |\rho|<1.$$

Then the marginal p.d.f. of X is given by

$$f_X(x) = \frac{21}{28\rho^2+12}(x^6+\rho^2x^2+\frac{\rho^2}{3}+\frac{1}{7}), -1\leq x\leq 1.$$

Again it can be shown $M \sim S(f_X)$ if and only if $a=0$ and $|b|=1$, or $|a|=1$ and $b=0$.

Remark 1.1. As an illustration, in the above corollary, we give the p.d.f. of M for the case $a^{-1}+b^{-1}=0$ in the following.
(1)If $0<m<2|a|$, then

$$f_M(m) = \frac{21(m(|a|-m)^6+2(|a|-m)^7/7+m(|a|-m)^4(5m^2+2a^2\rho))}{a^8(28\rho^2+12)}$$

$$
\begin{aligned}
&+ \quad \frac{21((|a|-m)^5(15m^2+4a^2\rho)/5+m^2(|a|-m)(m^4+a^4\rho)-a^6m+2|a|^7/7)}{a^8(28\rho^2+12)} \\
&+ \quad \frac{21(m(|a|-m)^2(3m^4+a^2m^2\rho+a^4\rho^2)+(|a|-m)^3(15m^4+6a^2m^2\rho+2a^4\rho^2)/3)}{a^8(28\rho^2+12)} \\
&- \quad \frac{21(a^4m(5m^2+2a^2\rho)-|a|^5(15m^2+4a^2\rho)/5-|a|m^2(m^4+a^4\rho))}{a^8(28\rho^2+12)} \\
&- \quad \frac{21(a^2m(3m^4+a^2m^2\rho+a^4\rho^2)-|a|^3(15m^4+6a^2m^2\rho+2a^4\rho^2)/3)}{a^8(28\rho^2+12)}.
\end{aligned}
$$

(2)If $-2|a| < m < 0$, then

$$
\begin{aligned}
f_M(m) &= \frac{21(a^6m+2|a|^7/7+a^4m(5m^2+2a^2\rho)+|a|^5(15m^2+4a^2\rho)/5)}{a^8(28\rho^2+12)} \\
&+ \quad \frac{21(|a|m^2(m^4+a^4\rho)-m(|a|+m)^6+2(|a|+m)^7/7-m(|a|+m)^4(5m^2+2a^2\rho))}{a^8(28\rho^2+12)} \\
&+ \quad \frac{21((|a|+m)^5(15m^2+4a^2\rho)/5+(|a|+m)(m^4+a^4\rho)m^2)}{a^8(28\rho^2+12)} \\
&+ \quad \frac{21(a^2m(3m^4+a^2m^2\rho+a^4\rho^2)+|a|^3(15m^4+6a^2m^2\rho+2a^4\rho^2)/3)}{a^8(28\rho^2+12)} \\
&- \quad \frac{21(m(|a|+m)^2(3m^4+a^2m^2\rho+a^4\rho^2)-(|a|+m)^3(15m^4+6a^2m^2\rho+2a^4\rho^2)/3)}{a^8(28\rho^2+12)}.
\end{aligned}
$$

1.4 Non-exchangeable symmetric random variables

Let X and Y be two independent continuous random variables with symmetric p.d.f.'s f_X and f_Y, respectively, and distribution functions F_X and F_Y, respectively. Then $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$ have p.d.f.'s

$$f_U(u) = f_X(u)F_Y(u) + f_Y(u)F_X(u),\ u \in \mathcal{R},$$

and

$$f_V(v) = f_X(v)(1 - F_Y(v)) + f_Y(v)(1 - F_X(v)),\ v \in \mathcal{R},$$

respectively. In other words, both U and V are mixture with equal weights of the two distributions one in $S(f_X)$ and one in $S(f_Y)$. Inspired by this observation, in the section, we will investigate the distributions of M, linear combinations of U and V, when X and Y are not identically distributed. It turns out when the joint distribution of (X, Y) has an elliptical form, M is the mixture with equal weights of two skew distributions αX and βY, respectively, for suitable constants α and β. Note that for bivariate normally distributed random variables, that U and V have this property has been pointed out by Loperfido (2002).

First, we give a theorem for the non-identically distributed situation, the proof is similar to Corollary 1.1 hence is omitted.

Theorem 1.4. Let (X, Y) be elliptical random variables with

$$\mathbf{A} = \begin{pmatrix} r & k \\ k & s \end{pmatrix}, \tag{29}$$

where $r, s > 0$, $k \in \mathcal{R}$, and $rs > k^2$, that is the joint p.d.f. of (X, Y) has the form

$$f(x, y) = \frac{1}{\sqrt{rs - k^2}} g\left(\frac{sx^2 + ry^2 - 2kxy}{rs - k^2}\right), \quad x, y \in \mathcal{R},$$

and g is a function from $\mathcal{R}^+$ to $\mathcal{R}^+$ satisfying $\int_0^\infty g(y)dy = 1/\pi$. We have
(i) if $a + b \neq 0$, then the p.d.f. of M is

$$f_M(m) = \frac{1}{\xi_1} \int_{-\infty}^{\alpha_1 m} g\left(w^2 + \frac{m^2}{\xi_1^2}\right) dw + \frac{1}{\xi_2} \int_{-\infty}^{\alpha_2 m} g\left(w^2 + \frac{m^2}{\xi_2^2}\right) dw, \ m \in \mathcal{R},$$

where

$$\begin{aligned} \xi_1 &= \sqrt{a^2 r + b^2 s + 2abk}, \ \xi_2 = \sqrt{a^2 s + b^2 r + 2abk}, \\ \alpha_1 &= \frac{(ar + bk - bs - ak)m}{\xi_1 \sqrt{rs - k^2}|a + b|}, \ \alpha_2 = \frac{(as + bk - br - ak)m}{\xi_2 \sqrt{rs - k^2}|a + b|} \ ; \end{aligned}$$

(ii) if $a + b = 0$, then $\xi_1 = \xi_2 = \sqrt{a^2(r + s - 2k)}$, and the p.d.f. of M is

$$f_M(m) = \frac{2}{\xi_1} \int_{-\infty}^{\infty} g\left(w^2 + \frac{m^2}{\xi_1^2}\right) dw, \ m/a > 0 \ ;$$

(iii) M is the mixture with equal weights of skew distributions of $(\xi_1/\sqrt{r})X$ and $(\xi_2/\sqrt{s})Y$. In particular for the case $a + b = 0$, $M \sim S((\xi_1/\sqrt{r})X)$.

The joint p.d.f. f of the following Corollary 1.7 corresponds to

$$A = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2, \end{pmatrix}$$

in (16) and A is given in (29) for Corollaries 1.8 and 1.9.

Corollary 1.7. Let (X, Y) be bivariate normally distributed with the joint p.d.f.

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} e^{-q/2}, \quad (x, y) \in \mathcal{R}^2, |\rho| < 1, \sigma_1, \sigma_2 > 0,$$

where

$$q = \frac{1}{(1-\rho^2)} \left(\frac{x^2}{\sigma_1^2} - \frac{2\rho xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2} \right).$$

Then the p.d.f. of M is given by

$$f_M(m) = \begin{cases} \frac{1}{|a|\sigma_1}\phi\left(\frac{m}{a\sigma_1}\right)\Phi\left(A_1 m\right) + \frac{1}{|a|\sigma_2}\phi\left(\frac{m}{a\sigma_2}\right)\Phi\left(B_1 m\right), & b=0, m \in \mathcal{R}, \\ \frac{1}{|b|\sigma_1}\phi\left(\frac{m}{b\sigma_1}\right)\Phi\left(A_2 m\right) + \frac{1}{|b|\sigma_2}\phi\left(\frac{m}{b\sigma_2}\right)\Phi\left(B_2 m\right), & a=0, m \in \mathcal{R}, \\ \frac{1}{\xi_1}\phi\left(\frac{m}{\xi_1}\right)\Phi\left(A_3 m\right) + \frac{1}{\xi_2}\phi\left(\frac{m}{\xi_2}\right)\Phi\left(B_3 m\right), & a^{-1}+b^{-1} > 0, m \in \mathcal{R}, \\ \frac{1}{\xi_1}\phi\left(\frac{m}{\xi_1}\right)\Phi\left(-A_3 m\right) + \frac{1}{\xi_2}\phi\left(\frac{m}{\xi_2}\right)\Phi\left(-B_3 m\right), & a^{-1}+b^{-1} < 0, m \in \mathcal{R}, \\ \frac{2}{|a|\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}}\phi\left(\frac{m}{|a|\sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}}\right) & a^{-1}+b^{-1} = 0, m/a > 0, \end{cases}$$

where

$$\begin{aligned} A_1 &= \frac{(\sigma_1 - \rho\sigma_2)}{a\sqrt{(1-\rho^2)}\sigma_1\sigma_2},\ B_1 = \frac{(\sigma_2 - \rho\sigma_1)}{a\sqrt{(1-\rho^2)}\sigma_1\sigma_2}, \\ A_2 &= -\frac{(\sigma_1 - \rho\sigma_2)}{b\sqrt{(1-\rho^2)}\sigma_1\sigma_2},\ B_2 = -\frac{(\sigma_2 - \rho\sigma_1)}{b\sqrt{(1-\rho^2)}\sigma_1\sigma_2}, \\ A_3 &= \frac{a\sigma_1^2 - \rho\sigma_1\sigma_2(a-b) - b\sigma_2^2}{\xi_1(a+b)\sigma_1\sigma_2\sqrt{1-\rho^2}}, B_3 = \frac{a\sigma_2^2 - \rho\sigma_1\sigma_2(a-b) - b\sigma_1^2}{\xi_2(a+b)\sigma_1\sigma_2\sqrt{1-\rho^2}}, \\ \xi_1 &= \sqrt{a^2\sigma_1^2 + 2ab\rho\sigma_1\sigma_2 + b^2\sigma_2^2},\ \xi_2 = \sqrt{a^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2 + b^2\sigma_1^2}. \end{aligned}$$

In the above corollary, it can be seen, M is the mixture with equal weights of skew distributions of $(\xi_1/\sigma_1)X$ and $(\xi_2/\sigma_2)Y$, where $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$. In particular for the case $a^{-1} + b^{-1} = 0$, $M \sim$ skew-$\mathcal{N}(0, a^2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2))$.

Corollary 1.8. Let (X, Y) be bivariate Cauchy distributed with the joint p.d.f.

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{rs-k^2}\left(1 + \frac{sx^2+ry^2-2kxy}{rs-k^2}\right)^{3/2}}, \quad (x,y) \in \mathcal{R}^2.$$

Then $X \sim \mathcal{C}(0, \sqrt{r}), Y \sim \mathcal{C}(0, \sqrt{s})$, and the p.d.f. of M is

$$f_M(m) = \begin{cases} \frac{|a|\sqrt{r}}{2\pi(m^2+a^2 r)}(1+A_1) + \frac{|a|\sqrt{s}}{2\pi(m^2+a^2 s)}(1+B_1), & b=0, m \in \mathcal{R}, \\ \frac{|b|\sqrt{r}}{2\pi(m^2+b^2 r)}(1-A_2) + \frac{|b|\sqrt{s}}{2\pi(m^2+b^2 s)}(1-B_2), & a=0, m \in \mathcal{R}, \\ \frac{\xi_1}{2\pi(m^2+\xi_1^2)}\left(1+A_3\right) + \frac{\xi_2}{2\pi(m^2+\xi_2^2)}\left(1+B_3\right), & a^{-1}+b^{-1} > 0, m \in \mathcal{R}, \\ \frac{\xi_1}{2\pi(m^2+\xi_1^2)}\left(1-A_3\right) + \frac{\xi_2}{2\pi(m^2+\xi_2^2)}\left(1-B_3\right), & a^{-1}+b^{-1} < 0, m \in \mathcal{R}, \\ \frac{2|a|\sqrt{r+s-2k}}{\pi(m^2+a^2(r+s-2k))}, & a^{-1}+b^{-1} = 0, m/a > 0, \end{cases}$$

where

$$
\begin{aligned}
A_1 &= \frac{(r-k)m}{\sqrt{r((r+s-2k)m^2+a^2(rs-k^2))}},\\
B_1 &= \frac{(s-k)m}{\sqrt{s((r+s-2k)m^2+a^2(rs-k^2))}},\\
A_2 &= \frac{(r-k)m}{\sqrt{r((r+s-2k)m^2+b^2(rs-k^2))}},\\
B_2 &= \frac{(s-k)m}{\sqrt{s((r+s-2k)m^2+b^2(rs-k^2))}},\\
A_3 &= \frac{(bs-ar+(b-a)k)m}{\xi_1\sqrt{(r+s-2k)m^2+(a+b)^2(rs-k^2)}},\\
B_3 &= \frac{(br-as+(b-a)k)m}{\xi_2\sqrt{(r+s-2k)m^2+(a+b)^2(rs-k^2)}},\\
\xi_1 &= \sqrt{a^2r+b^2s+2abk},\ \xi_2 = \sqrt{a^2s+b^2r+2abk}.
\end{aligned}
$$

In the above corollary, M is the mixture with equal weights of skew distributions of $(\xi_1/\sqrt{r})X$ and $(\xi_2/\sqrt{s})Y$. In particular if $a^{-1}+b^{-1}=0$, then $M \sim$ skew-$\mathcal{C}(0,\xi_1)$.

According to Johnson and Kotz (1970), the p.d.f. of Pearson Type VII distribution can be expressed in the following form

$$
f_X(x) = \frac{\Gamma(m)}{\sqrt{\pi}\Gamma(m-1/2)}\frac{c^{2m-1}}{((x-\lambda)^2+c^2)^m},\ m>0, c>0, x\in\mathcal{R}. \tag{30}
$$

In the next corollary, both X and Y have Pearson Type VII as their distributions with $m=3/2$, $\lambda=0$, and $c=r$ and s, respectively. That is

$$
f_X(x) = \frac{r}{(x^2+2r)^{3/2}},\ f_Y(y) = \frac{s}{(y^2+2s)^{3/2}},\ r,s>0, x,y\in\mathcal{R}. \tag{31}
$$

Corollary 1.9. Let (X,Y) be bivariate distributed with the joint p.d.f.

$$
f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{rs-k^2}\left(1+\frac{sx^2+ry^2-2kxy}{2(rs-k^2)}\right)^2},\quad (x,y)\in\mathcal{R}^2,
$$

Then the p.d.f. of M is

$$
f_M(m) = \begin{cases}
\frac{a^2r}{2(m^2+2a^2r)^{3/2}}(1+A_1) + \frac{a^2s}{2(m^2+2a^2s)^{3/2}}(1+B_1), & b=0, m\in\mathcal{R},\\
\frac{b^2r}{2(m^2+2b^2r)^{3/2}}(1-A_2) + \frac{b^2s}{2(m^2+2b^2s)^{3/2}}(1-B_2), & a=0, m\in\mathcal{R},\\
\frac{\xi_1^2}{2(m^2+2\xi_1^2)^{3/2}}(1+A_3) + \frac{\xi_2^2}{2(m^2+2\xi_2^2)^{3/2}}(1+B_3), & a^{-1}+b^{-1}>0, m\in\mathcal{R},\\
\frac{\xi_1^2}{2(m^2+2\xi_1^2)^{3/2}}(1-A_3) + \frac{\xi_2^2}{2(m^2+2\xi_2^2)^{3/2}}(1-B_3), & a^{-1}+b^{-1}<0, m\in\mathcal{R},\\
\frac{2a^2(r+s-2k)}{(m^2+2a^2(r+s-2k))^{3/2}}, & a^{-1}+b^{-1}=0, m/a>0,
\end{cases}
$$

where

$$A_1 = \frac{2m(r-k)(m^2+2a^2r)^{1/2}\sqrt{rs-k^2}}{\pi r((r+s-2k)m^2+2(rs-k^2)a^2)} + \frac{2\arctan\left(\frac{(r-k)m}{\sqrt{(m^2+2a^2r)(rs-k^2)}}\right)}{\pi},$$

$$B_1 = \frac{2m(s-k)(m^2+2a^2s)^{1/2}\sqrt{rs-k^2}}{\pi s((r+s-2k)m^2+2(rs-k^2)a^2)} + \frac{2\arctan\left(\frac{(s-k)m}{\sqrt{(m^2+2a^2s)(rs-k^2)}}\right)}{\pi},$$

$$A_2 = \frac{2m(r-k)(m^2+2a^2r)^{1/2}\sqrt{rs-k^2}}{\pi r((r+s-2k)m^2+2(rs-k^2)a^2)} + \frac{2\arctan\left(\frac{(r-k)m}{\sqrt{(m^2+2b^2r)(rs-k^2)}}\right)}{\pi},$$

$$B_2 = \frac{2m(s-k)(m^2+2a^2s)^{1/2}\sqrt{rs-k^2}}{\pi s((r+s-2k)m^2+2(rs-k^2)a^2)} + \frac{2\arctan\left(\frac{(s-k)m}{\sqrt{(m^2+2b^2s)(rs-k^2)}}\right)}{\pi},$$

$$A_3 = \frac{2m(bs(a+b)-ar(a+b)-(b^2-a^2)k)(m^2+2\xi_1^2)^{1/2}\sqrt{rs-k^2}}{\pi\xi_1^2((r+s-2k)m^2+2(a+b)^2(rs-k^2))} + \frac{2\arctan\left(\frac{(ar-bs-(a-b)k)m}{(a+b)\sqrt{(m^2+2\xi_1^2)(rs-k^2)}}\right)}{\pi},$$

$$B_3 = \frac{2m(br(a+b)-as(a+b)-(b^2-a^2)k)(m^2+2\xi_2^2)^{1/2}\sqrt{rs-k^2}}{\pi\xi_2^2((r+s-2k)m^2+2(a+b)^2(rs-k^2))} + \frac{2\arctan\left(\frac{(as-br-(a-b)k)m}{(a+b)\sqrt{(m^2+2\xi_2^2)(rs-k^2)}}\right)}{\pi},$$

$$\xi_1 = \sqrt{a^2r+b^2s+2abk},\ \xi_2 = \sqrt{a^2s+b^2r+2abk}.$$

Again, in the above corollary, M is the mixture with equal weights of skew distributions of $(\xi_1/\sqrt{r})X$ and $(\xi_2/\sqrt{s})Y$. In particular for the case $a^{-1}+b^{-1}=0$, $M \sim S((\xi_1/\sqrt{r})X)$.

Chapter 2

Ratio of generalized skew-normal and skew-t random variables

2.1 Introduction

It is known that for a bivariate random vector (X, Y), the distribution of the ratio X/Y is of interest in many areas. For different bivariate distributions, recently, there are many investigations to find the distributions of X/Y and XY. See, for example, Chamayou (2004), Nadarajah and Ali (2005, 2006), Nadarajah and El (2005), Nadarajah (2005, 2006), Gupta and Nadarajah (2006a, 2006b), Nadarajah and Kibria (2006), Nadarajah and Kotz (2006, 2007), Nadarajah and Gupta (2006, 2007), Coelho and Mexia (2007), and Sharafi et al. (2008), etc.

In this work, we will study the distribution of the ratio of two independent $GS\mathcal{N}(b_1, b_2)$ distributed random variables. Here following Jamalizadeha, et al. (2008), a random variable X is said to have a $GS\mathcal{N}(b_1, b_2)$ distribution, the so-called two-parameter generalized skew-normal distribution, if its p.d.f. is

$$\varphi(x; b_1, b_2) = c(b_1, b_2)\phi(x)\Phi(b_1 x)\Phi(b_2 x), \ x \in \mathcal{R}, \tag{1}$$

where ϕ and Φ are the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of $\mathcal{N}(0, 1)$ distribution, respectively, $b_1, b_2 \in \mathcal{R}$, and

$$c(b_1, b_2) = \left(\frac{1}{4} + \frac{1}{2\pi}\arctan\left(\frac{b_1 b_2}{\sqrt{1 + b_1^2 + b_2^2}}\right)\right)^{-1}, \tag{2}$$

According to Huang and Chen (2007), for a symmetric p.d.f f, X is said to have a skew distribution of f, if the p.d.f. of X is

$$f_X(x) = 2f(x)G(x), \quad x \in \mathcal{R}, \tag{3}$$

where G is a skew function, that is

$$0 \le G(x) \le 1 \ \text{ and } \ G(x) + G(-x) = 1, x \in \mathcal{R}. \tag{4}$$

For a common symmetric distribution, such as $\mathcal{N}(0,1)$ distribution, skew-$\mathcal{N}(0,1)$ distribution can be defined in a similar way.

Note that when $b_2 = 0$, then $\varphi(x; b_1, 0) = 2\phi(x)\Phi(b_1 x)$. Hence $GS\mathcal{N}(b_1, 0)$ is the usual $S\mathcal{N}(b_1)$ distribution. That is a distribution with the p.d.f. $2\phi(x)\Phi(\lambda x),\ x \in \mathcal{R}$, where $\lambda \in \mathcal{R}$. Similarly, $GS\mathcal{N}(0, b_2)$ is just $S\mathcal{N}(b_2)$. Except $b_1 = 0$ or $b_2 = 0$, $GS\mathcal{N}(b_1, b_2)$ distribution is not skew-$\mathcal{N}(0,1)$ distribution in the above sense. The following is an application of the p.d.f. given in (1). Let $X|Y = y$ be $S\mathcal{N}(\lambda y)$ distributed, Y be $S\mathcal{N}(\eta)$ distributed. Then

$$
\begin{aligned}
f_X(x|\lambda, \eta) &= 4\phi(x) \int_{-\infty}^{\infty} \phi(y)\Phi(\lambda x y)\Phi(\eta y) dy \\
&= 4\phi(x) \left(\frac{1}{4} + \frac{1}{2\pi} \arctan \frac{\lambda \eta x}{\sqrt{1 + \lambda^2 x^2 + \eta^2}} \right) \\
&= 2\phi(x) \left(\frac{1}{2} + \frac{1}{\pi} \arctan \frac{\lambda \eta x}{\sqrt{1 + \lambda^2 x^2 + \eta^2}} \right) = 2\phi(x)G(x),\ x \in \mathcal{R},
\end{aligned}
$$

where

$$
G(x) = \left(\frac{1}{2} + \frac{1}{\pi} \arctan \frac{\lambda \eta x}{\sqrt{1 + \lambda^2 x^2 + \eta^2}} \right),\ x \in \mathcal{R},
$$

is a skew function. Hence X has a skew-$\mathcal{N}(0,1)$ distribution. When $\lambda = 0$ or $\eta = 0$, X is $\mathcal{N}(0,1)$ distributed, and as $\eta \to \infty$,

$$
f_X(x|\lambda, \eta) \to 2\phi(x)\psi(\lambda x),
$$

where the skew function

$$
\psi(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x,\ x \in \mathcal{R},
$$

is the distribution function of $\mathcal{C}(0,1)$.

We are also interested in knowing when the distribution of the ratio of two $GS\mathcal{N}(b_1, b_2)$ distributed random variables will be skew-$\mathcal{C}(0,1)$ distributed.

2.2 The ratio of two GSN(b_1, b_2) distributions

In this section, we will find the distributions of the ratio of two generalized skew-normal distributed random variables U and V. The special case that both U and V are skew-normal distributed has been treated by Huang and Chen (2007). We give the result in the following.

Theorem 2.1. Let U and V be independent random variables distributed as $S\mathcal{N}(b_1)$ and $S\mathcal{N}(b_3)$, respectively, $b_1, b_3 \in \mathcal{R}$. Then $W \equiv X(b_1, b_3) = U/V$ has a skew-$\mathcal{C}(0,1)$ distribution with p.d.f.

$$
f_W(w) = \frac{2}{\pi(1 + w^2)} G(x), \tag{5}
$$

where

$$G(x) = \left(\frac{1}{2} + \frac{b_3 \arctan\left(b_1 w / \sqrt{1+b_3^2+w^2}\right)}{\pi\sqrt{1+b_3^2+w^2}} + \frac{b_1 w \arctan\left(b_3 / \sqrt{1+(1+b_1^2)w^2}\right)}{\pi\sqrt{1+(1+b_1^2)w^2}} \right),$$

$w, b_1, b_3 \in \mathcal{R}$, is a skew function.

For the ratio of two $GS\mathcal{N}$ distributed random variables, the general case is too cumbersome. But when one of the four parameters is zero, the p.d.f. of U/V can be obtained.

Theorem 2.2. Let U and V be independent random variables distributed as $GS\mathcal{N}(b_1, b_2)$ and $GS\mathcal{N}(b_3, 0)$, respectively, $b_1, b_2, b_3 \in \mathcal{R}$. Then $X \equiv X(b_1, b_2, b_3, 0) = U/V$ has the following p.d.f.

$$f_X(x) = \frac{1}{\pi(1+x^2)} \left(\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{b_1 b_2}{\sqrt{1+b_1^2+b_2^2}} \right) \right)^{-1} G_1(x), \quad x \in \mathcal{R}, \tag{6}$$

where

$$\begin{aligned} G_1(x) = \frac{1}{2} &+ \frac{b_1 x \left[\arctan\left(b_3/\sqrt{1+(1+b_1^2)x^2}\right) + \arctan\left(b_2 x/\sqrt{1+(1+b_1^2)x^2}\right)\right]}{\pi\sqrt{1+(1+b_1^2)x^2}} \\ &+ \frac{b_2 x \left[\arctan\left(b_3/\sqrt{1+(1+b_2^2)x^2}\right) + \arctan\left(b_1 x/\sqrt{1+(1+b_2^2)x^2}\right)\right]}{\pi\sqrt{1+(1+b_2^2)x^2}} \\ &+ \frac{b_3 \left[\arctan\left(b_1 x/\sqrt{1+b_3^2+x^2}\right) + \arctan\left(b_2 x/\sqrt{1+b_3^2+x^2}\right)\right]}{\pi\sqrt{1+b_3^2+x^2}}. \end{aligned} \tag{7}$$

The proof of Theorem 2.2 will be given in the Appendix. It can be seen easily that when $b_2 = 0$, (6) coincides with (5). Furthermore, we have

Theorem 2.3. In Theorem 2.2, U/V is skew-$\mathcal{C}(0,1)$ distributed if and only if b_1 or b_2 is zero.

Proof. Obviously, we only need to prove the "only if"part. In (6), by letting

$$K(x) = \left(1 + \frac{2}{\pi} \arctan\left(\frac{b_1 b_2}{\sqrt{1+b_1^2+b_2^2}} \right) \right)^{-1} G_1(x),$$

$f_X(x)$ can be rewritten as

$$f_X(x) = \frac{2}{\pi(1+x^2)} K(x) \quad x \in \mathcal{R}.$$

Hence U/V is Skew-$\mathcal{C}(0,1)$ distributed if and only if $K(x)$ is a skew function. In particular,

$$2K(0) = \left(1 + \frac{2}{\pi} \arctan\left(\frac{b_1 b_2}{\sqrt{1+b_1^2+b_2^2}} \right) \right)^{-1} = 1,$$

Since $G_1(0) = 1/2$. This in turn implies $b_1 = 0$ or $b_2 = 0$, and the proof follows.

The following theorem is a parallel result of Theorem 2.2. The proof is similar hence is omitted.

Theorem 2.4. Let S and T be independent random variables distributed as $GS\mathcal{N}(b_1, 0)$ and $GS\mathcal{N}(b_3, b_4)$, respectively, $b_1, b_3, b_4 \in \mathcal{R}$. Then the p.d.f. of $Y \equiv X(b_1, 0, b_3, b_4) = S/T$ is given by

$$f_Y(y) = \frac{1}{\pi(1+y^2)} \left(\frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{b_3 b_4}{\sqrt{1+b_3^2+b_4^2}} \right) \right)^{-1} G_2(y), \quad y \in \mathcal{R},$$

where

$$\begin{aligned} G_2(y) = \frac{1}{2} &+ \frac{b_1 y \left[\arctan \left(b_3/\sqrt{1+(1+b_1^2)y^2} \right) + \arctan \left(b_4/\sqrt{1+(1+b_1^2)y^2} \right) \right]}{\pi\sqrt{1+(1+b_1^2)y^2}} \\ &+ \frac{b_3 \left[\arctan \left(b_1 y/\sqrt{1+b_3^2+y^2} \right) + \arctan \left(b_4/\sqrt{1+b_3^2+y^2} \right) \right]}{\pi\sqrt{1+b_3^2+y^2}} \\ &+ \frac{b_4 \left[\arctan \left(b_1 y/\sqrt{1+b_4^2+y^2} \right) + \arctan \left(b_3/\sqrt{1+b_4^2+y^2} \right) \right]}{\pi\sqrt{1+b_4^2+y^2}}. \end{aligned}$$

With regard to the moments, we have the following result.

Corollary 2.1. In Theorem 2.2, $E(|X|^s)$ exists if and only if $|s| < 1$.

Proof. First it is known that

$$\int_{-\infty}^{\infty} |x|^s \frac{1}{\pi(1+x^2)} dx < \infty,$$

if and only if $|s| < 1$. As $0 \le G_1(x) < 4, \ \forall x \in \mathcal{R}$,

$$f_X(x) < \frac{1}{\pi(1+x^2)} \left(\frac{1}{8} + \frac{1}{4\pi} \arctan \left(\frac{b_1 b_2}{\sqrt{1+b_1^2+b_2^2}} \right) \right)^{-1}, \ x \in \mathcal{R}.$$

Hence $E(|X|^s) < \infty$ for $|s| < 1$. Next by using limit comparison test,

$$\begin{aligned} \lim_{x\to\infty} \frac{f_X(x)}{\frac{1}{\pi(1+x^2)}} = &\left(\frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{b_1 b_2}{\sqrt{1+b_1^2+b_2^2}} \right) \right)^{-1} \\ &\cdot \left(\frac{1}{2} + \frac{b_1 \arctan \left(\frac{b_2}{\sqrt{1+b_1^2}} \right)}{\pi\sqrt{1+b_2^2}} + \frac{b_2 \arctan \left(\frac{b_1}{\sqrt{1+b_2^2}} \right)}{\pi\sqrt{1+b_2^2}} \right) < \infty, \end{aligned}$$

consequently, $E(|X|^s) = \infty$ if $s > 1$. That $E(|X|^s) = \infty$ for $s < -1$ can be proved similarly. This completes the proof.

Corollary 2.1 has a version for the random variable Y in Theorem 2.4, as it is similar, we omit the statement of this result.

According to Sharafi and Behboodian (2008), a random variable X is said to have a $S\mathcal{N}B_n(b)$ distribution, if the p.d.f. of X is

$$f_n(x;\lambda) = c_n(b)\phi(x)\Phi^n(bx), \quad x \in \mathcal{R}, \quad b \in \mathcal{R}, \tag{8}$$

where

$$c_n(b) = \frac{1}{\int_{-\infty}^{\infty}\phi(x)\Phi^n(bx)dx}, \qquad n \geq 1. \tag{9}$$

As $S\mathcal{N}B_2(b_1)$ is exactly a $GS\mathcal{N}(b_1, b_1)$ distribution, we have the following immediate consequence of Theorem 2.2.

Corollary 2.2. Let P and Q be independent random variables distributed as $S\mathcal{N}B_2(b_1)$ and $GS\mathcal{N}(b_3, 0)$, respectively, $b_1, b_3 \in \mathcal{R}$. Then $Z \equiv Z(b_1, b_3) = P/Q$ has the following p.d.f.

$$f_Z(z) = \frac{1}{\pi(1+z^2)}\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{b_1^2}{\sqrt{1+2b_1^2}}\right)\right)^{-1} H(z), \quad z \in \mathcal{R},$$

where

$$\begin{aligned} H(z) = \frac{1}{2} &+ \frac{2b_1 z\left[\left(\arctan\left(b_3/\sqrt{1+(1+b_1^2)z^2}\right)\right)\right]}{\pi\sqrt{1+(1+b_1^2)z^2}} + \frac{2b_1 z\left[\left(\arctan\left(b_1 z/\sqrt{1+(1+b_1^2)z^2}\right)\right)\right]}{\pi\sqrt{1+(1+b_1^2)z^2}} \\ &+ \frac{2b_3\left[\left(\arctan\left(b_1 z/\sqrt{1+b_3^2+z^2}\right)\right)\right]}{\pi\sqrt{1+b_3^2+z^2}}. \end{aligned}$$

Finally we present some limiting results for the $X(b_1, b_2, b_3, 0)$ distribution.

Corollary 2.3. The following are the limiting p.d.f.'s of $X(b_1, b_2, b_3, 0)$ distribution.

$$\begin{array}{ll} \frac{1}{\pi(1+x^2)}(1+g_4(x))I(x \geq 0), & b_1, b_2 \to \infty, \\ \frac{1}{\pi(1+x^2)}(1-g_4(x))I(x < 0), & b_1, b_2 \to -\infty, \\ \frac{\lambda_1}{\pi(1+x^2)}(1+g_5(x))I(x \geq 0), & b_1, b_3 \to \infty, \\ \frac{\lambda_2}{\pi(1+x^2)}(1-g_5(x))I(x < 0), & b_1, b_3 \to -\infty, \\ \frac{\lambda_3}{\pi(1+x^2)}(1+g_6(x))I(x \geq 0), & b_2, b_3 \to \infty, \\ \frac{\lambda_4}{\pi(1+x^2)}(1-g_6(x))I(x < 0), & b_2, b_3 \to -\infty, \\ \frac{2}{\pi(1+x^2)}I(x \geq 0), & b_1, b_2, b_3 \to \infty, \\ \frac{2}{\pi(1+x^2)}I(x < 0), & b_1, b_2, b_3 \to -\infty, \end{array}$$

$$
\begin{array}{ll}
\frac{1}{\pi(1+x^2)} g_1(x) I(x \geq 0), & b_1 \to \infty, \\
\frac{1}{\pi(1+x^2)} g_{11}(x) I(x < 0), & b_1 \to -\infty, \\
\frac{1}{\pi(1+x^2)} g_2(x) I(x \geq 0), & b_2 \to \infty, \\
\frac{1}{\pi(1+x^2)} g_{21}(x) I(x < 0), & b_2 \to -\infty, \\
\frac{1}{\pi(1+x^2)} g_3(x) I(x \geq 0), & b_3 \to \infty, \\
\frac{1}{\pi(1+x^2)} g_{31}(x) I(x < 0), & b_3 \to -\infty,
\end{array}
$$

where

$$
\lambda_1 = \frac{\pi}{\arccos\left(-\frac{b_2}{\sqrt{1+b_2^2}}\right)}, \lambda_2 = \frac{\pi}{\arccos\left(\frac{b_2}{\sqrt{1+b_2^2}}\right)}, \lambda_3 = \frac{\pi}{\arccos\left(-\frac{b_1}{\sqrt{1+b_1^2}}\right)}, \lambda_4 = \frac{\pi}{\arccos\left(\frac{b_1}{\sqrt{1+b_1^2}}\right)},
$$

$$
g_1(x) = \frac{1}{\pi}\left[\pi + \frac{b_2 x\left(\pi + 2\arctan\left(\frac{b_3}{\sqrt{1+(1+b_2^2)x^2}}\right)\right)}{\sqrt{1+(1+b_2^2)x^2}} + \frac{b_3\left(\pi + 2\arctan\left(\frac{b_2 x}{\sqrt{1+b_3^2+x^2}}\right)\right)}{\sqrt{1+b_3^2+x^2}}\right],
$$

$$
g_{11}(x) = \frac{1}{\pi}\left[\pi + \frac{-b_2 x\left(\pi + 2\arctan\left(\frac{-b_3}{\sqrt{1+(1+b_2^2)x^2}}\right)\right)}{\sqrt{1+(1+b_2^2)x^2}} + \frac{-b_3\left(\pi + 2\arctan\left(\frac{-b_2 x}{\sqrt{1+b_3^2+x^2}}\right)\right)}{\sqrt{1+b_3^2+x^2}}\right],
$$

$$
g_2(x) = \frac{1}{\pi}\left[\pi + \frac{b_1 x\left(\pi + 2\arctan\left(\frac{b_3}{\sqrt{1+(1+b_1^2)x^2}}\right)\right)}{\sqrt{1+(1+b_1^2)x^2}} + \frac{b_3\left(\pi + 2\arctan\left(\frac{b_1 x}{\sqrt{1+b_3^2+x^2}}\right)\right)}{\sqrt{1+b_3^2+x^2}}\right],
$$

$$
g_{21}(x) = \frac{1}{\pi}\left[\pi + \frac{-b_1 x\left(\pi + 2\arctan\left(\frac{-b_3}{\sqrt{1+(1+b_1^2)x^2}}\right)\right)}{\sqrt{1+(1+b_1^2)x^2}} + \frac{-b_3\left(\pi + 2\arctan\left(\frac{-b_1 x}{\sqrt{1+b_3^2+x^2}}\right)\right)}{\sqrt{1+b_3^2+x^2}}\right],
$$

$$
g_3(x) = \frac{\left[\pi + \frac{b_1 x(\pi + 2\arctan(b_2 x/\sqrt{1+(1+b_1^2)x^2}))}{\sqrt{1+(1+b_1^2)x^2}} + \frac{b_2 x(\pi + 2\arctan(b_1 x/\sqrt{1+(1+b_2^2)x^2}))}{\sqrt{1+(1+b_2^2)x^2}}\right]}{\pi + 2\arctan\left(\frac{b_1 b_2}{\sqrt{1+b_1^2+b_2^2}}\right)},
$$

$$
g_{31}(x) = \frac{\left[\pi + \frac{-b_1 x(\pi + 2\arctan(-b_2 x/\sqrt{1+(1+b_1^2)x^2}))}{\sqrt{1+(1+b_1^2)x^2}} + \frac{-b_2 x(\pi + 2\arctan(-b_1 x/\sqrt{1+(1+b_2^2)x^2}))}{\sqrt{1+(1+b_2^2)x^2}}\right]}{\pi + 2\arctan\left(\frac{b_1 b_2}{\sqrt{1+b_1^2+b_2^2}}\right)},
$$

and

$$g_4(x) = \frac{b_3}{\sqrt{1+b_3^2+x^2}},\ g_5(x) = \frac{b_2 x}{\sqrt{1+(1+b_2^2)x^2}},\ g_6(x) = \frac{b_1 x}{\sqrt{1+(1+b_2^2)x^2}}.$$

2.3 Some figures of p.d.f. in Theorem 2.2

In this section, in Figure 2, we give some illustrations of the possible forms of the p.d.f. of the random variable $X(b_1, b_2, b_3, 0)$ in Theorem 2.2 under various choices of (b_1, b_2, b_3). It can be shown that the p.d.f. of the $X(b_1, b_2, b_3, 0)$ distribution may have one side heavier tail and one side thinner tail than the $\mathcal{C}(0,1)$ distribution.

Note that $X(b_1, b_2, b_3, 0)$ may not be unimodal. As an example, $X(50, 4, -1, 0)$ is bimodal and has three inflection points, see Figure 1.

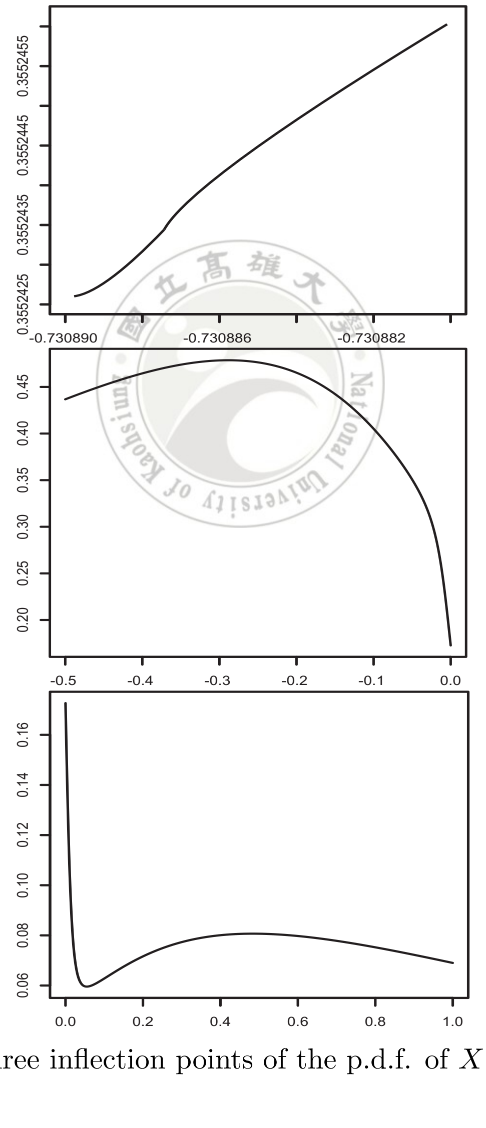

Figure 1. Three inflection points of the p.d.f. of $X(50, 4, -1, 0)$.

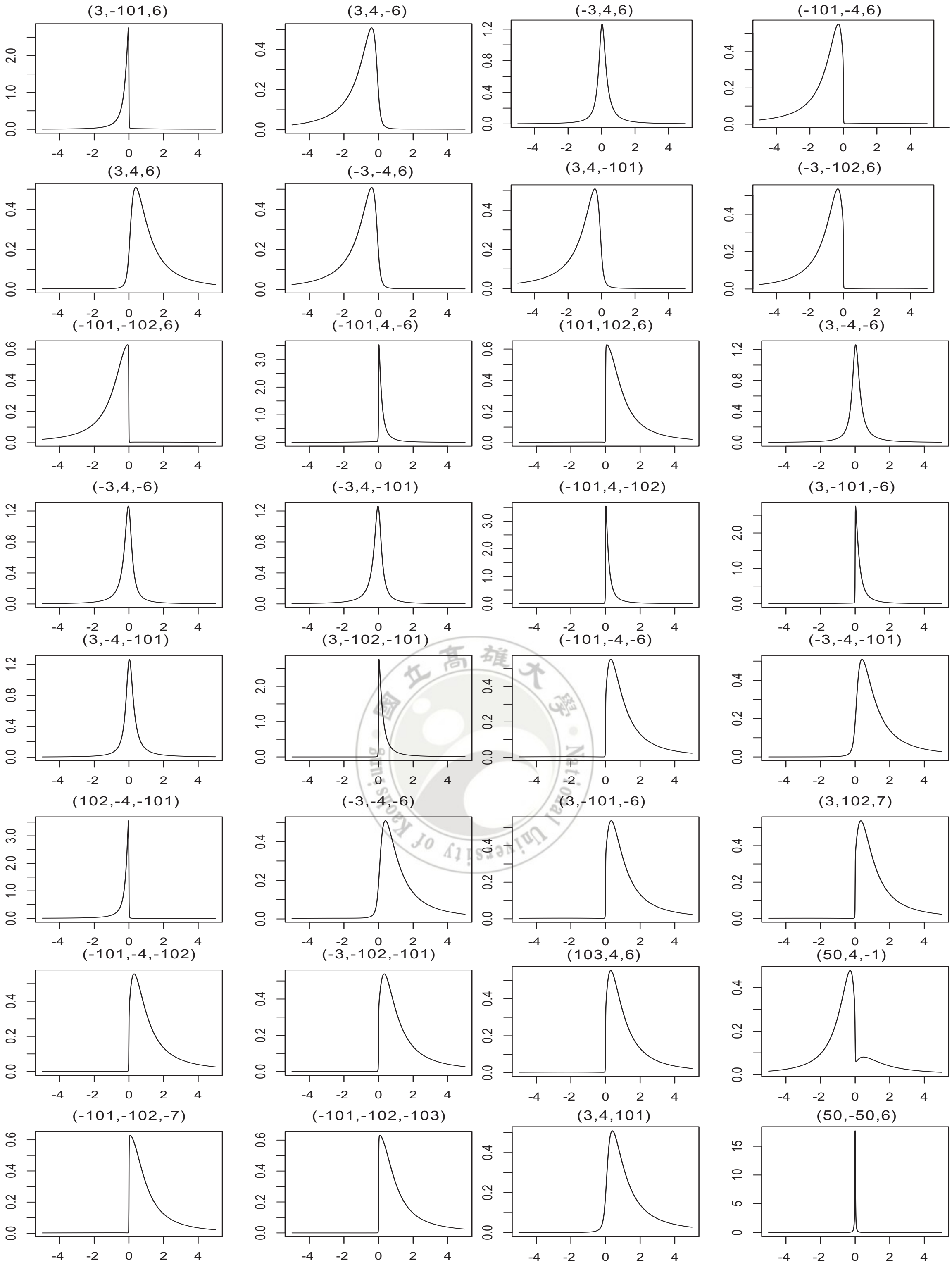

Figure 2. Probability density function of $X(b_1, b_2, b_3, 0)$ for several values of (b_1, b_2, b_3)

2.4 Appendix.

We prove Theorem 2.2 in the following. First we give two lemmas which can be found in Huang and Su (2008).

Lemma 2.1. Let $F_{\mathcal{T}_r}$ be the c.d.f. of $\mathcal{T}_r$ distributions. Then

$$F_{\mathcal{T}_r}(t) = \begin{cases} \frac{1}{2} + \frac{1}{\pi}\arctan\frac{t}{\sqrt{r}} + \frac{1}{2\sqrt{\pi}}\sum_{i=1}^{(r-1)/2}\frac{\Gamma(i)r^{i-1/2}}{\Gamma(i+1/2)}\frac{t}{(r+t^2)^i}, & \text{if r is odd,} \\ \frac{1}{2} + \frac{1}{2\sqrt{\pi}}\sum_{i=1}^{r/2}\frac{\Gamma(i-1/2)r^{i-1}}{\Gamma(i)}\frac{t}{(r+t^2)^{i-1/2}}, & \text{if r is even.} \end{cases} \tag{10}$$

Lemma 2.2. Let $a, b_1, b_2 \in \mathcal{R}, q \in \mathcal{N}$. If $a \neq 0$ and q is odd, we have

$$\int_0^\infty v^q\phi(av)\Phi(b_1v)\Phi(b_2v)dv = \frac{2^{q/2}}{4\pi a^{q+1}}\Gamma(\frac{q+1}{2})\sum_0^{(q+1)/2}\frac{\Gamma(i-1/2)}{\Gamma(i)}\left[\frac{b_1/|a|}{(1+b_1^2/a^2)^{i-1/2}}\right.$$
$$\left. F_{\mathcal{T}_{2i-1}}\left(\frac{b_2\sqrt{2i-1}}{|a|\sqrt{1+b_1^2/a^2}}\right) + \frac{b_2/|a|}{(1+b_2^2/a^2)^{i-1/2}}F_{\mathcal{T}_{2i-1}}\left(\frac{b_1\sqrt{2i-1}}{|a|\sqrt{1+b_2^2/a^2}}\right)\right] + \frac{2^{q/2-3}}{\sqrt{\pi}}\Gamma(\frac{q+1}{2}).$$

If $a \neq 0$ and q is even, we have

$$\int_0^\infty v^q\phi(av)\Phi(b_1v)\Phi(b_2v)dv = \frac{2^{q/2}}{4\pi|a|^{q+1}}\Gamma(\frac{q+1}{2})\sum_0^{q/2}\frac{\Gamma(i)}{\Gamma(i+1/2)}\left[\frac{b_1/|a|}{(1+b_1^2/a^2)^i}\right.$$
$$\left. F_{\mathcal{T}_{2i}}\left(\frac{b_2\sqrt{2i}}{|a|\sqrt{1+b_1^2/a^2}}\right) + \frac{b_2/|a|}{(1+b_2^2/a^2)^i}F_{\mathcal{T}_{2i}}\left(\frac{b_1\sqrt{2i-1}}{|a|\sqrt{1+b_2^2/a^2}}\right)\right]$$
$$+\frac{2^{q/2}}{|a|^{q+1}\sqrt{\pi}}\Gamma(\frac{q+1}{2})\left(\frac{1}{8} + \frac{1}{4\pi}(\arctan\frac{b_1}{|a|} + \arctan\frac{b_2}{|a|} + \arctan\frac{b_1b_2}{|a|\sqrt{a^2+b_1^2+b_2^2}})\right).$$

For the case $a = 0$ and b_1, b_2 are not both positive, we have

$$\int_0^\infty v^q\frac{1}{\sqrt{2\pi}}\Phi(b_1v)\Phi(b_2v)dv$$
$$= \frac{2^{(q+1)/2-1}\Gamma((q+2)/2)}{2\pi^{3/2}(q+1)}\left(\frac{-b_1}{|b_1|^{q+1}}F_{\mathcal{T}_{q+2}}(\frac{b_2\sqrt{q+2}}{|b_1|}) + \frac{-b_2}{|b_2|^{q+1}}F_{\mathcal{T}_{q+2}}(\frac{b_1\sqrt{q+2}}{|b_2|})\right).$$

In particularly, if $a \neq 0$, $q = 0$, then we have

$$\int_0^\infty \phi(av)\Phi(b_1v)\Phi(b_2v)dt$$
$$= \frac{1}{8|a|} + \frac{1}{4\pi|a|}(\arctan(b_1/|a|) + \arctan(b_2/|a|) + \arctan(b_1b_2/(|a|(a^2+b_1^2+b_2^2)^{1/2}))). \tag{11}$$

Lemma 2.3. Let $a, b_1, b_2, b_3 \in \mathcal{R}$. If $a \neq 0$,

$$
\begin{aligned}
&\int_0^\infty v\phi(av)\Phi(b_1 v)\Phi(b_2 v)\Phi(b_3 v)dv = \frac{1}{8(2\pi)^{1/2}a^2} \\
&+\frac{b_1}{\sqrt{2\pi}a^2}\left[\frac{1}{4\pi\sqrt{a^2+b_1^2}}\left(\arctan\left(\frac{b_3}{\sqrt{a^2+b_1^2}}\right)+\arctan\left(\frac{b_2 b_3}{\sqrt{(a^2+b_1^2+b_2^2+b_3^2)(a^2+b_1^2)}}\right)\right)\right. \\
&\left.+\frac{1}{8\sqrt{a^2+b_1^2}}+\frac{1}{4\pi\sqrt{a^2+b_1^2}}\arctan\left(\frac{b_2}{\sqrt{a^2+b_1^2}}\right)\right] \\
&+\frac{b_2}{\sqrt{2\pi}a^2}\left[\frac{1}{4\pi\sqrt{a^2+b_2^2}}\left(\arctan\left(\frac{b_3}{\sqrt{a^2+b_2^2}}\right)+\arctan\left(\frac{b_1 b_3}{\sqrt{(a^2+b_1^2+b_2^2+b_3^2)(a^2+b_2^2)}}\right)\right)\right. \\
&\left.+\frac{1}{8\sqrt{a^2+b_2^2}}+\frac{1}{4\pi\sqrt{a^2+b_2^2}}\arctan\left(\frac{b_1}{\sqrt{a^2+b_2^2}}\right)\right] \\
&+\frac{b_3}{\sqrt{2\pi}a^2}\left[\frac{1}{4\pi\sqrt{a^2+b_3^2}}\left(\arctan\left(\frac{b_1}{\sqrt{a^2+b_3^2}}\right)+\arctan\left(\frac{b_1 b_2}{\sqrt{(a^2+b_1^2+b_2^2+b_3^2)(a^2+b_3^2)}}\right)\right)\right. \\
&\left.+\frac{1}{8\sqrt{a^2+b_3^2}}+\frac{1}{4\pi\sqrt{a^2+b_3^2}}\arctan\left(\frac{b_2}{\sqrt{a^2+b_3^2}}\right)\right].
\end{aligned}
$$

For the case $a = 0$, we have

$$
\int_0^\infty v\frac{1}{\sqrt{2\pi}}\Phi(b_1 v)\Phi(b_2 v)\Phi(b_3 v)dv = \begin{cases} \infty, & b_1 \geq 0, b_2 \geq 0, b_3 \geq 0, \\ \frac{1}{16\sqrt{2\pi}b_i^2}, & b_i < 0, i = 1,2,3, b_j = b_k = 0, \\ \theta, & otherwise, \end{cases} \tag{13}
$$

where

$$
\begin{aligned}
\theta = &-\frac{\left[\frac{1}{4\pi|b_1|}(\arctan(b_2/|b_1|)+\arctan(b_3/|b_1|)+\arctan(b_2 b_3/(|b_1|(b_1^2+b_2^2+b_3^2)^{1/2})))\right]}{2\sqrt{2\pi}b_1} \\
&-\frac{\left(\left(\frac{b_2}{b_1^2+b_2^2}\left(1+\frac{b_3}{\sqrt{b_1^2+b_2^2+b_3^2}}\right)\right)+\left(\frac{b_3}{b_1^2+b_3^2}\left(1+\frac{b_2}{\sqrt{b_1^2+b_2^2+b_3^2}}\right)\right)\right)}{4(2\pi)^{3/2}b_1} \\
&-\frac{\left[\frac{1}{4\pi|b_2|}(\arctan(b_1/|b_2|)+\arctan(b_3/|b_2|)+\arctan(b_1 b_3/(|b_2|(b_1^2+b_2^2+b_3^2)^{1/2})))\right]}{2\sqrt{2\pi}b_2} \\
&-\frac{1}{2\sqrt{2\pi}b_1}\frac{1}{8|b_1|}-\frac{1}{2\sqrt{2\pi}b_2}\frac{1}{8|b_2|}-\frac{1}{2\sqrt{2\pi}b_3}\frac{1}{8|b_3|} \\
&-\frac{\left(\left(\frac{b_1}{b_1^2+b_2^2}\left(1+\frac{b_3}{\sqrt{b_1^2+b_2^2+b_3^2}}\right)\right)+\left(\frac{b_3}{b_2^2+b_3^2}\left(1+\frac{b_1}{\sqrt{b_1^2+b_2^2+b_3^2}}\right)\right)\right)}{4(2\pi)^{3/2}b_2} \\
&-\frac{\left[\frac{1}{4\pi|b_3|}(\arctan(b_1/|b_3|)+\arctan(b_2/|b_3|)+\arctan(b_1 b_2/(|b_3|(b_1^2+b_2^2+b_3^2)^{1/2})))\right]}{2\sqrt{2\pi}b_3}
\end{aligned}
$$

$$-\frac{\left(\left(\frac{b_1}{b_1^2+b_3^2}\left(1+\frac{b_2}{\sqrt{b_1^2+b_2^2+b_3^2}}\right)\right)+\left(\frac{b_2}{b_2^2+b_3^2}\left(1+\frac{b_1}{\sqrt{b_1^2+b_2^2+b_3^2}}\right)\right)\right)}{4(2\pi)^{3/2}b_3}. \tag{14}$$

Proof. We only prove the case $a \neq 0$. By the integration by parts, we have

$$\begin{aligned}
&\int_0^\infty v\phi(av)\Phi(b_1v)\Phi(b_2v)\Phi(b_3v)dv \\
&= \int_0^\infty v\frac{1}{\sqrt{2\pi}}e^{-a^2v^2/2}\Phi(b_1v)\Phi(b_2v)\Phi(b_3v)dv \\
&= \frac{\left[-\Phi(b_1v)\Phi(b_2v)\Phi(b_3v)e^{-a^2v^2/2}\Big|_0^\infty + \int_0^\infty e^{-a^2v^2/2}d(\Phi(b_1v)\Phi(b_2v)\Phi(b_3v))\right]}{\sqrt{2\pi}a^2} \\
&= \frac{\left[\frac{1}{8}+\int_0^\infty e^{-a^2v^2/2}[b_1\phi(b_1v)\Phi(b_2v)\Phi(b_3v)+b_2\phi(b_2v)\Phi(b_1v)\Phi(b_3v)+b_3\phi(b_3v)\Phi(b_1v)\Phi(b_2v)]dv\right]}{\sqrt{2\pi}a^2},
\end{aligned}$$

where the last integration can be obtained by (12), and the proof follows.

Proof of Theorem 2.2. First the joint p.d.f. of U and V is

$$\begin{aligned}
f_{U,V}(u,v) &= \left(\frac{1}{8}+\frac{1}{4\pi}\arctan\left(\frac{b_1b_2}{\sqrt{1+b_1^2+b_2^2}}\right)\right)^{-1}\phi(u)\phi(v)\Phi(b_1u)\Phi(b_2u)\Phi(b_3v) \\
&= \lambda\phi(u)\phi(v)\Phi(b_1u)\Phi(b_2u)\Phi(b_3v), \quad u,v\in\mathcal{R},
\end{aligned} \tag{15}$$

where

$$\lambda = \left(\frac{1}{8}+\frac{1}{4\pi}\arctan\left(\frac{b_1b_2}{\sqrt{1+b_1^2+b_2^2}}\right)\right)^{-1}. \tag{16}$$

Hence the p.d.f. of X is

$$f_X(x) = \lambda\int_{-\infty}^\infty |v|\phi(xv)\phi(v)\Phi(b_1xv)\Phi(b_2xv)\Phi(b_3v)dv$$

$$= \lambda\int_0^\infty v\phi(xv)\phi(v)\Phi(b_1xv)\Phi(b_2xv)\Phi(b_3v)dv + \lambda\int_0^\infty v\phi(xv)\phi(v)\Phi(-b_1xv)\Phi(-b_2xv)\Phi(-b_3v)dv.$$

Let S=$\{(1,2),(2,1)\}$. By Lemma 2.3, we have

$$\begin{aligned}
f_X(x) = \frac{\lambda}{16\pi(1+x^2)}+\frac{\lambda}{16\pi(1+x^2)}+\sum_{(i,j)\in S}\frac{b_ix\lambda}{2\pi(1+x^2)}\Bigg[&\frac{1}{8\sqrt{1+(1+b_i^2)x^2}} \\
+\frac{\arctan\left(\frac{b_jx}{\sqrt{1+(1+b_i^2)x^2}}\right)}{4\pi\sqrt{1+(1+b_i^2)x^2}} &+ \frac{\arctan\left(\frac{b_3}{\sqrt{1+(1+b_i^2)x^2}}\right)+\arctan\left(\frac{b_jxb_3}{\sqrt{1+(1+b_i^2)x^2}\sqrt{1+(1+b_i^2+b_j^2)x^2+b_3^2}}\right)}{4\pi\sqrt{1+(1+b_i^2)x^2}}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{b_3\lambda}{2\pi(1+x^2)}\left[\frac{1}{8\sqrt{1+b_3^2+x^2}}+\frac{\arctan\left(\frac{b_1x}{\sqrt{1+b_3^2+x^2}}\right)}{4\pi\sqrt{1+b_3^2+x^2}}\right.\\
&\left.+\frac{\arctan\left(\frac{b_2x}{\sqrt{1+b_3^2+x^2}}\right)+\arctan\left(\frac{b_1xb_2x}{\sqrt{1+b_3^2+x^2}\sqrt{1+(1+b_1^2+b_2^2)x^2+b_3^2}}\right)}{4\pi\sqrt{1+b_3^2+x^2}}\right]\\
&-\sum_{(i,j)\in S}\frac{b_ix\lambda}{2\pi(1+x^2)}\left[\frac{1}{8\sqrt{1+(1+b_i^2)x^2}}+\frac{\arctan\left(\frac{-b_jx}{\sqrt{1+(1+b_i^2)x^2}}\right)}{4\pi\sqrt{1+(1+b_i^2)x^2}}\right.\\
&\left.+\frac{\arctan\left(\frac{-b_3}{\sqrt{1+(1+b_i^2)x^2}}\right)+\arctan\left(\frac{b_jxb_3}{\sqrt{1+(1+b_i^2)x^2}\sqrt{1+(1+b_i^2+b_j^2)x^2+b_3^2}}\right)}{4\pi\sqrt{1+(1+b_i^2)x^2}}\right]\\
&-\frac{b_3\lambda}{2\pi(1+x^2)}\left[\frac{1}{8\sqrt{1+b_3^2+x^2}}+\frac{\arctan\left(\frac{-b_1x}{\sqrt{1+b_3^2+x^2}}\right)}{4\pi\sqrt{1+b_3^2+x^2}}\right.\\
&\left.+\frac{\arctan\left(\frac{-b_2x}{\sqrt{1+b_3^2+x^2}}\right)+\arctan\left(\frac{b_1xb_2x}{\sqrt{1+b_3^2+x^2}\sqrt{1+(1+b_1^2+b_2^2)x^2+b_3^2}}\right)}{4\pi\sqrt{1+b_3^2+x^2}}\right]\\
&=\left(\frac{\lambda}{(1+x^2)8\pi}\right)\\
&+\frac{b_1x\lambda}{1+x^2}\frac{\left[\arctan\left(b_3/\sqrt{1+(1+b_1^2)x^2}\right)+\arctan\left(b_2x/\sqrt{1+(1+b_1^2)x^2}\right)\right]}{(2\pi)^2\sqrt{1+(1+b_1^2)x^2}}\\
&+\frac{b_2x\lambda}{1+x^2}\frac{\left[\arctan\left(b_3/\sqrt{1+(1+b_2^2)x^2}\right)+\arctan\left(b_1x/\sqrt{1+(1+b_2^2)x^2}\right)\right]}{(2\pi)^2\sqrt{1+(1+b_2^2)x^2}}\\
&+\frac{b_3\lambda}{1+x^2}\frac{\left[\arctan\left(b_1x/\sqrt{1+b_3^2+x^2}\right)+\arctan\left(b_2x/\sqrt{1+b_3^2+x^2}\right)\right]}{(2\pi)^2\sqrt{1+b_3^2+x^2}}\\
&=\frac{1}{\pi(1+x^2)}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{b_1b_2}{\sqrt{1+b_1^2+b_2^2}}\right)\right)^{-1}G_1(x),\quad x\in\mathcal{R},
\end{aligned}
$$

where

$$G_1(x) = \frac{1}{2} \; + \frac{b_1 x \left[\arctan\left(b_3/\sqrt{1+(1+b_1^2)x^2}\right) + \arctan\left(b_2 x/\sqrt{1+(1+b_1^2)x^2}\right)\right]}{\pi\sqrt{1+(1+b_1^2)x^2}}$$
$$+ \frac{b_2 x \left[\arctan\left(b_3/\sqrt{1+(1+b_2^2)x^2}\right) + \arctan\left(b_1 x/\sqrt{1+(1+b_2^2)x^2}\right)\right]}{\pi\sqrt{1+(1+b_2^2)x^2}}$$
$$+ \frac{b_3 \left[\arctan\left(b_1 x/\sqrt{1+b_3^2+x^2}\right) + \arctan\left(b_2 x/\sqrt{1+b_3^2+x^2}\right)\right]}{\pi\sqrt{1+b_3^2+x^2}}.$$

Chapter 3

Identities for negative moments of quadratic forms in skew normal variables

3.1 Introduction

For non-negative random variables, it is known that negative moments are usually difficult to compute. Among others, Chao and Straederman (1972) studied the problem of finding the expected value of functions of a random variable X of the form $f(X) = (X+A)^{-n}$, where $X+A > 0$, and n is a non-negative integer. Wu et al. (2009) proved that under suitable conditions, for some special sequences of r.v.'s $\{X_n, n \geq 1\}$, $E(a + X_n)^{-\alpha} \sim (a + E(X_n))^{-\alpha}$, as $n \to \infty$, where "$\sim$"denotes asymptotically equal. On the other hand, quadratic forms of multivariate normal random variables appear in many areas of statistics, such as time series, hypothesis testing, and general linear model, etc. There are many investigations dedicated to evaluation of moments of quadratic forms, as well as moments of ratios of quadratic forms. Magnas (1986,1990) provided some numerical estimators about $E(U)^s$, $E(U^s(X'CX))$ and $E(U^s(a'X))$, where $s = 1, 2, \cdots$, $U = X'AX/X'BX$, X is $\mathcal{N}_r(\mu, \Sigma)$(the r-dimensional normal distribution with mean vector μ and correlation matrix Σ) distributed, A and C are symmetric matrices, B is a symmetric and non-negative matrix, and $a \in \mathcal{R}^r$. Mathai and Provost (1992) gave a compendium of formulas for inverse moments of quadratic forms in multivariate normal variables in terms of Lauricella's function. Gupta and Kabe (1998) provided a method to obtain the exact moments of ratios of quadratic forms $X'AX/X'X$ where A is a positive definite matrix, and X is $\mathcal{N}_r(\mu, \Sigma)$ distributed. By assuming that the ratio and its own denominator of the quadratic form $X'AX/X'BX$ are independent, where X is $\mathcal{N}_r(0, \Sigma)$ distributed, $B = aI_r$, $a > 0$, I_r is an $r \times r$ identity matrix, and A is a positive definite matrix, Conniffe and Spencer (2001) solved some hydrology problems. Paolella (2003) gave several numerical methods for computing the moments of a ratio of quadratic forms in multivariate normal variables.

Recently under the assumption of normality, by using two key lemmas of Meng (2005), Rukhin (2009) provided some formulas relating negative central moments of the quadratic form defined by a positive definite matrix to those determined by the inverse matrix. He also gave similar relationships for ratios of quadratic forms. These results can be used to check the numerical accuracy of different algorithms for evaluation of these moments(see Magnas (1986,1990), and Paolella (2003)).

In this note, under two multivariate skew normal distributions, we obtain some results parallel to Rukhin (2009).

3.2 Main results

In this section, we provide some formulas relating negative central moments of the quadratic forms defined by a positive definite matrix to those determined by the inverse matrix. According to Huang and Chen (2006), first we give a definition.

Definition 3.1. A random variable Z is said to be multivariate skew normal distributed if its probability density function (p.d.f.) has the form

$$f_Z(z) = 2\phi_r(z;\Omega)G(\alpha' z),\ z \in \mathcal{R}^r, \tag{1}$$

where $\Omega > 0$, $\alpha \in \mathcal{R}^r$, $\phi_r(z;\Omega)$ is the p.d.f. of $\mathcal{N}_r(0,\Omega)$ distribution, and $G(\cdot)$ is a skew function, that is $0 \le G(x) \le 1$, $G(x) + G(-x) = 1$, $\forall x \in \mathcal{R}$.

Throughout this section, assume Z has the p.d.f. given in (1). The moment generating function (m.g.f.) of the quadratic form $Z'AZ$ was given by Huang and Chen (2006).

Lemma 3.1. Let $Q = Z'AZ$, where A is an $r \times r$ symmetric matrix A. Then the m.g.f. of Q is given by

$$E(e^{tQ}) = |I_r - 2tA\Omega|^{-1/2},\ I_r - 2tA\Omega > 0,\ t \in \mathcal{R}.$$

By using Lemma 3.1, the following lemma can be obtained immediately.

Lemma 3.2. Let $Q_1 = Z'AZ$, and $Q_2 = Z'BZ$, where A and B are $r \times r$ symmetric matrices. Then the joint m.g.f. of Q_1 and Q_2 is given by

$$E(e^{tQ_1+sQ_2}) = |I_r - (2tA + 2sB)\Omega|^{-1/2},\ I_r - (2tA + 2sB)\Omega > 0,\ t, s \in \mathcal{R}.$$

We also need the following two lemmas by Meng (2005). Let

$$M_{X,Y}(t_1,t_2) = E\left(e^{t_1X+t_2Y}\right),\ t_1,t_2 \in \mathcal{R},$$

denote the joint m.g.f. of X and Y.

Lemma 3.3. Suppose a is a non-negative integer and $b > 0$, $P(Y > 0) = 1$, $M_{X,Y}(t_1, 0)$ exists in a neighborhood of $t_1 = 0$, and X^a/Y^b is quasi-integrable with respect to P, where P is the probability measure of (X, Y). Then $M_{X,Y}^{(a,0)}(0, -t_2)t_2^{b-1}$ is quasi-integrable with respect to $\mathcal{R}^+$ and the identity

$$E\left(\frac{X^a}{Y^b}\right) = \frac{1}{\Gamma(b)}\int_0^\infty t^{b-1}M_{X,Y}^{(a,0)}(0, -t)dt = \frac{1}{\Gamma(b)}\int_0^\infty t^{b-1}E(X^a exp(-tY))dt$$

holds, where the values $\pm\infty$ are allowed.

Lemma 3.4. Suppose a is a positive noninteger with $a = [a] + (a)$, where $[a]$ is the largest integer not exceeding a, $(a) = a - [a]$, and $b > 0$ and $P(X \geq 0, Y > 0) = 1$. Then

$$\begin{aligned} E\left(\frac{X^a}{Y^b}\right) &= \frac{1}{\Gamma((a))\Gamma(b)}\int_0^\infty\int_0^\infty t_1^{(a)-1}t_2^{b-1}M_{X,Y}^{([a],0)}(-t_1, -t_2)dt_1dt_2 \\ &= \frac{1}{\Gamma((a))\Gamma(b)}\int_0^\infty\int_0^\infty t_1^{(a)-1}t_2^{b-1}E(X^{[a]}exp(-t_1X - t_2Y))dt_1dt_2, \end{aligned}$$

and one side is finite if and only if the other side is.

When $G(x) = 1/2,\ x \in \mathcal{R}$, that is Z is $\mathcal{N}_r(0, \Omega)$ distributed, Theorems 2.1 and 2.2 of Rukhin (2009) gave formulas for $E(Z'AZ)^{-q}$ and $E((Z'BZ)^p/(Z'AZ)^q)$, respectively. As Lemmas 3.1 and 3.2 indicate that the distribution of $Z'AZ$, as well as the joint distribution of $Z'AZ$ and $Z'BZ$, both are independent of G, consequently, the results of Rukhin (2009) also hold for our multivariate skew normal distribution. We state the results as Theorems 3.1 and 3.2 below.

Theorem 3.1. Let A be an $r \times r$ positive definite matrix. Then if $0 < q < r/2$,

$$E(Z'AZ)^{-q} = \frac{\Gamma(r/2 - q)}{2^{2q-r/2}\Gamma(q)|A\Omega|^{1/2}}E(Z'CZ)^{q-r/2},$$

where $C = \Omega^{-1/2}A^{-1/2}\Omega^{-1}A^{-1/2}\Omega^{-1/2}$.

The following are some immediate consequences.

Corollary 3.1. If $\Omega = I_r$, then

$$E(Z'AZ)^{-q} = \frac{\Gamma(r/2 - q)}{2^{2q-r/2}\Gamma(q)|A|^{1/2}}E(Z'A^{-1}Z)^{q-r/2}.$$

Corollary 3.2. If $A = \Omega^{-1}$, then

$$\frac{E(Z'\Omega^{-1}Z)^{-q}}{E(Z'\Omega^{-1}Z)^{q-r/2}} = \frac{\Gamma(r/2 - q)}{2^{2q-r/2}\Gamma(q)}.$$

Corollary 3.3. If $q = r/4$, then

$$\frac{E(Z'AZ)^{-r/4}}{E(Z'CZ)^{-r/4}} = |A\Omega|^{-1/2}.$$

Corollary 3.4.

$$\frac{E(Z'AZ)^{-q}}{E(Z'Z)^{-q}} = |A|^{-1/2}\frac{E(Z'CZ)^{q-r/2}}{E(Z'\Omega^{-1}\Omega^{-1}Z)^{q-r/2}} \xrightarrow[q \to r/2]{} |A|^{-1/2}.$$

The next theorem gives the expectation of the ratio of powers of two quadratic forms.

Theorem 3.2. Let A be an $r\times r$ positive definite matrix, B an $r\times r$ symmetric and non-negative definite matrix, also assume $0 < q < p + r/2$ with $p \geq 0$. Then

$$E\frac{(Z'BZ)^p}{(Z'AZ)^q} = \frac{\Gamma(r/2+p-q)}{2^{2q-p-r/2}\Gamma(q)|A\Omega|^{1/2}} E\frac{(Z'\Omega^{-1/2}A^{-1/2}BA^{-1/2}\Omega^{-1/2}Z)^p}{(Z'\Omega^{-1/2}A^{-1/2}\Omega^{-1}A^{-1/2}\Omega^{-1/2}Z)^{p-q+r/2}}.$$

Corollary 3.5. If $\Omega = I_r$, then

$$E\frac{(Z'BZ)^p}{(Z'AZ)^q} = \frac{\Gamma(r/2+p-q)}{2^{2q-p-r/2}\Gamma(q)|A|^{1/2}} E\frac{(Z'A^{-1/2}BA^{-1/2}Z)^p}{(Z'A^{-1}Z)^{p-q+r/2}}.$$

Corollary 3.6. If $A = \Omega^{-1}$, then

$$E\frac{(Z'BZ)^p}{(Z'AZ)^q} = \frac{\Gamma(r/2+p-q)}{2^{2q-p-r/2}\Gamma(q)} E\frac{(Z'BZ)^p}{(Z'\Omega^{-1}Z)^{p-q+r/2}}.$$

Corollary 3.7. If $q = r/4 + p/2$, then

$$|A\Omega|^{1/2} E\frac{(Z'BZ)^p}{(Z'AZ)^{r/4+p/2}} = E\frac{(Z'\Omega^{-1/2}A^{-1/2}BA^{-1/2}\Omega^{-1/2}Z)^p}{(Z'\Omega^{-1/2}A^{-1/2}\Omega^{-1}A^{-1/2}\Omega^{-1/2}Z)^{r/4+p/2}}.$$

As mentioned it before, Gupta and Kabe (1998) gave a method to obtain the exact moments of ratios of quadratic forms $X'AX/X'X$, where X is $\mathcal{N}_r(\mu, \Sigma)$ distributed and A is an $r \times r$ positive definite matrix. That is

$$E\left(\frac{X'AX}{X'X}\right)^p = \frac{R(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}; \lambda_1, \cdots, \lambda_r)}{R(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}; \theta_1, \cdots, \theta_r)},$$

where $\lambda_1, \cdots, \lambda_r$ are roots of ΩA, and $\theta_1, \cdots, \theta_r$ are roots of Ω, and

$$R\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}; \lambda_1, \cdots, \lambda_r\right) = \int \left(\sum_{i=1}^{r} \lambda_i y_i\right)^p \pi^{-r/2}(y_1 y_2 \cdots y_r)^{-1/2} dy_1 \cdots dy_{r-1}, \quad \sum_{i=1}^{r} y_i = 1.$$

The following special case of Theorem 3.2 can be compared to Gupta and Kabe (1998).

Corollary 3.8. If $A = I_r$ and $p = q$, then

$$E\left(\frac{Z'BZ}{Z'Z}\right)^p = \frac{\Gamma(r/2)}{2^{p-r/2}\Gamma(p)|\Omega|^{1/2}} E\frac{(Z'\Omega^{-1/2}B\Omega^{-1/2}Z)^p}{(Z'\Omega^{-2}Z)^{r/2}}.$$

Let B_1 and B_2 be two $r \times r$ symmetric matrices, Huang and Chen (2006) proved that $Z'B_1Z$ and $Z'B_2Z$ are independent if and only if $B_1\Omega B_2 = 0$. Here both A and Ω are nonsingular, hence $A\Omega B = 0$ implies $B = 0$. Consequently, for nonzero B, the two quadratic forms $Z'AZ$ and $Z'BZ$ are not independent.

3.3 An extension

The following definition of a more general multivariate skew normal distribution is due to Wang, et al. (2009).

Definition 3.2. Let the p.d.f. of X be given by

$$f_X(x) = 2\phi_r(x; I_r)\Phi(\alpha' x), \ x \in \mathcal{R}^r, \tag{2}$$

where $\alpha \in \mathcal{R}^r$, and $\Phi(\cdot)$ is the standard normal cumulative distribution function(c.d.f.). Then $Y = \mu + \beta' X$ is said to be $\mathcal{SN}_r(\mu, \beta, \alpha)$ distributed, where $\mu \in \mathcal{R}^n$ is the location parameter, the $r \times n$ matrix β is the scale parameter and α is the shape parameter.

Wang, et al. (2009) gave some properties of the $\mathcal{SN}_r(\mu, \beta, \alpha)$ distribution. In the following, we consider a more general model with $\mu = 0$. More precisely, let $V = \beta' U$, where the p.d.f. of U is given by

$$f_U(u) = 2\phi_r(u; I_r)G(\alpha' u), \ u \in \mathcal{R}^r,$$

where again $G(\cdot)$ is a skew function. Note that if β is non-singular, the p.d.f. of V is identical to the p.d.f. of Z given in Definition 3.1 with $\beta'\beta = \Omega$. Results similar to Section 2 are given in the following.

Lemma 3.5. Let $Q = V'AV$, where A is an $n \times n$ symmetric matrix. Then the m.g.f. of Q is given by

$$E(e^{tQ}) = |I_r - 2t\beta A\beta'|^{-1/2}, \ I_r - 2t\beta A\beta' > 0, \ t \in \mathcal{R}.$$

Proof.

$$\begin{aligned}
&E(e^{tQ}) = E(exp(t(U'\beta)A(\beta' U)))dU \\
&= \frac{2}{(2\pi)^{r/2}} \int_{\mathcal{R}^r} exp\left(t\left(U'\beta A\beta' U\right) - \frac{U'U}{2}\right) G(\alpha' U)dU \\
&= 2\int_{\mathcal{R}^r} (2\pi)^{-r/2} exp\left(-\frac{1}{2}U'(I_r - 2t\beta A\beta')U\right) G(\alpha' U)dU \\
&= 2|I_r - 2t\beta A\beta'|^{-1/2} E_U(G(\alpha'(I_r - 2t\beta A\beta')^{-1/2}U)) = |I_r - 2t\beta A\beta'|^{-1/2}.
\end{aligned}$$

For the present multivariate skew normal distribution, we have the following theorems.

Theorem 3.3. Let A be an $n \times n$ positive definite matrix, and β a full row rank matrix. Then if $0 < q < r/2$,

$$E(V'AV)^{-q} = \frac{\Gamma(r/2-q)}{2^{2q-r/2}\Gamma(q)|\beta A\beta'|^{1/2}} E(V'DV)^{q-r/2},$$

where $D = \beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1}\beta$.

Proof. The theorem will be proved upon noting

$$2^q\Gamma(q)E(V'AV)^{-q} = \int_0^\infty t^{q-1}E\left(e^{-tV'AV/2}\right)dt = \int_0^\infty \frac{t^{q-1}}{|I_r + t\beta A\beta'|^{1/2}}dt$$
$$= |\beta A\beta'|^{-1/2}\int_0^\infty u^{r/2-q+1}E\left(e^{-uV'DV/2}\right)du = \frac{\Gamma(r/2-q)}{2^{q-r/2}|\beta A\beta'|^{1/2}}E(V'DV)^{q-r/2},$$

where Lemmas 3.3 and 3.5 are used in order to obtain the first and second equations, respectively.

The next theorem concerns the ratio of powers of two quadratic forms.

Theorem 3.4. Let A be an $n \times n$ positive definite matrix, B an $n \times n$ symmetric and non-negative definite matrix, and β a full row rank matrix. Also assume $0 < q < p + r/2$ with $p \geq 0$. Then

$$E\frac{(V'BV)^p}{(V'AV)^q} = \frac{\Gamma(r/2+p-q)}{2^{2q-p-r/2}\Gamma(q)|\beta A\beta'|^{1/2}} E\frac{(V'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2}(\beta\beta')^{-1}\beta V)^p}{(V'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1}\beta V)^{p-q+r/2}}.$$

Proof. First let p be a non-negative integer. Then

$$2^q\Gamma(q)E\frac{(V'BV)^p}{(V'AV)^q} = \int_0^\infty t^{q-1}E\left((V'BV)^p e^{-tV'AV/2}\right)dt$$
$$= (-2)^p\int_0^\infty t^{q-1}\frac{d^p}{ds^p}E\left(e^{-V'(sB+tA)V/2}\right)\Big|_{s=0}dt$$
$$= (-2)^p\int_0^\infty t^{q-1}\frac{d^p}{ds^p}\frac{1}{|I_r + s\beta B\beta' + t\beta A\beta'|^{1/2}}\Big|_{s=0}dt$$
$$= \frac{(-2)^p}{|\beta A\beta'|^{1/2}}\int_0^\infty u^{r/2-q-1}\frac{d^p}{ds^p}\frac{1}{|I_r + su(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2} + u(\beta A\beta')^{-1}|^{1/2}}\Big|_{s=0}du$$
$$= \frac{(-2)^p}{|\beta A\beta'|^{1/2}}\int_0^\infty u^{r/2-q-1}$$
$$\cdot\frac{d^p}{ds^p}E\left(e^{-V'(su\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2}(\beta\beta')^{-1}\beta + u\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1})\beta V/2}\right)\Big|_{s=0}du$$
$$= \frac{1}{|\beta A\beta'|^{1/2}}\int_0^\infty u^{r/2+p-q-1}E\Big((V'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2}(\beta\beta')^{-1}\beta V)^p$$
$$\cdot e^{-uV'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1}\beta V/2}\Big)du$$
$$= \frac{\Gamma(r/2+p-q)}{2^{q-p-r/2}|\beta A\beta'|^{1/2}}E\frac{(V'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2}(\beta\beta')^{-1}\beta V)^p}{(V'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1}\beta V)^{p-q+r/2}},$$

where Lemmas 3.3 and 3.5 are used to obtain the first and third equations, respectively.

Next consider the case p is not an integer. Again we have

$$
\begin{aligned}
&2^q\Gamma((p))\Gamma(q)E\frac{(V'BV)^p}{(V'AV)^q} = \frac{1}{2^{(p)}}\int_0^\infty\int_0^\infty s^{(p)-1}t^{q-1}E\left((V'BV)^{[p]}e^{-sV'BV/2-tV'AV/2}\right)dsdt \\
&= \frac{(-2)^{[p]}}{2^{(p)}}\int_0^\infty\int_0^\infty s^{(p)-1}t^{q-1}\frac{d^{[p]}}{ds^{[p]}}E\left(e^{-V'(sB+tA)V/2}\right)dsdt \\
&= \frac{(-2)^{[p]}}{2^{(p)}}\int_0^\infty\int_0^\infty s^{(p)-1}t^{q-1}\frac{d^{[p]}}{ds^{[p]}}\frac{1}{|I_r+s\beta B\beta'+t\beta A\beta'|^{1/2}}dsdt \\
&= \frac{(-2)^{[p]}}{2^{(p)}|\beta A\beta'|^{1/2}}\int_0^\infty\int_0^\infty s^{(p)-1}u^{r/2-q-1} \\
&\quad\cdot\frac{d^{[p]}}{ds^{[p]}}E\left(e^{-V'(u\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1}\beta+su\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2})(\beta\beta')^{-1}\beta V/2}\right)dsdu \\
&= \frac{(-2)^{[p]}}{2^{(p)}|\beta A\beta'|^{1/2}}\int_0^\infty\int_0^\infty m^{(p)-1}u^{r/2+p-q-1} \\
&\quad\cdot\frac{d^{[p]}}{dm^{[p]}}E\left(e^{-V'(u\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1}\beta+m\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2})(\beta\beta')^{-1}\beta V/2}\right)dmdu \\
&= \frac{\Gamma((p))\Gamma(r/2+p-q)}{2^{q-p-r/2}|\beta A\beta'|^{1/2}}E\frac{(V'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1/2}\beta B\beta'(\beta A\beta')^{-1/2}(\beta\beta')^{-1}\beta V)^p}{(V'\beta'(\beta\beta')^{-1}(\beta A\beta')^{-1}(\beta\beta')^{-1}\beta V)^{p-q+r/2}}.
\end{aligned}
$$

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