# A Study of the Skew-Symmetric Models 

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# 偏斜對稱模型的研究 

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## 摘要

自從 Azzalini $(1985,1986)$ 提出關於偏斜常態分佈的一些基本的性質開始，就有許多基於某些常見對稱分佈的偏斜分佈之研究出現。這類偏斜－對稱分佈不僅包含原本的對稱分佈，往往也有一些與原本對稱分佈相同的性質。

在本論文裡，我們考慮雨個有關偏斜－對稱分佈的主題。在第一章中，我們研究多變量偏斜常態－對稱分佈的二次形式。基於 Gupta and Chang（2003）這篇論文，我們推導出多變量偏斜常態－對稱分佈，即考慮機率密度函數具有下列形式 $f_{\boldsymbol{Z}}(\boldsymbol{z})=2 \phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega}) G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right)$ ，其中 $\boldsymbol{\Omega}>0, \boldsymbol{\alpha} \in \mathbb{R}^{p}, \phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega})$ 為一p維常態分佈的機率密度函数，期望向量為 $\mathbf{0}$ 且相關係數矩陣為 $\Omega>0$ ，而 $G$ 為霂足 $G^{\prime}$ 對稱於 0 的絕對連續分佈函數。我們先推導出某些二次形式的動差母函數，並發現一些二次形式的分佈與 $G$ 獨立。其次我們推尊出一個線性組合與一個二次形式，及兩個二次形式的聯合動差母函數，並且給出一些它們相互獨立的條件。最後我們將 $G$ 分別取為常態，隻指數，logistic及均匀分佈的分佈函數，來看其各自的一些特別的二次形式之分佈。

在第二章中，我們研究的是單變量的廣義偏斜－柯西分佈。我們先推導廣義偏斜－對稱分佈。假設 $Y$ 是一絕對連續且對稱於 0 的隨機變數，其機率密度函數為 $f$ ，而分佈函數為 $F$ 。若一隨機變數 $X$ 満足 $X^{2} \stackrel{d}{=} Y^{2}$ ，則 $X$ 被稱為一 $F($ or $f)$ 之廣義偏斜分佈。接下來我們考慮廣義偏斜－柯西分佈，並給一些此分佈之特別的例子。其中有些例子是經由廣義偏斜－常態分佈或廣義偏斜－$t$ 分佈所產生的。

關鍵字：柯西分佈，獨立性，動差母函數，非常態模型，二次形式，偏斜柯西分佈，偏斜常態分佈，偏斜對稱分佈，偏斜 $t$ 分佈

# A Study of the Skew-Symmetric Models 

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#### Abstract

Since Azzalini $(1985,1986)$ introduced the fundamental properties of the skew-normal distribution, there are many investigations about the skew distributions based on certain symmetric probability density functions. These classes of the skew-symmetric distributions include the original symmetric distribution and have some properties like the original one and yet is skew.

In this thesis, we consider two topics of the skew-symmetric models. In Chapter 1, we study the quadratic forms of multivariate skew normal-symmetric distributions. Following the paper by Gupta and Chang (2003) we generalize a multivariate skew normal-symmetric distribution with p.d.f. of the form $f_{\boldsymbol{Z}}(\boldsymbol{z})=2 \phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega}) G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right)$, where $\boldsymbol{\Omega}>0, \boldsymbol{\alpha} \in \mathbb{R}^{p}$, $\phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega})$ is the $p$-dimensional normal p.d.f. with zero mean vector and correlation matrix $\boldsymbol{\Omega}$, and $G$ is taken to be an absolutely continuous distribution function such that $G^{\prime}$ is symmetric about 0 . First we obtain the moment generating function of certain quadratic forms. It is interesting to find that the distributions of some quadratic forms are independent of $G$. Then the joint moment generating functions of a linear compound and a quadratic form, and two quadratic forms, and conditions for their independence are given. Finally we take $G$ to be one of normal, Laplace, logistic or uniform distribution, and determine the distribution of a special quadratic form for each case.

In Chapter 2, we study the generalized skew-Cauchy distributions. We investigate the generalized skew-symmetric distributions. Suppose $Y$ is an absolutely continuous random variable symmetric about 0 with probability density function $f$ and cumulative distribution function $F$. If a random variable $X$ satisfies $X^{2} \stackrel{d}{=} Y^{2}$, then $X$ is said to have a generalized skew distribution of $F$ (or $f$ ). The generalized skew-Cauchy (GSC) distribution are considered and special examples of GSC distribution are presented. Some of these examples are generated from generalized skew-normal or generalized skew- $t$ distributions.


Keywords: Chi-square distribution, independence, moment generating function, non-normal models, quadratic form, Skew-Cauchy distribution, skew-normal distribution, skew-symmetric distribution, skew- $t$ distribution.

## Chapter 1

## Quadratic Forms of Multivariate Skew Normal-Symmetric Distributions

### 1.1 Introduction

The univariate skew normal distribution was introduced by Azzalini (1985) and (1986), and Gupta et al. (2004b), and its multivariate version by Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Gupta and Kollo (2000), and Gupta et al. (2004a). These classes of distributions include the normal distribution and have some properties like the normal and yet is skew. They are useful in studying robustness. Here a $p$-dimensional random vector $\boldsymbol{Z}$ is said to have a multivariate skew normal distribution if it is continuous and its probability density function (p.d.f.) is given by

$$
\begin{equation*}
f_{\boldsymbol{Z}}(\boldsymbol{z})=2 \phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega}) \Phi\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right) \tag{1.1}
\end{equation*}
$$

where $\boldsymbol{\Omega}>0, \boldsymbol{\alpha} \in \mathbb{R}^{p}, \phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega})$ is the p.d.f. of $N_{p}(\mathbf{0}, \boldsymbol{\Omega})$ distribution (the $p$-dimensional normal distribution with zero mean vector and correlation matrix $\boldsymbol{\Omega}$ ), and $\Phi(\cdot)$ is the standard normal cumulative distribution function (c.d.f.). It is denoted by $\boldsymbol{Z} \sim S N_{p}(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, to mean that the random vector $\boldsymbol{Z}$ has $p$-variate skew normal p.d.f. (1.1). Quadratic forms of skew normal random vectors have been studied by many authors, including Azzalini (1985), Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Loperfido (2001), Genton et al. (2001), and Gupta and Huang (2002).

The classes of univariate symmetric p.d.f.s which depend on a skewness parameter have also been studied by Gupta et al. (2002), and Arellano-Valle et al. (2004). In particular the skew normal, uniform, Student's $t$, Cauchy, Laplace, and logistic distributions are given and some of their properties are explored. The multivariate skew-Cauchy distribution and multivariate skew $t$-distribution are studied by Arnold and Beaver (2000), and Gupta (2003), respectively. Following Gupta et al. (2002), Nadarajah and Kotz (2003) studied univariate
skewed distributions generated by the normal kernel. More precisely, they generated skew p.d.f.s of the form $2 f(u) G(\lambda u)$, where $f$ is taken to be a normal p.d.f. with zero mean, while the c.d.f. $G$ is taken to come from one of normal, Student's $t$, Cauchy, Laplace, logistic or uniform distribution. A class of multivariate skew distributions has been introduced by Gupta and Chang (2003).

Following Gupta and Chang (2003), the general form of multivariate skew-symmetric distribution is given by the following lemma.

Lemma 1.1. Let $f$ be a p.d.f. of a $p$-dimensional random vector symmetric about $\mathbf{0}$, and $G$ an absolutely continuous distribution function such that $G^{\prime}$ is symmetric about 0 . Then

$$
\begin{equation*}
f_{\boldsymbol{Z}}(\boldsymbol{z} ; \boldsymbol{\alpha})=2 f(\boldsymbol{z}) G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right), \quad \boldsymbol{z} \in \mathbb{R}^{p} \tag{1.2}
\end{equation*}
$$

is a p.d.f. of a random $p$-vector $\boldsymbol{Z}$ for any $\boldsymbol{\alpha} \in \mathbb{R}^{p}$.

Note that in Lemma 1.1, for $f_{\boldsymbol{Z}}(\boldsymbol{z})$ being a p.d.f., the condition for $G$ may not be needed. For some $k>0,0 \leq G(x) \leq k, G^{\prime}(x)$ exists, and $G(x)+G(-x)=1, \forall x \in \mathbb{R}$, is enough. For example, if $G(x)=1 / 2, \forall x \in \mathbb{R}$, then $f_{\boldsymbol{Z}}(\boldsymbol{z} ; \boldsymbol{\alpha})=f(\boldsymbol{z}), \boldsymbol{z} \in \mathbb{R}^{p}$, is a p.d.f.. Also the multivariate skew normal distribution is one special case of the general form, which is obtained by taking $f \equiv \phi_{p}$ and $G \equiv \Phi$ in (1.2).

In this paper, we consider a class of multivariate skew-symmetric distributions generated by the normal kernel. We only take $f \equiv \phi_{p}$ in (1.2), and let $G$ be defined as in Lemma 1.1. Namely, we say $\boldsymbol{Z}$ has a multivariate skew normal-symmetric distribution, if the p.d.f. of $\boldsymbol{Z}$ is given by

$$
\begin{equation*}
f_{\boldsymbol{Z}}(\boldsymbol{z})=2 \phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega}) G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right), \quad \boldsymbol{z} \in \mathbb{R}^{p} \tag{1.3}
\end{equation*}
$$

for some $\boldsymbol{\alpha} \in \mathbb{R}^{p}$.
Gupta and Huang (2002) have studied quadratic forms of multivariate skew normal-normal model. That is they consider the case $G \equiv \Phi$ in (1.3). In this paper, we will obtain some parallel results for the class of multivariate skew normal-symmetric distributions. In Section 2, we discuss the moment generating function (m.g.f.) of the quadratic form, $\boldsymbol{Q}=(\boldsymbol{Z}-\boldsymbol{a})^{\prime} A(\boldsymbol{Z}-\boldsymbol{a})$, where $\boldsymbol{a} \in \mathbb{R}^{p}, \boldsymbol{A}$ is a $p \times p$ symmetric matrix, and $\boldsymbol{Z}$ has a multivariate skew normal-symmetric distribution with p.d.f. given in (1.3). In Section 3, the independence of a linear compound and a quadratic form, and two quadratic forms are studied. Then in the following sections, we discuss some skewed models generated by a normal kernel: the multivariate skew normal-normal model (Section 4), the multivariate skew normal-Laplace model (Section 5), the multivariate skew normal-logistic model (Section 6) and the multivariate skew normal-uniform model (Section 7). The reason that we do not consider the normal- $t$ and normal-Cauchy models, which are also studied in Nadarajah and Kotz (2003), is that the closed form of the m.g.f. of $\boldsymbol{Q}$ cannot be obtained in either case.

Note that when $\boldsymbol{Z}$ has the p.d.f. (1.3), the m.g.f. of $\boldsymbol{Z}$ is

$$
\begin{equation*}
M_{\boldsymbol{Z}}(\boldsymbol{t})=E\left(e^{t^{\prime} \boldsymbol{Z}}\right)=2 \exp \left\{\frac{1}{2} \boldsymbol{t}^{\prime} \boldsymbol{\Omega} \boldsymbol{t}\right\} E_{\boldsymbol{U}}\left[G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{U}+\boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{t}\right)\right], \quad \boldsymbol{t} \in \mathbb{R}^{p} \tag{1.4}
\end{equation*}
$$

and the m.g.f. of the linear form $\boldsymbol{h}^{\prime} \boldsymbol{Z}, \boldsymbol{h} \in \mathbb{R}^{p}$, is

$$
\begin{equation*}
M_{\boldsymbol{h}^{\prime} \boldsymbol{Z}}(t)=E\left[e^{t \boldsymbol{h}^{\prime} \boldsymbol{Z}}\right]=M_{\boldsymbol{Z}}[t \boldsymbol{h}]=2 \exp \left\{\frac{1}{2} t^{2} \boldsymbol{h}^{\prime} \boldsymbol{\Omega} \boldsymbol{h}\right\} E_{\boldsymbol{U}}\left[G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{U}+t \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{h}\right)\right], \quad t \in \mathbb{R} \tag{1.5}
\end{equation*}
$$

where $\boldsymbol{U} \sim N_{p}(\mathbf{0}, \boldsymbol{\Omega})$.
Throughout the rest of this paper, let $\boldsymbol{a} \in \mathbb{R}^{p}, \boldsymbol{A}^{\prime}=\boldsymbol{A}$, a $p \times p$ matrix.

### 1.2 Moment generating functions of certain quadratic forms

In this section, let $\boldsymbol{Z}$ be a multivariate skew normal-symmetric distribution with p.d.f. given in (1.3). First we derive the m.g.f. of the quadratic form $\boldsymbol{Q}=(\boldsymbol{Z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{Z}-\boldsymbol{a})$.

Theorem 2.1. The m.g.f. of $\boldsymbol{Q}$ is given by

$$
\begin{align*}
M_{\boldsymbol{Q}^{( }}(t)= & \frac{2 \exp \left\{\boldsymbol{a}^{\prime}\left[t \boldsymbol{A}+2 t^{2} \boldsymbol{A}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A}\right] \boldsymbol{a}\right\}}{|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{1 / 2}} \\
& \times E_{\boldsymbol{U}_{1}}\left[G\left(-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}+\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1 / 2} \boldsymbol{U}_{1}\right)\right], \quad t \in \mathbb{R} \tag{1.6}
\end{align*}
$$

where $\boldsymbol{U}_{1} \sim N_{p}(\mathbf{0}, \boldsymbol{I})$.

Proof. For $t \in \mathbb{R}$, the m.g.f. of $\boldsymbol{Q}$ is

$$
\begin{aligned}
M_{\boldsymbol{Q}}(t)= & E\left(e^{t \boldsymbol{Q}}\right) \\
= & 2 \int_{\mathbb{R}^{p}} \exp \left\{t(\boldsymbol{z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{z}-\boldsymbol{a})\right\} \phi_{p}(\boldsymbol{z} ; \boldsymbol{\Omega}) G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right) d \boldsymbol{z} \\
= & 2 \int_{\mathbb{R}^{p}} \exp \left\{-\frac{1}{2}\left(\boldsymbol{z}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{z}-2 t(\boldsymbol{z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{z}-\boldsymbol{a})\right)\right\} G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right) d \boldsymbol{z} \\
= & \frac{2 \exp \left\{\boldsymbol{a}^{\prime}\left[t \boldsymbol{A}+2 t^{2} \boldsymbol{A}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A}\right] \boldsymbol{a}\right\}}{|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{1 / 2}} \\
& \times E_{\boldsymbol{U}_{1}}\left[G\left(-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}+\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1 / 2} \boldsymbol{U}_{1}\right)\right], \quad t \in \mathbb{R} .
\end{aligned}
$$

We also need the following two preliminary lemmas.

Lemma 2.2. Let $X$ be a normally distributed random variable with mean zero. Let $F(y)$ be a continuous function which satisfies $F(y)+F(-y)=1, y \in \mathbb{R}$. Then

$$
E[F(X)]=\frac{1}{2}
$$

Lemma 2.3. If $\boldsymbol{X} \sim N_{n}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{C}$ is an $n \times m$ constant matrix, $m \leq n$, and $\boldsymbol{V}$ is an $m \times 1$ constant vector, then $\boldsymbol{Y}=\boldsymbol{C}^{\prime} \boldsymbol{X}+\boldsymbol{V} \sim N_{m}\left(\boldsymbol{C}^{\prime} \boldsymbol{\mu}+\boldsymbol{V}, \boldsymbol{C}^{\prime} \boldsymbol{\Sigma} \boldsymbol{C}\right)$. Specially, if $m=1$, i.e. $\boldsymbol{C}=\boldsymbol{c}$ is an $n \times 1$ column vector, and let $\boldsymbol{V}=\mathbf{0}$, then $\boldsymbol{Y}=\boldsymbol{c}^{\prime} \boldsymbol{X} \sim N\left(\boldsymbol{c}^{\prime} \boldsymbol{\mu}, \boldsymbol{c}^{\prime} \boldsymbol{\Sigma} \boldsymbol{c}\right)$.

By using the above results, the following corollary gives the distribution of four special quadratic forms.

Corollary 2.4. Let $M_{\boldsymbol{Q}_{i}}(t)=E\left(e^{t} \boldsymbol{Q}_{i}\right), t \in \mathbb{R}, i=1,2,3,4$.
(i). Let $\boldsymbol{Q}_{1}=\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z}$, where $\boldsymbol{A} \boldsymbol{\Omega}=\operatorname{diag}\left(\delta_{1}, \cdots, \delta_{p}\right)$. Then $\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z} \backsim \sum_{j=1}^{p} \delta_{j} X_{j}$, where $X_{j} \backsim$ $\chi_{1}^{2}, j=1, \cdots, p$, are independent and identically distributed (i.i.d.).
(ii). Let $\boldsymbol{Q}_{2}=\boldsymbol{Z}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{Z}$. Then $\boldsymbol{Z}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{Z} \backsim \chi_{p}^{2}$.
(iii). Let $\boldsymbol{Q}_{3}=(\boldsymbol{Z}-\boldsymbol{a})^{\prime} \boldsymbol{\Omega}^{-1}(\boldsymbol{Z}-\boldsymbol{a})$. Then

$$
M_{\boldsymbol{Q}_{3}}(t)=\frac{2 \exp \left\{\frac{t}{1-2 t} \boldsymbol{a}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{a}\right\}}{(1-2 t)^{p / 2}} \times E_{\boldsymbol{U}_{1}}\left[G\left(\frac{-2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}+\frac{1}{(1-2 t)^{1 / 2}} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega}^{1 / 2} \boldsymbol{U}_{1}\right)\right], t \in \mathbb{R}
$$

(iv). Let $\boldsymbol{Q}_{4}=\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z}$. Then

$$
M_{\boldsymbol{Q}_{4}}(t)=|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{-1 / 2}, \boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}>0, t \in \mathbb{R}
$$

## Proof.

(i). Substituting $\boldsymbol{a}=\mathbf{0}$, and $\boldsymbol{A} \boldsymbol{\Omega}=\operatorname{diag}\left(\delta_{1}, \cdots, \delta_{p}\right)$ in (1.6), we obtain the m.g.f. of $\boldsymbol{Q}_{1}$ :

$$
\begin{aligned}
M_{\boldsymbol{Q}_{1}}(t) & =\frac{2}{|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{1 / 2}} \times E_{\boldsymbol{U}_{1}}\left[G\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1 / 2} \boldsymbol{U}_{1}\right)\right] \\
& =\left[\prod_{j=1}^{p}\left(1-2 t \delta_{j}\right)\right]^{-1 / 2}=\prod_{j=1}^{p}\left(1-2 t \delta_{j}\right)^{-1 / 2}, t \in \mathbb{R}
\end{aligned}
$$

where $E_{\boldsymbol{U}_{1}}\left[G\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1 / 2} \boldsymbol{U}_{1}\right)\right]=1 / 2$ is obtained by using Lemma 2.2.
(ii). By case (i), substituting $\boldsymbol{A}=\boldsymbol{\Omega}^{-1}$ such that $\boldsymbol{A} \boldsymbol{\Omega}=\boldsymbol{\Omega}^{-1} \boldsymbol{\Omega}=\boldsymbol{I}$, i.e. $\delta_{j}=1, j=1, \ldots, p$, we obtain the m.g.f. of $\boldsymbol{Q}_{2}$ :

$$
M_{\boldsymbol{Q}_{2}}(t)=\prod_{j=1}^{p}\left(1-2 t \delta_{j}\right)^{-1 / 2}=(1-2 t)^{-p / 2}, t \in \mathbb{R}
$$

(iii). Substituting $\boldsymbol{A}=\boldsymbol{\Omega}^{-1}$ in (1.6), we obtain the m.g.f. of $\boldsymbol{Q}_{3}$ :

$$
\begin{aligned}
M_{\boldsymbol{Q}_{3}}(t)= & \frac{2 \exp \left\{\boldsymbol{a}^{\prime}\left[t \boldsymbol{\Omega}^{-1}+2 t^{2} \boldsymbol{\Omega}^{-1}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{\Omega}^{-1}\right)^{-1} \boldsymbol{\Omega}^{-1}\right] \boldsymbol{a}\right\}}{\left|\boldsymbol{I}-2 t \boldsymbol{\Omega}^{-1} \boldsymbol{\Omega}\right|^{1 / 2}} \\
& \times E_{\boldsymbol{U}_{1}}\left[G\left(-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{\Omega}^{-1}\right)^{-1} \boldsymbol{\Omega}^{-1} \boldsymbol{a}+\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{\Omega}^{-1}\right)^{-1 / 2} \boldsymbol{U}_{1}\right)\right] \\
= & \frac{2 \exp \left\{\frac{t}{1-2 t} \boldsymbol{a}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{a}\right\}}{(1-2 t)^{p / 2}} \times E_{\boldsymbol{U}_{1}}\left[G\left(\frac{-2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}+\frac{1}{(1-2 t)^{1 / 2}} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega}^{1 / 2} \boldsymbol{U}_{1}\right)\right], t \in \mathbb{R} .
\end{aligned}
$$

(iv). Substituting $\boldsymbol{a}=\mathbf{0}$ in (1.6), we obtain the m.g.f. of $\boldsymbol{Q}_{4}$ :

$$
M_{\boldsymbol{Q}_{4}}(t)=|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{-1 / 2}, \boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}>0, t \in \mathbb{R}
$$

Note that $M_{\boldsymbol{Q}_{1}}(t), M_{\boldsymbol{Q}_{2}}(t)$ and $M_{\boldsymbol{Q}_{4}}(t)$ are the same as those in Gupta and Huang (2002), respectively, where multivariate skew normal-normal model is considered. In other words, the distributions of the three quadratic forms $\boldsymbol{Q}_{1}, \boldsymbol{Q}_{2}$ and $\boldsymbol{Q}_{4}$, are independent of $G$. In particular, the distribution of $\boldsymbol{Q}_{4}=\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z}$ is the same as that of the corresponding quadratic form where $\boldsymbol{Z} \sim N_{p}(\mathbf{0}, \boldsymbol{\Omega})$. Also from (iv), it is easy to obtain the following more general result than (ii) (see proposition 7 of Azzalini and Capitanio (1999)).

Corollary 2.5. Let $\boldsymbol{B}$ be a symmetric positive semidefinite $p \times p$ matrix of rank $k$ such that $\boldsymbol{B} \boldsymbol{\Omega} \boldsymbol{B}=\boldsymbol{B}$. Then $\boldsymbol{Z}^{\prime} \boldsymbol{B} \boldsymbol{Z} \sim \chi_{k}^{2}$.

### 1.3 Independence of linear forms and quadratic forms

Theorem 3.1. For $\boldsymbol{h} \in \mathbb{R}^{p}$, if $\boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{h}=0$ and $\boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{\alpha}=0$, the linear form $\boldsymbol{h}^{\prime} \boldsymbol{Z}$ and the quadratic form $\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z}$ are independent.

Proof. The joint m.g.f. of $\boldsymbol{h}^{\prime} \boldsymbol{Z}$ and $\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z}$ is

$$
\begin{aligned}
M_{1}\left(t_{1}, t_{2}\right)= & 2 \int_{\mathbb{R}^{p}} \frac{\exp \left\{-\frac{1}{2}\left[\boldsymbol{z}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{z}-2 t_{1} \boldsymbol{h}^{\prime} \boldsymbol{z}-2 t_{2} \boldsymbol{z}^{\prime} \boldsymbol{A} \boldsymbol{z}\right]\right\}}{(2 \pi)^{p / 2}|\boldsymbol{\Omega}|^{1 / 2}} G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right) d \boldsymbol{z} \\
= & \frac{2 \exp \left\{\frac{1}{2} t_{1}^{2} \boldsymbol{h}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1} \boldsymbol{h}\right\}}{\left|\boldsymbol{I}-2 t_{2} \boldsymbol{A} \boldsymbol{\Omega}\right|^{1 / 2}} \\
& \times E_{\boldsymbol{U}_{1}}\left[G\left(t_{1} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1} \boldsymbol{h}+\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1 / 2} \boldsymbol{U}_{1}\right)\right], t_{1}, t_{2} \in \mathbb{R} .(1.7)
\end{aligned}
$$

By Lemma 2.3, we have

$$
\begin{align*}
t_{1} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1} \boldsymbol{h}+\boldsymbol{\alpha}^{\prime} & \left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1 / 2} \boldsymbol{U}_{1} \\
& \sim N\left(t_{1} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1} \boldsymbol{h}, \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right) \tag{1.8}
\end{align*}
$$

Now note that

$$
\begin{equation*}
\left(\boldsymbol{\Omega}^{-1}-2 t_{2} \boldsymbol{A}\right)^{-1}=\boldsymbol{\Omega} \sum_{j=0}^{\infty}\left(2 t_{2}\right)^{j}(\boldsymbol{A} \boldsymbol{\Omega})^{j} \tag{1.9}
\end{equation*}
$$

for $\left\|2 t_{2} \boldsymbol{A} \boldsymbol{\Omega}\right\|<1$, where $\|\cdot\|$ is a matrix norm. Hence the expansion (1.9) is always valid in the neighborhood of $t_{2}=0$ (see Horn and Johnson (1996, p.301)). Finally from (1.7), (1.8) and (1.9), if $\boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{h}=0$, and $\boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{\alpha}=0$, it yields

$$
\begin{equation*}
M_{1}\left(t_{1}, t_{2}\right)=\frac{2 \exp \left\{\frac{1}{2} t_{1}^{2} \boldsymbol{h}^{\prime} \boldsymbol{\Omega} \boldsymbol{h}\right\}}{\left|\boldsymbol{I}-2 t_{2} \boldsymbol{A} \boldsymbol{\Omega}\right|^{1 / 2}} \times E_{\boldsymbol{U}_{1}}\left[G\left(t_{1} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{h}+\boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega}^{-1 / 2} \boldsymbol{U}_{1}\right)\right], t_{1}, t_{2} \in \mathbb{R} \tag{1.10}
\end{equation*}
$$

which in turn implies $\boldsymbol{h}^{\prime} \boldsymbol{Z}$ and $\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z}$ are independent.

For the normal-normal model, it is shown in Gupta and Huang (2002), $\boldsymbol{h}^{\prime} \boldsymbol{Z}$ and $\boldsymbol{Z}^{\prime} \boldsymbol{A} \boldsymbol{Z}$ are independent if and only if $\boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{h}=0$ and $\boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{\alpha}=0$. Yet for the general model, we only can show the if part. The following theorem nevertheless is in if and only if form.

Theorem 3.2. Let $\boldsymbol{B}_{1}$ and $\boldsymbol{B}_{2}$ be $p \times p$ symmetric matrices. The quadratic forms $\boldsymbol{Z}^{\prime} \boldsymbol{B}_{1} \boldsymbol{Z}$ and $\boldsymbol{Z}^{\prime} \boldsymbol{B}_{2} \boldsymbol{Z}$ are independent if and only if $\boldsymbol{B}_{1} \boldsymbol{\Omega} \boldsymbol{B}_{2}=0$.

Proof. The joint m.g.f. of $\boldsymbol{Z}^{\prime} \boldsymbol{B}_{1} \boldsymbol{Z}$ and $\boldsymbol{Z}^{\prime} \boldsymbol{B}_{2} \boldsymbol{Z}$ is

$$
\begin{align*}
M_{2}\left(t_{1}, t_{2}\right) & =2 \int_{\mathbb{R}^{p}} \frac{\exp \left\{-\frac{1}{2}\left[\boldsymbol{z}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{z}-2 t_{1} \boldsymbol{z}^{\prime} \boldsymbol{B}_{1} \boldsymbol{z}-2 t_{2} \boldsymbol{z}^{\prime} \boldsymbol{B}_{2} \boldsymbol{z}\right]\right\}}{(2 \pi)^{p / 2}|\boldsymbol{\Omega}|^{1 / 2}} G\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{z}\right) d \boldsymbol{z} \\
& =\frac{2}{|\boldsymbol{\Omega}|^{1 / 2}\left|\boldsymbol{\Omega}^{-1}-2 t_{1} \boldsymbol{B}_{1}-2 t_{2} \boldsymbol{B}_{2}\right|^{1 / 2}} E_{\boldsymbol{U}_{1}}\left(G\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t_{1} \boldsymbol{B}_{1}-2 t_{2} \boldsymbol{B}_{2}\right)^{-1 / 2} \boldsymbol{U}_{1}\right)\right) \\
& =\left|\boldsymbol{I}-2\left(t_{1} \boldsymbol{B}_{1}+t_{2} \boldsymbol{B}_{2}\right) \boldsymbol{\Omega}\right|^{-1 / 2}, t_{1}, t_{2} \in \mathbb{R} . \tag{1.11}
\end{align*}
$$

Again the last step is obtained by using Lemma 2.2. Now from (1.11) for independence we get the condition $\boldsymbol{B}_{1} \boldsymbol{\Omega} \boldsymbol{B}_{2}=0$.

Corollary 3.3. Let $\boldsymbol{B}_{1}, \cdots, \boldsymbol{B}_{n}$ be $p \times p$ symmetric matrices. The quadratic forms $\boldsymbol{Z}^{\prime} \boldsymbol{B}_{i} \boldsymbol{Z}$, $i=1, \cdots, n$, are mutually independent if and only if $\boldsymbol{B}_{i} \boldsymbol{\Omega} \boldsymbol{B}_{j}=0, i \neq j$.

By (1.11), $M_{2}\left(t_{1}, t_{2}\right)$ is independent of $\boldsymbol{\alpha}$. Hence as in Proposition 2 of Loperfido (2001), we have the following consequence.

Corollary 3.4. Let $\boldsymbol{B}_{1}, \cdots, \boldsymbol{B}_{n}$ be $p \times p$ symmetric matrices. The joint distribution of the quadratic forms $\left(\boldsymbol{Z}^{\prime} \boldsymbol{B}_{1} \boldsymbol{Z}, \cdots, \boldsymbol{Z}^{\prime} \boldsymbol{B}_{n} \boldsymbol{Z}\right), i=1, \cdots, n$, does not depend on $\boldsymbol{\alpha}$.

### 1.4 Multivariate skew normal-normal model

Let $G=\Phi$, then we obtain the multivariate skew normal-normal distribution for $\boldsymbol{Z}$. As mentioned it before, this distribution has been studied by Azzalini and Dalla Valle (1996), its applications are given in Azzalini and Capitanio (1999), and its quadratic form has been studied by Gupta and Huang (2002).

### 1.4.1 M.G.F. of $(Z-a)^{\prime} A(Z-a)$

In this section we derive the m.g.f. of the quadratic form $\boldsymbol{Q}=(\boldsymbol{Z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{Z}-\boldsymbol{a})$. For this we need the following lemma (see Zacks (1981), pp. 53-59).

Lemma 4.1. Let $\boldsymbol{U} \sim N_{p}(\mathbf{0}, \boldsymbol{\Omega})$. Then, for any scalar $u$ and $\boldsymbol{v} \in \mathbb{R}^{p}$, we have

$$
\begin{equation*}
E\left[\Phi\left(u+\boldsymbol{v}^{\prime} \boldsymbol{U}\right)\right]=\Phi\left\{\frac{u}{\left(1+\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right\} . \tag{1.12}
\end{equation*}
$$

Using the above lemma, we obtain the following theorem (Theorem 2 of Gupta and Huang (2002)).

Theorem 4.2. The m.g.f. of $\boldsymbol{Q}$ is given by

$$
\begin{align*}
M_{\boldsymbol{Q}}(t)= & \frac{2 \exp \left\{\boldsymbol{a}^{\prime}\left[t \boldsymbol{A}+2 t^{2} \boldsymbol{A}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A}\right] \boldsymbol{a}\right\}}{|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{1 / 2}} \\
& \times \Phi\left[-\frac{2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}}{\left(1+\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right], \quad t \in \mathbb{R} . \tag{1.13}
\end{align*}
$$

### 1.4.1.1. Special case

Case (iii). The m.g.f. of $Q_{3}$ is

$$
M_{\boldsymbol{Q}_{3}}(t)=\frac{2 \exp \left\{\frac{t}{1-2 t} \boldsymbol{a}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{a}\right\}}{(1-2 t)^{p / 2}} \times \Phi\left[\frac{-2 t}{1-2 t} \frac{\boldsymbol{\alpha}^{\prime} \boldsymbol{a}}{\left(1+\boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha} /(1-2 t)\right)^{1 / 2}}\right], t \in \mathbb{R}
$$

### 1.5 Multivariate skew normal-Laplace model

Let $G=G_{2}$ be the c.d.f. of a Laplace distribution, namely

$$
G_{2}(x)= \begin{cases}1-\frac{1}{2} \exp (-x) & , x \geq 0 \\ \frac{1}{2} \exp (x) & , x<0\end{cases}
$$

we obtain the multivariate skew normal-Laplace distribution for $\boldsymbol{Z}$.

### 1.5.1 M.G.F. of $(\boldsymbol{Z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{Z}-\boldsymbol{a})$

In the following we derive the m.g.f. of the quadratic form $\boldsymbol{Q}=(\boldsymbol{Z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{Z}-\boldsymbol{a})$. For this we need the following lemma which can be obtained by routine computation.

Lemma 5.1. Let $\boldsymbol{U} \sim N_{p}(\mathbf{0}, \boldsymbol{\Omega})$. Then, for any scalar $u$ and $\boldsymbol{v} \in \mathbb{R}^{p}$, we have

$$
\begin{align*}
E\left[G_{2}\left(u+\boldsymbol{v}^{\prime} \boldsymbol{U}\right)\right]= & \frac{1}{2} \exp \left\{u+\frac{1}{2} \boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right\} \Phi\left(\frac{-\left(u+\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)}{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right)+\Phi\left(\frac{u}{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right) \\
& -\frac{1}{2} \exp \left\{-u+\frac{1}{2} \boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right\} \Phi\left(\frac{u-\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}}{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right) . \tag{1.14}
\end{align*}
$$

Substituting (1.14) in (1.6) yields the following theorem.

Theorem 5.2. The m.g.f. of $\boldsymbol{Q}$ is given by

$$
\begin{align*}
M_{\boldsymbol{Q}}(t)= & \frac{2 \exp \left\{\boldsymbol{a}^{\prime}\left[t \boldsymbol{A}+2 t^{2} \boldsymbol{A}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A}\right] \boldsymbol{a}\right\}}{|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{1 / 2}} \\
& \times\left[\frac{1}{2} \exp \left\{-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}+\frac{1}{2} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right\}\right. \\
& \times \Phi\left(\frac{2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}-\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}}{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right)+\Phi\left(\frac{-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}}{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right) \\
& -\frac{1}{2} \exp \left\{2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}+\frac{1}{2} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right\} \\
& \left.\times \Phi\left(\frac{-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}-\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}}{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right)\right], t \in \mathbb{R} . \tag{1.15}
\end{align*}
$$

### 1.5.1.1. Special case

Case (iii). The m.g.f. of $\boldsymbol{Q}_{3}$ is

$$
\begin{aligned}
M_{\boldsymbol{Q}_{3}}(t)= & \frac{2 \exp \left\{\frac{t}{1-2 t} \boldsymbol{a}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{a}\right\}}{(1-2 t)^{p / 2}} \times\left[\Phi\left(\frac{\frac{-2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}}{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right)^{1 / 2}}\right)\right. \\
& +\frac{1}{2} \exp \left\{-\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}+\frac{1}{2(1-2 t)} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right\} \Phi\left(\frac{\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}-\frac{1}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}}{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right)^{1 / 2}}\right) \\
& \left.-\frac{1}{2} \exp \left\{\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}+\frac{1}{2(1-2 t)} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right\} \Phi\left(\frac{\frac{-2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}-\frac{1}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}}{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right)^{1 / 2}}\right)\right], t \in \mathbb{R} .
\end{aligned}
$$

### 1.6 Multivariate skew normal-logistic model

Let $G=G_{3}$ be the c.d.f. of a logistic distribution, namely

$$
G_{3}(x)=\frac{1}{1+\exp (-x / \beta)}, \quad-\infty<x<\infty
$$

we obtain the multivariate skew normal-logistic distribution for $\boldsymbol{Z}$. Using the Taylor series expansion for $(1+w)^{-1}$, then

$$
G_{3}(x)= \begin{cases}\sum_{j=0}^{\infty}\binom{-1}{j} \exp \left(-\frac{j x}{\beta}\right) & , x \geq 0 \\ \exp \left(\frac{x}{\beta}\right) \sum_{j=0}^{\infty}\binom{-1}{j} \exp \left(\frac{j x}{\beta}\right) & , x<0\end{cases}
$$

### 1.6.1 M.G.F. of $(Z-a)^{\prime} A(Z-a)$

In the following we derive the m.g.f. of the quadratic form $\boldsymbol{Q}=(\boldsymbol{Z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{Z}-\boldsymbol{a})$. First we need a lemma given by Nadarajah and Kotz (2003).

Lemma 6.1. Let $\boldsymbol{U} \sim N_{p}(\mathbf{0}, \boldsymbol{\Omega})$. Then, for any scalar $u$ and $\boldsymbol{v} \in \mathbb{R}^{p}$, we have

$$
\begin{align*}
E\left[G_{3}\left(u+\boldsymbol{v}^{\prime} \boldsymbol{U}\right)\right]= & \sum_{j=0}^{\infty}\binom{-1}{j}\left[\exp \left\{-\frac{j}{\beta} u+\frac{j^{2}}{2 \beta^{2}} \boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right\} \Phi\left(\frac{u-\frac{j}{\beta} \boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}}{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right)\right. \\
& \left.+\exp \left\{\frac{(j+1)}{\beta} u+\frac{(j+1)^{2}}{2 \beta^{2}} \boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right\} \Phi\left(\frac{-u-\frac{(j+1)}{\beta} \boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}}{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right)\right] . \tag{1.16}
\end{align*}
$$

Substituting (1.16) into (1.6), we obtain the following theorem.

Theorem 6.2. The m.g.f. of $\boldsymbol{Q}$ is given by

$$
\begin{align*}
M_{\boldsymbol{Q}}(t)= & \frac{2 \exp \left\{\boldsymbol{a}^{\prime}\left[t \boldsymbol{A}+2 t^{2} \boldsymbol{A}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A}\right] \boldsymbol{a}\right\}}{|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{1 / 2}} \\
& \times \sum_{j=0}^{\infty}\binom{-1}{j}\left[\exp \left\{\frac{2 t j}{\beta} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}+\frac{j^{2}}{2 \beta^{2}} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right\}\right. \\
& \times \Phi\left(\frac{\left.-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}-\frac{j}{\beta} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)}{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right) \\
& +\exp \left\{-\frac{2 t(j+1)}{\beta} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}+\frac{(j+1)^{2}}{2 \beta^{2}} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right\} \\
& \left.\times \Phi\left(\frac{2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}-\frac{(j+1)}{\beta} \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}}{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right)\right], t \in \mathbb{R} . \tag{1.17}
\end{align*}
$$

### 1.6.1.1. Special case

Case (iii). The m.g.f. of $\boldsymbol{Q}_{3}$ is

$$
\begin{aligned}
M_{\boldsymbol{Q}_{3}}(t)= & \frac{2 \exp \left\{\frac{t}{1-2 t} \boldsymbol{a}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{a}\right\}}{(1-2 t)^{p / 2}} \\
& \times\left[\sum _ { j = 0 } ^ { \infty } ( \begin{array} { c } 
{ - 1 } \\
{ j }
\end{array} ) \left[\exp \left\{\frac{2 t j}{(1-2 t) \beta} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}+\frac{j^{2}}{2(1-2 t) \beta^{2}} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right\} \times \Phi\left(\frac{-\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}-\frac{j}{(1-2 t) \beta} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}}{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\boldsymbol{\Omega}} \boldsymbol{\Omega}\right)^{1 / 2}}\right)\right.\right. \\
& \left.\left.+\exp \left\{-\frac{2 t(j+1)}{(1-2 t) \beta} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}+\frac{(j+1)^{2}}{2(1-2 t) \beta^{2}} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right\} \times \Phi\left(\frac{\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}-\frac{j+1}{(1-2 t) \beta} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}}{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right)^{1 / 2}}\right)\right]\right], t \in \mathbb{R} .
\end{aligned}
$$

### 1.7 Multivariate skew normal-uniform model

Let $G=G_{4}$ be the c.d.f. of a uniform distribution, namely

$$
G_{4}(x)=\left\{\begin{array}{cl}
0 & , x<-h \\
\frac{x+h}{2 h} & ,-h \leq x<h \\
1 & , x \geq h
\end{array}\right.
$$

we obtain the multivariate skew normal-uniform distribution for $\boldsymbol{Z}$.

### 1.7.1 M.G.F. of $(\boldsymbol{Z}-a)^{\prime} \boldsymbol{A}(\boldsymbol{Z}-a)$

In the following we derive the m.g.f. of the quadratic form $\boldsymbol{Q}=(\boldsymbol{Z}-\boldsymbol{a})^{\prime} \boldsymbol{A}(\boldsymbol{Z}-\boldsymbol{a})$. Again first we give a lemma.

Lemma 7.1. Let $\boldsymbol{U} \sim N_{p}(\mathbf{0}, \boldsymbol{\Omega})$. Then, for any scalar $u$ and $\boldsymbol{v} \in \mathbb{R}^{p}$,

$$
\begin{align*}
E\left[G_{4}\left(u+\boldsymbol{v}^{\prime} \boldsymbol{U}\right)\right]= & \frac{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}{2 h \sqrt{2 \pi}} \exp \left\{-\frac{(h+u)^{2}}{2 \boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}}\right\}-\frac{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}{2 h \sqrt{2 \pi}} \exp \left\{-\frac{(h-u)^{2}}{\left.2 \boldsymbol{v}^{\boldsymbol{\Omega} \boldsymbol{\Omega} \boldsymbol{v}}\right\}}\right. \\
& +\left(\frac{u}{2 h}-\frac{1}{2}\right)\left[\Phi\left(\frac{h-u}{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right)-1\right]+\left(\frac{u}{2 h}+\frac{1}{2}\right) \Phi\left(\frac{h+u}{\left(\boldsymbol{v}^{\prime} \boldsymbol{\Omega} \boldsymbol{v}\right)^{1 / 2}}\right) . \tag{1.18}
\end{align*}
$$

Substituting (1.18) into (1.6), we obtain the following theorem.
Theorem 7.2. The m.g.f. of $\boldsymbol{Q}$ is given by

$$
\begin{align*}
& M_{\boldsymbol{Q}}(t)= \frac{2}{} \exp \left\{\boldsymbol{a}^{\prime}\left[t \boldsymbol{A}+2 t^{2} \boldsymbol{A}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A}\right] \boldsymbol{a}\right\} \\
&|\boldsymbol{I}-2 t \boldsymbol{A} \boldsymbol{\Omega}|^{1 / 2} \\
& \times\left[\frac{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}{2 h \sqrt{2 \pi}} \exp \left\{-\frac{\left(h-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}\right)^{2}}{2\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)}\right\}\right. \\
&-\frac{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}{2 h \sqrt{2 \pi}} \exp \left\{-\frac{\left(h+2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}\right)^{2}}{2\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)}\right\} \\
&+\left(\frac{-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}}{2 h}-\frac{1}{2}\right)\left[\Phi\left(\frac{h+2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}}{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right)-1\right]  \tag{1.19}\\
&\left.+\left(\frac{-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}}{2 h}+\frac{1}{2}\right) \Phi\left(\frac{h-2 t \boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{A} \boldsymbol{a}}{\left(\boldsymbol{\alpha}^{\prime}\left(\boldsymbol{\Omega}^{-1}-2 t \boldsymbol{A}\right)^{-1} \boldsymbol{\alpha}\right)^{1 / 2}}\right)\right], t \in \mathbb{R} .(1
\end{align*}
$$

### 1.7.1.1. Special case

Case (iii). The m.g.f. of $\boldsymbol{Q}_{3}$ is

$$
M_{\boldsymbol{Q}_{3}}(t)=\frac{2 \exp \left\{\frac{t}{1-2 t} \boldsymbol{a}^{\prime} \boldsymbol{\Omega}^{-1} \boldsymbol{a}\right\}}{(1-2 t)^{p / 2}}
$$

$$
\begin{aligned}
& \times\left\{\frac{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}\right)^{1 / 2}}{2 h \sqrt{2 \pi}}\left(\exp \left\{-\frac{\left(h-\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}\right)^{2}}{\frac{2}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}}\right\}-\exp \left\{-\frac{\left(h+\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}\right)^{2}}{\frac{2}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{\Omega} \boldsymbol{\alpha}}\right\}\right)\right. \\
& \quad+\left(\frac{\frac{-2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}}{2 h}-\frac{1}{2}\right)\left[\Phi\left(\frac{h+\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}}{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\mathbf{}} \boldsymbol{\Omega} \boldsymbol{\alpha}\right)^{1 / 2}}\right)-1\right] \\
& \\
& \left.+\left(\frac{\frac{-2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}}{2 h}+\frac{1}{2}\right) \Phi\left(\frac{h-\frac{2 t}{1-2 t} \boldsymbol{\alpha}^{\prime} \boldsymbol{a}}{\left(\frac{1}{1-2 t} \boldsymbol{\alpha}^{\mathbf{\Omega}} \boldsymbol{\alpha}\right)^{1 / 2}}\right)\right\}, t \in \mathbb{R} .
\end{aligned}
$$

## Chapter 2

## Generalized Skew-Cauchy Distribution

### 2.1 Introduction

The univariate skew-normal distribution has been studied by many authors, see e.g. Azzalini (1985,1986), Henze (1986), Chiogna (1998) and Gupta et al. (2004b). Following Azzalini (1985), a random variable $X$ is said to have a skew-normal distribution with parameter $\lambda$, denoted by $X \sim \mathcal{S N}(\lambda)$, if the probability density function (p.d.f.) is given by

$$
\begin{equation*}
f_{X}(x)=2 \phi(x) \Phi(\lambda x), \quad \lambda, x \in \mathbb{R} \tag{2.1}
\end{equation*}
$$

where $\phi$ and $\Phi$ are the p.d.f. and cumulative distribution function (c.d.f.) of the standard normal distribution, respectively.

By letting the p.d.f. of the random variable $X$ be

$$
\begin{equation*}
f_{X}(x)=2 \phi(x) \Phi\left(\frac{\lambda_{1} x}{\sqrt{1+\lambda_{2} x^{2}}}\right), \lambda_{1}, x \in \mathbb{R}, \lambda_{2} \geq 0 \tag{2.2}
\end{equation*}
$$

Arellano-Valle et al. (2004) defined a so-called skew-generalized normal distribution, they denoted this distribution by $\mathcal{S G \mathcal { N }}\left(\lambda_{1}, \lambda_{2}\right)$.

The multivariate skew-normal distribution has also been considered by Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Gupta and Kollo (2003), and Gupta et al. (2004a). Here a $p$-dimensional random vector $\boldsymbol{X}$ is said to have a multivariate skew-normal distribution, denoted by $\boldsymbol{X} \sim \mathcal{S N}_{p}(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, if it is continuous and its p.d.f. is given by

$$
\begin{equation*}
f_{\boldsymbol{X}}(\boldsymbol{x})=2 \phi_{p}(\boldsymbol{x} ; \boldsymbol{\Omega}) \Phi\left(\boldsymbol{\alpha}^{\prime} \boldsymbol{x}\right) \tag{2.3}
\end{equation*}
$$

where $\boldsymbol{\Omega}>0, \boldsymbol{\alpha} \in \mathbb{R}^{p}, \phi_{p}(\boldsymbol{x} ; \boldsymbol{\Omega})$ is the p.d.f. of $\mathcal{N}_{p}(\mathbf{0}, \boldsymbol{\Omega})$ distribution (the $p$-dimensional normal distribution with zero mean vector and correlation matrix $\boldsymbol{\Omega}$ ). Quadratic forms of skew-normal random vectors have been studied by Azzalini (1985), Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Loperfido (2001), Genton et al. (2001), and Gupta and Huang (2002).

Based on Gupta and Huang (2002), some parallel results for the class of multivariate skew normal-symmetric distributions have also been obtained by Huang and Chen (2006).

If the p.d.f. of a random variable $X$ has the form

$$
\begin{equation*}
f_{X}(x)=2 f(x) G(x), x \in \mathbb{R} \tag{2.4}
\end{equation*}
$$

where $f$ is a p.d.f. of a random variable symmetric about 0 , and $G$ is a Lebesque measurable function satisfying $0 \leq G(x) \leq 1$ and $G(x)+G(-x)=1$ a.e. on $\mathbb{R}$, then $X$ is said to have the so-called skew-symmetric distribution. Gupta et al. (2002) studied the models in which $f$ is taken to be the p.d.f. from one of the following distributions: normal, Student's $t$, Cauchy, Laplace, logistic, and uniform distribution, and $G$ is a distribution function such that $G^{\prime}$ is symmetric about 0. Nadarajah and Kotz (2003) considered the models that $f$ is taken to be a normal p.d.f. with zero mean, while $G$ is taken to come from one of the above continuous symmetric distributions. Multivariate skew-symmetric distributions have also been studied by Gupta and Chang (2003) and Wang et al. (2004a). The multivariate skew-Cauchy distribution and multivariate skew $t$-distribution are studied by Arnold and Beaver (2000), and Gupta (2003), respectively.

It is known that the square of each of the $\mathcal{N}(0,1), \mathcal{S N}(\lambda)$ and $\mathcal{S G \mathcal { N }}\left(\lambda_{1}, \lambda_{2}\right)$ distribution is $\chi_{1}^{2}$ distributed. Based on this observation, in this paper, first we introduce the generalized skewsymmetric model in Section 2. Then in Section 3, we introduce the generalized skew-Cauchy (GSC) distribution. In Section 4, some examples as well as their p.d.f.s of GSC distribution generated by the ratio of two independent generalized skew normally distributed random variables will be given. Finally, in Section 5, several of the possible shapes of the p.d.f. of a main example in Section 4 under various choices of parameters will be illustrated.

### 2.2 Generalized skew distributions

First we give a definition.

Definition 1. Suppose $Y$ is an absolutely continuous random variable symmetric about 0 with p.d.f. $f$ and c.d.f. $F$. Assume random variable $X$ satisfies

$$
\begin{equation*}
X^{2} \stackrel{d}{=} Y^{2} \tag{2.5}
\end{equation*}
$$

Then $X$ is said to have a generalized skew distribution of $F$ (or $f$ ).

In the above definition, if $Y$ has a common distribution, such as $\mathcal{N}(0,1)$ distribution, then $X$ is said to have a generalized skew- $\mathcal{N}(0,1)$ distribution. The p.d.f. of a generalized skew distribution can be obtained by using the following lemma.

Lemma 1.(Huang et al. (2005)) Let $n$ be a positive integer, and $h(t), t \in A$, a continuous p.d.f. Also assume $A \subset[0, \infty)$, when $n$ is even. Then $X^{n}$ has $h$ as its p.d.f., if and only if the p.d.f. of
$X$ is

$$
f_{X}(x)= \begin{cases}n x^{n-1} h\left(x^{n}\right) & , n \text { is odd }  \tag{2.6}\\ n|x|^{n-1} h\left(x^{n}\right) G(x) & , n \text { is even }\end{cases}
$$

where $x \in B=\left\{x \mid x \in \mathbb{R}, x^{n} \in A\right\}$, and $G(x)$ is a Lebesgue measurable function which satisfies $0 \leq G(x) \leq 1$ and $G(x)+G(-x)=1$ a.e., $\forall x \in B$.

According to the above lemma, the following theorem is obtained immediately.

Theorem 1. Assume the random variable $Y$ is defined as in Definition 1, and $X$ has a generalized skew distribution of $f$. Then the p.d.f. of $X$ is

$$
\begin{equation*}
f_{X}(x)=2 f(x) G(x), x \in \mathbb{R} \tag{2.7}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
f_{X}(x)=f(x)(1+H(x)), x \in \mathbb{R} \tag{2.8}
\end{equation*}
$$

where $G$, the skew function, is a Lebesque measurable function satisfying

$$
\begin{equation*}
0 \leq G(x) \leq 1 \text { and } G(x)+G(-x)=1 \text { a.e. on } \mathbb{R}, \tag{2.9}
\end{equation*}
$$

and $H(x)=2 G(x)-1$, satisfying

$$
\begin{equation*}
-1 \leq H(x) \leq 1 \text { and } H(-x)=-H(x) \text { a.e. on } \mathbb{R} \tag{2.10}
\end{equation*}
$$

Proof. Let $h(t)$ be the p.d.f. of $X^{2}$. Then $h(t)=t^{-1 / 2} f\left(t^{1 / 2}\right), t>0$. By Lemma 1,

$$
f_{X}(x)=2|x| \frac{1}{|x|} f(|x|) G(x)=2 f(x) G(x), x \in \mathbb{R}
$$

as required, where $f(|x|)=f(x)$ is by the fact that $f$ is symmetric about 0 . The rest of the proof is obvious hence is omitted.

There are infinitely many functions satisfy (2.9). For example $G$ is the distribution function corresponding to a symmetric random variable (in particular $G$ can be taken as $F$ ), $G(x)=$ $(1+\sin x) / 2$ (hence $G$ is not necessary to be increasing), $G(x) \equiv 1 / 2$ (in this case $f_{X}(x)=$ $f(x), x \in \mathbb{R})$, etc. The p.d.f. given in (2.7) has the same form as in (2.4). In fact, Arnold and Lin (2004) have used $f_{X}$ in (2.7) with $f=\phi$ to define the generalized skew- $\mathcal{N}(0,1)$ distribution. $\mathcal{S N}(\lambda), \mathcal{S G \mathcal { N }}\left(\lambda_{1}, \lambda_{2}\right)$, the skew normal-symmetric models of Nadarajah and Kotz (2003) all belong to the class of generalized skew- $\mathcal{N}(0,1)$ distribution.

Let $\left(Y_{1}, Y_{2}\right)$ be $\mathcal{B} \mathcal{V} \mathcal{N}(0,0,1,1, \rho)$ distributed, $|\rho| \neq 1$. Denote $Y_{(1)}=\min \left\{Y_{1}, Y_{2}\right\}$ and $Y_{(2)}=\max \left\{Y_{1}, Y_{2}\right\}$. Loperfido (2002) pointed out that $Y_{(1)} \sim \mathcal{S N}(-\gamma)$ and $Y_{(2)} \sim \mathcal{S N}(\gamma)$, where $\gamma=[(1-\rho) /(1+\rho)]^{1 / 2}$. For the minimum and maximum of a random sample of size two,
we have the following result.
Proposition 1. Suppose $X_{1}$ and $X_{2}$ are two independent and identically distributed random variables with the common absolutely continuous c.d.f. $F$ and p.d.f. $f$, where $f$ is assumed to be symmetric about 0 . Let $X_{(1)}=\min \left\{X_{1}, X_{2}\right\}, X_{(2)}=\max \left\{X_{1}, X_{2}\right\}$. Then $X_{(1)}$ and $X_{(2)}$ are both generalized skew distributions of $f$ with p.d.f.s

$$
\begin{equation*}
f_{X_{(1)}}\left(y_{1}\right)=2 f\left(y_{1}\right) F\left(-y_{1}\right), y_{1} \in \mathbb{R} \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{X_{(2)}}\left(y_{2}\right)=2 f\left(y_{2}\right) F\left(y_{2}\right), y_{2} \in \mathbb{R}, \tag{2.12}
\end{equation*}
$$

respectively. Also $\left|X_{(1)}\right| \stackrel{d}{=}\left|X_{(2)}\right| \stackrel{d}{=}\left|X_{1}\right|$.
Proof. For independent and identically distributed random variables, the marginal p.d.f.s of $X_{(1)}$ and $X_{(2)}$ can be obtained immediately. By using $1-F\left(y_{1}\right)=F\left(-y_{1}\right), y_{1} \in \mathbb{R}$, it yields (2.11). The rest of this proposition is obvious.

It can be seen easily, that in the above proposition, neither the minimum nor the maximum has a generalized skew distribution of $f$, if the sample size of random variables is greater than two. Also when $\left(X_{1}, X_{2}\right)$ is $\mathcal{B} \mathcal{V}(0,0,1,1,0)$ distributed, namely $X_{1}$ and $X_{2}$ are independent $\mathcal{N}(0,1)$ distributed, then Proposition 1 implies $X_{(1)}$ and $X_{(2)}$ are $\mathcal{S N}(-1)$ and $\mathcal{S N}(1)$ distributed, respectively, which coincides with the result by Loperfido (2002).

The next property for generalized skew- $\mathcal{N}\left(0, \sigma^{2}\right)$ distribution is also immediate.

Proposition 2. Let $X_{1}, \cdots, X_{n+m}, n, m \geq 1$, be independent random variables each has a generalized skew- $\mathcal{N}\left(0, \sigma^{2}\right)$ distribution. Then

$$
\frac{\sum_{i=1}^{n} X_{i}^{2} / n}{\sum_{i=n+1}^{n+m} X_{i}^{2} / m} \sim \mathcal{F}_{n, m},
$$

where $\mathcal{F}_{n, m}$ has an $\mathcal{F}$ distribution with $n$ and $m$ degrees of freedom.
Note that it is allowed that the random variables $X_{1}, \cdots, X_{n+m}$ in the above proposition are not necessary to be identically distributed. The following is an equivalent condition to (2.5).

Proposition 3.(Wang et al. (2004b)) If $X \sim 2 f(x) G(x)$ and $Y \sim 2 \tilde{f}(x) \tilde{G}(x)$, where $2 f(x) G(x)$ and $2 \tilde{f}(x) \tilde{G}(x)$ are two p.d.f.s of generalized skew distributions, then

$$
\begin{aligned}
f(x)=\tilde{f}(x) & \Leftrightarrow \tau(X) \stackrel{d}{=} \tau(Y), \text { for every even function } \tau, \\
& \Leftrightarrow X^{2} \stackrel{d}{=} Y^{2} .
\end{aligned}
$$

It should be mentioned here, one even function $\tau$, such that $\tau(X) \stackrel{d}{=} \tau(Y)$ is enough to imply $X^{2} \stackrel{d}{=} Y^{2}$. We give a simple proposition below, which can be compared with Proposition 3 of

Arellano-Valle et al. (2004).

Proposition 4. Let $X$ be generalized skew- $\mathcal{N}\left(0, \sigma^{2}\right)$ distributed, $Y$ be $\mathcal{N}\left(0, \sigma^{2}\right)$ distributed, and $Z$ be $\chi_{1}^{2}$ distributed, $\sigma>0$. Then $|X| \stackrel{d}{=}|Y| \stackrel{d}{=} \sigma \sqrt{Z} \sim \mathcal{H} \mathcal{N}\left(0, \sigma^{2}\right)$, where $\mathcal{H} \mathcal{N}\left(0, \sigma^{2}\right)$ denotes the half-normal distribution with parameter $\sigma$.

Although there are some parallel properties between non-skew and skew distributions, there also have many properties hold for non-skew distributions but not for skew distributions. We list some examples below:

Let $X_{1}$ and $X_{2}$ be independent and identically distributed random variables with $\mathcal{N}\left(0, \sigma^{2}\right)$ being their common distribution. Then
(i). $X_{1}^{2}+X_{2}^{2}$ and $X_{1} / \sqrt{X_{1}^{2}+X_{2}^{2}}$ are independent,
(ii). $X_{1}^{2}+X_{2}^{2}$ and $X_{1} / X_{2}$ are independent,
(iii). $X_{1}-X_{2}$ and $X_{1}+X_{2}$ are independent.

But none of these properties hold for any other generalized skew- $\mathcal{N}\left(0, \sigma^{2}\right)$ distributions.

### 2.3 The GSC models

We now use Definition 1 to define the generalized skew-Cauchy distribution.
$X$ is said to have a generalized skew- $\mathcal{C}(0, \sigma)$ distribution, denoted by $\mathcal{G S C}(\sigma)$, where $\sigma>0$, if $X^{2} \stackrel{d}{=} Y^{2}$, where $Y$ has a $\mathcal{C}(0, \sigma)$ distribution. That is $X^{2}$ has the p.d.f.

$$
\begin{equation*}
h(t)=\frac{\sigma}{\pi\left[\sqrt{t}\left(\sigma^{2}+t\right)\right]}, t \geq 0, \sigma>0 \tag{2.13}
\end{equation*}
$$

Denote the distribution of $X^{2}$ by $\mathcal{C}^{2}(0, \sigma)$.
By Theorem 1, $X$ has a $\mathcal{G S C}(\sigma)$ distribution, if and only if the p.d.f. of $X$ has either of the following forms

$$
\begin{equation*}
f_{X}(x)=\frac{2 \sigma}{\pi\left(\sigma^{2}+x^{2}\right)} G(x), x \in \mathbb{R}, \sigma>0 \tag{2.14}
\end{equation*}
$$

or

$$
\begin{equation*}
f_{X}(x)=\frac{\sigma}{\pi\left(\sigma^{2}+x^{2}\right)}(1+H(x)), x \in \mathbb{R}, \sigma>0 \tag{2.15}
\end{equation*}
$$

where $G$ and $H$ are Lebesque measurable functions satisfying (2.9) and (2.10), respectively. There are some simple properties for the distribution of $\mathcal{G S C}(\sigma)$.

## Proposition 5.

(i). The only symmetric $\mathcal{G S C}(\sigma)$ distribution is $\mathcal{C}(0, \sigma)$ distribution.
(ii). Let $X \sim \mathcal{G S C}(\sigma)$, and $r \in \mathbb{R}$. Then $E|X|^{r}$ exists if and only if $|r|<1$.
(iii). $X \sim \mathcal{G S C}(\sigma) \Leftrightarrow X^{2} \sim \mathcal{C}^{2}(0, \sigma) \Leftrightarrow \frac{1}{X^{2}} \sim \mathcal{C}^{2}\left(0, \frac{1}{\sigma}\right) \Leftrightarrow \frac{1}{X} \sim \mathcal{G S C}\left(\frac{1}{\sigma}\right)$.

Gupta et al. (2002) gave three examples of GSC distribution. The first example is defined in a similar way as the skew normal distribution defined by Azzalini $(1985,1986)$. That is the p.d.f. of $X$ is $2 f(x) F(\lambda x)$, where $f(\cdot)$ and $F(\cdot)$ are the p.d.f. and c.d.f. of $\mathcal{C}(0, \sigma)$ distribution, respectively. More precisely, the p.d.f. of $X$ is given by

$$
\begin{equation*}
f_{1}(x)=\frac{\sigma}{\pi\left(\sigma^{2}+x^{2}\right)}\left[1+\frac{2 \arctan (\lambda x / \sigma)}{\pi}\right], \lambda, x \in \mathbb{R}, \sigma>0 . \tag{2.16}
\end{equation*}
$$

As $\mathcal{C}(0,1)$ distribution is exactly the $\mathcal{T}_{1}$ distribution, inspired by this, the second example of GSC distribution is based on the skew- $\mathcal{T}_{1}$ distribution, the latter is defined in a similar way as $t$ distribution.

Example 1. Let $X=U / \sqrt{W}$, where $U$ has a generalized skew- $\mathcal{N}(0,1)$ distribution and $W$ independent of $U$ is $\chi_{1}^{2}$ distributed. Then $X$ has a $\mathcal{G S C}(1)$ distribution.

Note that the random variable $X$ given above satisfies $X^{2} \stackrel{d}{=} X_{1}^{2}$, where $X_{1}=U_{1} / \sqrt{W_{1}}$, $U_{1}$ has a $\mathcal{N}(0,1)$ distribution, and $W_{1}$ independent of $U_{1}$ is $\chi_{1}^{2}$ distributed. That is $X_{1}$ is $\mathcal{T}_{1}$ distributed. Hence $\mathcal{G S C}(1)$ is also a generalized skew- $\mathcal{T}_{1}$ distribution.

For a special case, let $U$ have a $\mathcal{S N}\left(\lambda_{1}\right)$ distribution. Then the p.d.f. of $X$ is given by

$$
\begin{equation*}
f_{2}(x)=\frac{1}{\pi\left(1+x^{2}\right)}\left[1+\frac{\lambda_{1} x}{\sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}}\right], \lambda_{1}, x \in \mathbb{R} \tag{2.17}
\end{equation*}
$$

which is the second example of GSC distribution given by Gupta et al. (2002).

### 2.4 More examples of GSC distribution

First we give another GSC example below, which is a slight generalization of the second example given by Gutpa et al. (2002).

Example 2. Let $U$ and $V$ be two independent random variables both are generalized skew$N\left(0, \sigma^{2}\right)$ distributed. Then $X=U /|V|$ has a $\mathcal{G S C}(1)$ distribution.

In particular, let $U$ be $\mathcal{S N}(\lambda)$ distributed, and $V$ be generalized skew- $\mathcal{N}(0,1)$ distributed. Then $X=U /|V|$ has the p.d.f. given in (2.17).

The reason that the two $X$ 's defined in Example 1 and this example are equally distributed is due to Proposition 4.

Suppose that $U$ and $V$ are two independent random variables and both are $\mathcal{N}\left(0, \sigma^{2}\right)$ distributed, $\sigma>0$. It is known that not only $U /|V|$ but also $U / V$ is $\mathcal{C}(0,1)$ distributed. The next
example indicates similar result holds for generalized skew-normal distributions. This example nevertheless is a slight generalization of Examples 1 and 2. Note that both $\sqrt{W}$ in Example 1 and $|V|$ in Example 2 are generalized skew- $\mathcal{N}(0,1)$ distributed.

Example 3. Let $U$ and $V$ be two independent random variables both distributed as generalized skew- $N\left(0, \sigma^{2}\right)$ distribution. Then $X=U / V$ has a $\mathcal{G S C}(1)$ distribution.

The third way of Gupta et al. (2002) to define GSC distribution is by letting $X=U / V$, where $U$ and $V$ are independent random variables both distributed as $\mathcal{S N}(\lambda)$. Obviously $X$ has a $\mathcal{G S C}(1)$ distribution. Although Gupta et al. (2002) failed to obtain the closed form of the p.d.f. of $X$, the p.d.f. actually can be obtained. The following theorem indicates that the closed form of the p.d.f. of $X$ can be derived, even under a more general setting.

Theorem 2. Let $U$ and $V$ be independent random variables distributed as $\mathcal{S N}\left(\lambda_{1}\right)$ and $\mathcal{S N}\left(\lambda_{2}\right)$, respectively, $\lambda_{1}, \lambda_{2} \in \mathbb{R}$. Then $X \equiv X\left(\lambda_{1}, \lambda_{2}\right)=U / V$ has a $\mathcal{G S C}(1)$ distribution with p.d.f.

$$
\begin{align*}
f_{X}(x)= & \frac{1}{\pi\left(1+x^{2}\right)}\left(1+\frac{2 \lambda_{2} \arctan \left(\lambda_{1} x / \sqrt{1+\lambda_{2}^{2}+x^{2}}\right)}{\pi \sqrt{1+\lambda_{2}^{2}+x^{2}}}+\frac{2 \lambda_{1} x \arctan \left(\lambda_{2} / \sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}\right)}{\pi \sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}}\right) \\
& x \in \mathbb{R} . \tag{2.18}
\end{align*}
$$

Before proving this theorem, we give some preliminary results below. The first lemma can be found in Gupta and Brown (2001).

Lemma 2. For any $b \in \mathbb{R}$,

$$
\begin{equation*}
\int_{0}^{\infty} \phi(t) \Phi(b t) d t=\frac{1}{4}+\frac{1}{2 \pi} \arctan (b) . \tag{2.19}
\end{equation*}
$$

Lemma 3. For $s \geq 0$, integer $r \geq 1$ and $a_{1}, \cdots, a_{r} \in \mathbb{R}, \sum_{i=1}^{r} a_{i}^{2} \neq 0$,

$$
\begin{equation*}
\int_{0}^{\infty} v^{s} \phi\left(a_{1} v\right) \cdots \phi\left(a_{r} v\right) d v=\frac{\Gamma((s+1) / 2) 2^{(s-1) / 2}}{(2 \pi)^{r / 2}\left(\sum_{i=1}^{r} a_{i}^{2}\right)^{(s+1) / 2}} \tag{2.20}
\end{equation*}
$$

Proof. Since $\phi\left(a_{1} v\right) \cdots \phi\left(a_{r} v\right)=(\sqrt{2 \pi})^{-(r-1)} \phi\left(\left(\sum_{i=1}^{r} a_{i}^{2}\right)^{1 / 2} v\right)$, without loss of generality, it suffices to prove the case $r=1$. Now by letting $t=a_{1}^{2} v^{2}$, we have

$$
\begin{aligned}
\int_{0}^{\infty} v^{s} \phi\left(a_{1} v\right) d v & =\frac{1}{\sqrt{2 \pi}} \int_{0}^{\infty} v^{s} e^{-a_{1}^{2} v^{2} / 2} d v \\
& =\frac{1}{2 \sqrt{2 \pi}\left(a_{1}^{2}\right)^{(s+1) / 2}} \int_{0}^{\infty} t^{(s-1) / 2} e^{-t / 2} d t \\
& =\frac{\Gamma((s+1) / 2) 2^{(s-1) / 2}}{\sqrt{2 \pi}\left(a_{1}^{2}\right)^{(s+1) / 2}}
\end{aligned}
$$

as desired.

The next lemma is an extension of the above two lemmas.

Lemma 4. For $s \geq 2$, integer $r \geq 1$ and $a_{1}, \cdots, a_{r}, b \in \mathbb{R}, \sum_{i=1}^{r} a_{i}^{2} \neq 0$, we have the following recursive formula

$$
\begin{align*}
& \int_{0}^{\infty} v^{s} \phi\left(a_{1} v\right) \cdots \phi\left(a_{r} v\right) \Phi(b v) d v \\
= & \frac{b \Gamma(s / 2) 2^{s / 2-1}}{(2 \pi)^{(r+1) / 2}\left(\sum_{i=1}^{r} a_{i}^{2}\right)\left(\sum_{i=1}^{r} a_{i}^{2}+b^{2}\right)^{s / 2}}+\frac{s-1}{\sum_{i=1}^{r} a_{i}^{2}} \int_{0}^{\infty} v^{s-2} \phi\left(a_{1} v\right) \cdots \phi\left(a_{r} v\right) \Phi(b v) d v(.2
\end{align*}
$$

Also

$$
\begin{equation*}
\int_{0}^{\infty} \phi\left(a_{1} v\right) \cdots \phi\left(a_{r} v\right) \Phi(b v) d v=\frac{1}{(2 \pi)^{(r+1) / 2}\left(\sum_{i=1}^{r} a_{i}^{2}\right)^{1 / 2}}\left(\frac{\pi}{2}+\arctan \left(\frac{b}{\left(\sum_{i=1}^{r} a_{i}^{2}\right)^{1 / 2}}\right)\right) \tag{2.22}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{\infty} v \phi\left(a_{1} v\right) \cdots \phi\left(a_{r} v\right) \Phi(b v) d v=\frac{1}{2(2 \pi)^{r / 2}\left(\sum_{i=1}^{r} a_{i}^{2}\right)}\left(1+\frac{b}{\left(\sum_{i=1}^{r} a_{i}^{2}+b^{2}\right)^{1 / 2}}\right) \tag{2.23}
\end{equation*}
$$

Proof. Again it suffices to prove the case $r=1$. For $s \geq 2$, by integration by parts and Lemma 3 , it yields

$$
\begin{aligned}
& \int_{0}^{\infty} v^{s} \phi\left(a_{1} v\right) \Phi(b v) d v \\
= & \int_{0}^{\infty} v^{s} \frac{1}{\sqrt{2 \pi}} e^{-a_{1}^{2} v^{2} / 2} \Phi(b v) d v \\
= & \frac{-1}{\sqrt{2 \pi} a_{1}^{2}}\left[\left.v^{s-1} \Phi(b v) e^{-\frac{a_{1}^{2} v^{2}}{2}}\right|_{0} ^{\infty}-\int_{0}^{\infty} e^{-\frac{a_{1}^{2} v^{2}}{2}} d\left(v^{s-1} \Phi(b v)\right)\right] \\
= & \frac{b}{\sqrt{2 \pi} a_{1}^{2}} \int_{0}^{\infty} v^{s-1} e^{-a_{1}^{2} v^{2} / 2} \phi(b v) d v+\frac{s-1}{\sqrt{2 \pi} a_{1}^{2}} \int_{0}^{\infty} v^{s-2} e^{-a_{1}^{2} v^{2} / 2} \Phi(b v) d v \\
= & \frac{b}{a_{1}^{2}} \int_{0}^{\infty} v^{s-1} \phi\left(a_{1} v\right) \phi(b v) d v+\frac{s-1}{a_{1}^{2}} \int_{0}^{\infty} v^{s-2} \phi\left(a_{1} v\right) \Phi(b v) d v \\
= & \frac{b \Gamma(s / 2) 2^{s / 2-1}}{2 \pi a_{1}^{2}\left(a_{1}^{2}+b^{2}\right)^{s / 2}}+\frac{s-1}{a_{1}^{2}} \int_{0}^{\infty} v^{s-2} \phi\left(a_{1} v\right) \Phi(b v) d v .
\end{aligned}
$$

This proves (2.21) for the case $r=1$.
Next by letting $t=\left|a_{1}\right| v$, from Lemma 2 we have

$$
\int_{0}^{\infty} \phi\left(a_{1} v\right) \Phi(b v) d v=\frac{1}{\left|a_{1}\right|} \int_{0}^{\infty} \phi(t) \Phi\left(\frac{b}{\left|a_{1}\right|} t\right) d t=\frac{1}{4\left|a_{1}\right|}+\frac{1}{2 \pi\left|a_{1}\right|} \arctan \left(\frac{b}{\left|a_{1}\right|}\right)
$$

this is exactly (2.22) for $r=1$.

Finally, again by integration by parts and Lemma 3,

$$
\begin{aligned}
& \int_{0}^{\infty} v \phi\left(a_{1} v\right) \Phi(b v) d v \\
= & \frac{-1}{\sqrt{2 \pi} a_{1}^{2}}\left[\left.\Phi(b v) e^{-a_{1}^{2} v^{2} / 2}\right|_{0} ^{\infty}-\int_{0}^{\infty} e^{-a_{1}^{2} v^{2} / 2} d \Phi(b v)\right] \\
= & \frac{-1}{\sqrt{2 \pi} a_{1}^{2}}\left[-\frac{1}{2}-b \int_{0}^{\infty} e^{-a_{1}^{2} v^{2} / 2} \phi(b v) d v\right] \\
= & \frac{1}{2 \sqrt{2 \pi} a_{1}^{2}}+\frac{b}{a_{1}^{2}} \int_{0}^{\infty} \phi(a v) \phi(b v) d v \\
= & \frac{1}{2 \sqrt{2 \pi} a_{1}^{2}}+\frac{b}{a_{1}^{2}} \frac{\Gamma(1 / 2) 2^{-1 / 2}}{2 \pi\left(a_{1}^{2}+b^{2}\right)^{1 / 2}}=\frac{1}{2 \sqrt{2 \pi} a_{1}^{2}}\left[1+\frac{b}{\left(a_{1}^{2}+b^{2}\right)^{1 / 2}}\right] .
\end{aligned}
$$

This completes the proof of this lemma.

We also have an extended corollary.

Corollary 1. For integer $r \geq 1$, and $a_{1}, \cdots, a_{r}, b_{1}, b_{2} \in \mathbb{R}, \sum_{i=1}^{r} a_{i}^{2} \neq 0$,

$$
\begin{align*}
& \int_{0}^{\infty} v \phi\left(a_{1} v\right) \cdots \phi\left(a_{r} v\right) \Phi\left(b_{1} v\right) \Phi\left(b_{2} v\right) d v \\
= & \frac{1}{2(2 \pi)^{(r+2) / 2}\left(\sum_{i=1}^{r} a_{i}^{2}\right)}\left[\pi+\frac{b_{1}\left(\pi+2 \arctan \left(b_{2} /\left(\sum_{i=1}^{r} a_{i}^{2}+b_{1}^{2}\right)^{1 / 2}\right)\right)}{\left(\sum_{i=1}^{r} a_{i}^{2}+b_{1}^{2}\right)^{1 / 2}}\right. \\
& \left.+\frac{b_{2}\left(\pi+2 \arctan \left(b_{1} /\left(\sum_{i=1}^{r} a_{i}^{2}+b_{2}^{2}\right)^{1 / 2}\right)\right)}{\left(\sum_{i=1}^{r} a_{i}^{2}+b_{2}^{2}\right)^{1 / 2}}\right] . \tag{2.24}
\end{align*}
$$

Proof. Again it suffices to prove the case $r=1$. By integration by parts and Lemma 4, it yields

$$
\begin{aligned}
& \int_{0}^{\infty} v \phi\left(a_{1} v\right) \Phi\left(b_{1} v\right) \Phi\left(b_{2} v\right) d v \\
= & \int_{0}^{\infty} v \frac{1}{\sqrt{2 \pi}} e^{-a_{1}^{2} v^{2} / 2} \Phi\left(b_{1} v\right) \Phi\left(b_{2} v\right) d v \\
= & \frac{-1}{\sqrt{2 \pi} a_{1}^{2}}\left[\left.\Phi\left(b_{1} v\right) \Phi\left(b_{2} v\right) e^{-a_{1}^{2} v^{2} / 2}\right|_{0} ^{\infty}-\int_{0}^{\infty} e^{-a_{1}^{2} v^{2} / 2} d\left(\Phi\left(b_{1} v\right) \Phi\left(b_{2} v\right)\right)\right] \\
= & \frac{-1}{\sqrt{2 \pi} a_{1}^{2}}\left[-\frac{1}{4}-\int_{0}^{\infty} e^{-a_{1}^{2} v^{2} / 2}\left[b_{1} \phi\left(b_{1} v\right) \Phi\left(b_{2} v\right)+b_{2} \phi\left(b_{2} v\right) \Phi\left(b_{1} v\right)\right] d v\right] \\
= & \frac{1}{4 \sqrt{2 \pi} a_{1}^{2}}+\frac{b_{1}}{a_{1}^{2}} \int_{0}^{\infty} \phi\left(a_{1} v\right) \phi\left(b_{1} v\right) \Phi\left(b_{2} v\right) d v+\frac{b_{2}}{a_{1}^{2}} \int_{0}^{\infty} \phi\left(a_{1} v\right) \phi\left(b_{2} v\right) \Phi\left(b_{1} v\right) d v \\
= & \frac{1}{4 \sqrt{2 \pi} a_{1}^{2}}+\frac{b_{1}}{a_{1}^{2}}\left[\frac{1}{(2 \pi)^{3 / 2}\left(a_{1}^{2}\right.}+b_{1}^{2}\right)^{1 / 2} \\
& +\frac{b_{2}}{a_{1}^{2}}\left[\frac{1}{2}+\arctan \left(\frac{b_{2}}{(2 \pi)^{3 / 2}\left(a_{1}^{2}+b_{2}^{2}\right)^{1 / 2}}\left(\frac{\pi}{2}+\arctan \left(\frac{\left.b_{1}^{2}\right)^{1 / 2}}{\left(a_{1}^{2}+b_{2}^{2}\right)^{1 / 2}}\right)\right)\right]\right.
\end{aligned}
$$

$$
=\frac{1}{2(2 \pi)^{3 / 2} a_{1}^{2}}\left[\pi+\frac{b_{1}\left(\pi+2 \arctan \left(b_{2} /\left(a_{1}^{2}+b_{1}^{2}\right)^{1 / 2}\right)\right)}{\left(a_{1}^{2}+b_{1}^{2}\right)^{1 / 2}}+\frac{b_{2}\left(\pi+2 \arctan \left(b_{1} /\left(a_{1}^{2}+b_{2}^{2}\right)^{1 / 2}\right)\right)}{\left(a_{1}^{2}+b_{2}^{2}\right)^{1 / 2}}\right]
$$

## Proof of Theorem 2.

That $X$ has a $\mathcal{G S C}(1)$ distribution is obvious. We derive the p.d.f. of $X$ in the following. First the joint p.d.f. of $U$ and $V$ is

$$
f_{U, V}(u, v)=4 \phi(u) \phi(v) \Phi\left(\lambda_{1} u\right) \Phi\left(\lambda_{2} v\right), u, v \in \mathbb{R}
$$

Hence the p.d.f. of $X$ is

$$
\begin{aligned}
f_{X}(x) & =4 \int_{-\infty}^{\infty}|v| \phi(x v) \phi(v) \Phi\left(\lambda_{1} x v\right) \Phi\left(\lambda_{2} v\right) d v \\
& =4 \int_{0}^{\infty} v \phi(x v) \phi(v) \Phi\left(\lambda_{1} x v\right) \Phi\left(\lambda_{2} v\right) d v+4 \int_{0}^{\infty} v \phi(x v) \phi(v) \Phi\left(-\lambda_{1} x v\right) \Phi\left(-\lambda_{2} v\right) d v
\end{aligned}
$$

By using Corollary 1, it yields

$$
\begin{aligned}
f_{X}(x)= & 4 \cdot \frac{1}{8 \pi^{2}\left(1+x^{2}\right)}\left[\pi+\frac{\lambda_{2}\left(\pi+2 \arctan \left(\lambda_{1} x / \sqrt{1+\lambda_{2}^{2}+x^{2}}\right)\right)}{\sqrt{1+\lambda_{2}^{2}+x^{2}}}\right. \\
& \left.+\frac{\lambda_{1} x\left(\pi+2 \arctan \left(\lambda_{2} / \sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}\right)\right)}{\sqrt{1+\left(1+\lambda_{1}^{2} x^{2}\right.}}\right] \\
& +4 \cdot \frac{1}{8 \pi^{2}\left(1+x^{2}\right)}\left[\pi+\frac{-\lambda_{2}\left(\pi+2 \arctan \left(-\lambda_{1} x / \sqrt{1+\lambda_{2}^{2}+x^{2}}\right)\right)}{\sqrt{1+\lambda_{2}^{2}+x^{2}}}\right. \\
& \left.+\frac{-\lambda_{1} x\left(\pi+2 \arctan \left(-\lambda_{2} / \sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}\right)\right)}{\sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}}\right] \\
= & \frac{1}{\pi\left(1+x^{2}\right)}\left(1+\frac{2 \lambda_{2} \arctan \left(\lambda_{1} x / \sqrt{1+\lambda_{2}^{2}+x^{2}}\right)}{\pi \sqrt{1+\lambda_{2}^{2}+x^{2}}}+\frac{2 \lambda_{1} x \arctan \left(\lambda_{2} / \sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}\right)}{\pi \sqrt{1+\left(1+\lambda_{1}^{2}\right) x^{2}}}\right) \\
& \lambda_{1}, \lambda_{2}, x \in \mathbb{R},
\end{aligned}
$$

as desired.

We give another special case of Example 3.

Example 4. Let $X=U / V$, where $U$ is $\mathcal{N}(0,1)$ distributed, $V$ is $\mathcal{S N}(\lambda)$ distributed, and $U$ and $V$ are independent. By noting $\mathcal{S} \mathcal{N}(0) \stackrel{d}{=} \mathcal{N}(0,1)$, from (2.18) we obtain immediately

$$
f_{X}(x)=\frac{1}{\pi\left(1+x^{2}\right)}, x \in \mathbb{R}
$$

Consequently, $X$ is $\mathcal{C}(0,1)$ distributed and independent of $\lambda$. Being $\mathcal{C}(0,1)$ distributed, $X$ and $1 / X$ have the same distribution. Hence $X_{1}=V / U$ is also $\mathcal{C}(0,1)$ distributed.

The following is an extension of Example 4.

Example 5. Let $U$ be $\mathcal{N}\left(0, \sigma^{2}\right)$ distributed, and $V$ be generalized skew- $\mathcal{N}\left(0, \sigma^{2}\right)$ distributed. Then $X=U / V$ is $\mathcal{C}(0,1)$ distributed.

Proof. First the joint p.d.f. of $U$ and $V$ is

$$
f_{U, V}(u, v)=\frac{2}{\sigma^{2}} \phi\left(\frac{u}{\sigma}\right) \phi\left(\frac{v}{\sigma}\right) G(v), u, v \in \mathbb{R}, \sigma>0
$$

where $G(v)$ is a Lebesque measurable function satisfying condition (2.9). Hence by letting $t=v / \sigma$, the p.d.f. of $X$ is

$$
\begin{aligned}
f_{X}(x) & =\int_{-\infty}^{\infty} \frac{2}{\sigma^{2}} \phi\left(\frac{x v}{\sigma}\right) \phi\left(\frac{v}{\sigma}\right) G(v)|v| d v \\
& =\int_{-\infty}^{\infty} 2 \phi(x t) \phi(t) G(\sigma t)|t| d t \\
& =2 \int_{0}^{\infty} t \phi(x t) \phi(t) G(\sigma t) d t+2 \int_{0}^{\infty} t \phi(x t) \phi(t) G(-\sigma t) d t \\
& =2 \int_{0}^{\infty} t \phi(x t) \phi(t)[G(\sigma t)+G(-\sigma t)] d t \\
& =2 \int_{0}^{\infty} t \phi(x t) \phi(t) d t=\frac{1}{\pi\left(1+x^{2}\right)}, x \in \mathbb{R}
\end{aligned}
$$

as desired.
Obviously the result still holds true if $U$ is generalized skew $-\mathcal{N}\left(0, \sigma^{2}\right)$ distributed and $V$ is $\mathcal{N}\left(0, \sigma^{2}\right)$ distributed.

Finally, we give some limiting distributions for the random variable $X\left(\lambda_{1}, \lambda_{2}\right)$ defined in Theorem 2. The proof of the following proposition is easy, hence is omitted.

## Proposition 6.

(i). $\lim _{\lambda_{1} \rightarrow 0} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=} \lim _{\lambda_{2} \rightarrow 0} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=} \mathcal{C}(0,1)$,
(ii). $\lim _{\lambda_{1} \rightarrow \infty} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=} T_{1}, \lim _{\lambda_{1} \rightarrow-\infty} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=}-T_{1}$,
(iii). $\lim _{\lambda_{2} \rightarrow \infty} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=} T_{2}, \lim _{\lambda_{2} \rightarrow-\infty} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=}-T_{2}$,
(iv). $\lim _{\lambda_{1}, \lambda_{2} \rightarrow \infty} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=} \lim _{\lambda_{1}, \lambda_{2} \rightarrow-\infty} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=} \mathcal{H C}(0,1)$,
(v). $\lim _{\substack{\lambda_{1} \rightarrow \infty \\ \lambda_{2} \rightarrow-\infty}} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=} \lim _{\substack{\lambda_{1} \rightarrow-\infty \\ \lambda_{2} \rightarrow \infty}} X\left(\lambda_{1}, \lambda_{2}\right) \stackrel{d}{=}-\mathcal{H C}(0,1)$,
where $\mathcal{H C}(0,1)$ denotes the half- $\mathcal{C}(0,1)$ distribution, $T_{1}$ has the following p.d.f.

$$
\begin{equation*}
f_{3}(x)=\frac{1}{\pi\left(1+x^{2}\right)}\left[1+\frac{\lambda_{2} \operatorname{sgn}(x)}{\sqrt{1+\left(1+\lambda_{2}\right)}}\right], x \in \mathbb{R} \tag{2.25}
\end{equation*}
$$

where $\operatorname{sgn}(x)=1$, if $x>0,0$, if $x=0,-1$, if $x<0$, and $T_{2}$ has the p.d.f. given in (2.17).

### 2.5 Some figures of the p.d.f. of the GSC distribution

In this section, several of the possible shapes of the p.d.f. of the random variable $X\left(\lambda_{1}, \lambda_{2}\right)$ in Theorem 2 under various choices of $\left(\lambda_{1}, \lambda_{2}\right)$ are illustrated. Figures (e)-(h) in Figure 2 demonstrate some limiting behaviors given in Proposition 6. From Figure 2, it seems the p.d.f. of the $X\left(\lambda_{1}, \lambda_{2}\right)$ distribution may have one side heavier tail and one side thinner tail than the $\mathcal{C}(0,1)$ distribution. However, it can be seen easily that for any $\lambda_{1}, \lambda_{2} \in \mathbb{R}$, the ratio of the p.d.f. of $X\left(\lambda_{1}, \lambda_{2}\right)$ to the p.d.f. of $\mathcal{C}(0,1)$ distribution tends to 1 as $x \rightarrow \infty$ or $-\infty$.

In general, GSC distribution may not have the same tail heaviness as the $\mathcal{C}(0,1)$ distribution also may not be unimodal. As an example let $\sigma=1$ and $H(x)=\sin x$ in (2.15), Figure 1 depicts this p.d.f. curve. Yet our conjecture is for any $\lambda_{1}, \lambda_{2} \in \mathbb{R}$, the p.d.f. curve of $X\left(\lambda_{1}, \lambda_{2}\right)$ is unimodal. This and some other related problems will be studied in the future.


Figure 1. Probability density function of $f(x)=(1+\sin x) /\left(\pi\left(1+x^{2}\right)\right), x \in \mathbb{R}$


Figure 2. Probability density function of $X\left(\lambda_{1}, \lambda_{2}\right)$ for several values of $\left(\lambda_{1}, \lambda_{2}\right)$

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## 小傳

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