Some Characterization Results Based on Certain Conditional Expectations

by

Chia-Hua Liu

Advisor

Wen-Jang Huang

Institute of Statistics,

National University of Kaohsiung

Kaohsiung, Taiwan 811 R.O.C.

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Contents

關於某些條件期望值的刻劃

指導教授：黃文璋博士
國立高雄大學應用數學系

學生：劉嘉樺
國立高雄大學統計學研究所

摘　要

給定X,Y為二獨立且非退化的隨機變數, Lukacs (1995) 證明 $X/(X+Y)$ 和 $X+Y$ 獨立, 若且唯若 X,Y 均有 gamma 分佈, 且有相同的尺度參數 。

本文中, 在 X/U 和 U 獨立且 X/U 為 $\mathcal{B}e(p,q)$ 分佈的假設之下, 我們利用 $E(h(U,X)|X) = b$ 的條件來刻劃 (U,X) 的分佈 。其中, h 被允許為 $U-X$ 的指數函數或是 $U-X$ 的三角函數 。我們的結果之一是: 假如 $q=1$, 且對於某一正整數 n, $E(\sum_{i=1}^{n} e^{i(U-X)}|X) = b$, 其中 b 為一常數, 則 (U,X) 的分佈可以決定 。一些相關的其他結果也將被提出 。

關鍵詞: Beta分佈、刻劃、常數迴歸、條件期望值、gamma分佈、混合分佈

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Advisor: Dr. Wen-Jang Huang
Department of Applied Mathematics
National University of Kaohsiung

Student: Chia-Hua Liu
Institute of Statistics
National University of Kaohsiung

ABSTRACT

Given two independent non-degenerate positive random variables X and Y, Lukacs (1955) proved that $X/(X+Y)$ and $X+Y$ are independent if and only if X and Y are gamma distributed with the same scale parameter.

In this work, under the assumption X/U and U are independent, and X/U has a $\mathcal{B}e(p,q)$ distribution, we characterize the distribution of (U,X) by the condition $E(h(U,X)|X) = b$, where h is allowed to be an exponential function or trigonometric function of $U-X$. Among others, we prove if $q=1$, and for some positive integer n, $E(\sum_{i=1}^{n} e^{i(U-X)}|X) = b$, where b is a constant, then the distribution of (U,X) can be determined. Some other related results are also presented.

Keywords: Beta distribution, characterization, constant regression, conditional expectation, gamma distribution, mixture distributions.

1. Introduction

It is known that if X and Y are independent gamma random variables with the same scale parameter, i.e. X has a $\Gamma(p,r)$ distribution, Y has a $\Gamma(q,r)$ distribution, for some constants $p,q,r>0$, then the two random variables

$$X+Y \quad and \quad \frac{X}{X+Y}$$

are mutually independent and have $\Gamma(p+q,r)$ and $\mathcal{B}e(p,q)$ distributions, respectively. Here the notation $\Gamma(p,r)$, $p,r>0$, and $\mathcal{B}e(p,q)$, $p,q>0$, denote the gamma distribution and beta distribution having the probability density functions (p.d.f.)

$$f_1(x)=\frac{x^{p-1}e^{-x/r}}{\Gamma(p)r^p}, \quad x>0,$$

and

$$f_2(x)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1}=\frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \quad 0<x<1,$$

respectively, where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}dx, \quad t>0,$$

and $B(\cdot,\cdot)$ is the beta function defined by

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \ p,q>0.$$

Lukacs (1955) showed that the above property can be used to characterize the gamma distributions in the following sense. If X and Y are independent non-degenerate positive random variables and $X+Y$ and $X/(X+Y)$ are mutually independent, then X and Y must have gamma distributions with the same scale parameter, but possibly with different values of the shape parameter.

By setting $U=X+Y$ and $W=X/(X+Y)$ in Lukacs type characterization, we get another form of characterization using the independence of U and W, and independence of UW and $U(1-W)$. Note that $X=UW$, X,U have gamma distributions, and W has beta distribution in this case.

Given X and Y are independent, Bolger and Harkness (1965), Hall and Simons (1969), Wesolowski (1989,1990) and Li et al. (1994), Huang and Su (1997), Bobecka and Wesolowski (2002), Chou and Huang (2003), Huang and Chou (2004) and many others characterized the distribution of X and Y by weaken the independent condition of $X/(X+Y)$ and $X/(X+Y)$ to the so-called constant regression.

Instead of weakening the independent condition of $X/(X+Y)$ and $X+Y$ to constant regressions, conditions which are weaker than the independence of X and Y, and replacing the independence of $X/(X+Y)$ and $X+Y$ by the stronger assumption: X/U and U are independent and X/U is $\mathcal{B}e(p,q)$ distributed, Gupta and Wesolowski (1997), Huang and Wong (1998), Gupta and Wesolowski (2001), and Huang and Chang (2005) characterized the distribution of U by using $E(h(U,X)|X) = b$, where h is some function of (U,X) and b is a constant. In particular, Huang and Chang proved if $q = 1$, and for some integer $n \geq 1$, $E(\sum_{i=1}^{n} a_i(U-X)^i|X) = b$, where $a_1, \cdots, a_n, b$, are real constants such that $a_1^2 + \cdots + a_n^2 \neq 0$ and $b \neq 0$, or for some real number $n > 0$, $E((U-X)^n|X) = b$, where $b > 0$ is a constant, then the distribution of (U,X) can be determined.

In this work, we characterize the distribution of (U,X) by the form of the condition $E(h(U,X)|X) = b$, yet for h instead of polynomial functions as considered in Huang and Chang (2005), we allow h to be exponential functions or trigonometric functions of $U-X$.

2. Preliminaries

Let (X,Y) have the p.d.f.

$$f_{X,Y}(x,y) = \sum_{i=1}^{k} c_i \frac{x^{p-1}e^{-x/r_i}}{\Gamma(p)r_i^p} \frac{y^{q-1}e^{-y/r_i}}{\Gamma(q)r_i^q}\ ,\ x,y>0, \tag{1}$$

where $k \geq 1$, $p,q > 0$, $r_1, \cdots, r_k > 0$, $c_1, \cdots, c_k > 0$, $\sum_{i=1}^{k} c_i = 1$. The distribution of (X,Y) is the mixture of k distributions $F_1(x,y), \cdots, F_k(x,y)$, where $F_i(x,y),\ i = 1, \cdots, k$, is the joint distribution function of two independent random variables with $\Gamma(p, r_i)$, $\Gamma(q, r_i)$ distributions, respectively. Obviously when (X,Y) has the p.d.f. given in (1), the marginal distributions of X and Y are also mixed gamma distributions. Let $U = X+Y$, and $W = X/(X+Y)$. Then it is easy to see that

$$f_{U,W} = \left(\sum_{i=1}^{k} c_i \frac{u^{p+q-1}e^{-u/r_i}}{\Gamma(p+q)r_i^{p+q}}\right) \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} w^{p-1}(1-w)^{q-1},\quad u>0, 0<w<1.$$

Hence for the mixed case, U and W are still independent, the distribution of U is the mixture of k distributions $\Gamma(p+q, r_1), \cdots, \Gamma(p+q, r_k)$, and W has a $\mathcal{B}e(p,q)$ distribution. This is an example for $X/(X+Y)$ and $X+Y$ being independent, and $X/(X+Y)$ has a beta distribution, yet X and Y are not independent and neither of the marginal distribution of X and Y is gamma.

Conversely let X and U be two random variables. Assume X/U and U are independent, X/U has a $\mathcal{B}e(p,q)$ distribution. Then (X,U) has the p.d.f.

$$f_{X,U}(x,u) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}u^{1-p-q}(u-x)^{q-1}f_U(u),\quad 0<x<u<T\leq\infty, \tag{2}$$

where $f_U(u)$, $0 < u < T$, is the p.d.f. of U, $T = \inf\{u : F_U(u) = 1\}$, and $F_U(u)$, $u \in \mathbb{R}$, is the distribution function of U. From (2), the marginal p.d.f. of X, and the conditional p.d.f. of

U given X can be determined while knowing f_U. That is

$$f_X(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1} \int_x^T u^{1-p-q}(u-x)^{q-1} f_U(u)du, \quad 0 < x < T,$$

and

$$f_{U|X}(u|x) = \frac{u^{1-p-q}(u-x)^{q-1} f_U(u)}{\int_x^T u^{1-p-q}(u-x)^{q-1} f_U(u)du}, \quad 0 < x < u < T. \tag{3}$$

For example,

(i) if U has a $\Gamma(p+1, r)$ distribution, then X has a $\Gamma(p, r)$ distribution;

(ii) if

$$f_U(u) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \frac{u^{p+j-1} e^{-u/r_i}}{\Gamma(p+j) r_i^{p+j}}\ , \ u > 0,$$

where $\sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} = 1$, such that $f_U(u) \geq 0$, $\forall u > 0$, then

$$f_X(x) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \frac{x^{p+j-2} e^{-x/r_i}}{\Gamma(p+j-1) r_i^{p+j-1}}\ , \ x > 0;$$

The above two distributions of U will appear in the next section.

3. Main results

We now present a lemma, which will be used to prove Theorem 1.

Lemma 1

(i) Let $p > 0$, $0 < r_1 < r_2$ and $c_1 + c_2 = 1$. The function

$$f_3(u) = c_1 \frac{u^{p-1} e^{-u/r_1}}{\Gamma(p) r_1^p} + c_2 \frac{u^{p-1} e^{-u/r_2}}{\Gamma(p) r_2^p}, \ u > 0, \tag{4}$$

is a p.d.f. if and only if

$$0 \leq c_2 \leq \frac{r_2^p}{r_2^p - r_1^p}\ . \tag{5}$$

(ii) Let $0 < p_2 < p_1$, $0 < r_2 \leq r_1$ and $c_1 + c_2 = 1$. The function

$$f_4(u) = c_1 \frac{u^{p_1-1} e^{-u/r_1}}{\Gamma(p_1) r_1^{p_1}} + c_2 \frac{u^{p_2-1} e^{-u/r_2}}{\Gamma(p_2) r_2^{p_2}}, \ u > 0,$$

is a p.d.f. if and only if $0 \leq c_2 \leq 1$.

Proof. (i). First we show the sufficiency. That $\int_0^\infty f_3(u)du = 1$ is obvious, for f_3 given in (4). We now prove $f_3(u) \geq 0,\ \forall u > 0$. From (5), we obtain

$$\frac{c_2}{r_2^p} \geq \frac{c_2 - 1}{r_1^p} .$$

Consequently,

$$\frac{c_2}{r_2^p} \geq \frac{(c_2 - 1)e^{-(1/r_1 - 1/r_2)u}}{r_1^p} , \ \forall u > 0, \tag{6}$$

which is equivalent to

$$(1 - c_2)\frac{u^{p-1}e^{-u/r_1}}{\Gamma(p)r_1^p} + c_2\frac{u^{p-1}e^{-u/r_2}}{\Gamma(p)r_2^p} \geq 0 , \ \forall u > 0.$$

Next we show the necessity. Being a p.d.f., $f_3(u) \geq 0,\ \forall u > 0$, hence (6) holds. If $c_2 < 0$, then (6) in turn implies

$$\frac{(1 - c_2)e^{-(1/r_1 - 1/r_2)u}}{r_1^p} > -\frac{c_2}{r_2^p} \geq 0,\ \forall u > 0. \tag{7}$$

As $r_2 > r_1$ and $1 - c_2 > 0$, the left hand side of (7) tends to 0 as $u \to \infty$. The contradiction implies $c_2 \geq 0$. On the other hand, assume c_2 can be greater than $r_2^p/(r_2^p - r_1^p)$. It turns out there exists some $h > 0$, such that (4) can be a p.d.f. when $c_2 = (r_2^p + h)/(r_2^p - r_1^p)$. Substituting this c_2 into (6), yields

$$r_1^p r_2^p + r_1^p h \geq (r_1^p r_2^p + r_2^p h)e^{-(1/r_1 - 1/r_2)u},\ \forall u > 0. \tag{8}$$

Again the right side of (8) tends to $r_1^p r_2^p + r_2^p h$ as $u \to 0$, which is greater than the left hand side of (8). The contradiction implies $c_2 \leq r_2^p/(r_2^p - r_1^p)$.

(ii). The sufficiency is obvious, we only need to prove the necessity. Being a p.d.f., $f_4(u) \geq 0,\ \forall u > 0$, hence

$$(1 - c_2)\Gamma(p_2)r_2^{p_2}u^{p_1 - p_2}e^{-(1/r_1 - 1/r_2)u} + c_2\Gamma(p_1)r_1^{p_1} \geq 0,\ \forall u > 0, \tag{9}$$

As $p_1 > p_2$, $r_1 \geq r_2$, if $c_2 > 1$, the left hand side of (9) tends to $-\infty$ as $u \to \infty$; if $c_2 < 0$, the left side of (9) tends to $c_2\Gamma(p_1)r_1^{p_1} < 0$ as $u \to 0$. The contradiction implies $0 \leq c_2 \leq 1$.

This completes the proof of this theorem.

Throughout the rest of this section, assume X/U and U are independent and X/U is $\mathcal{B}e(p, 1)$ distributed. In the following theorem, without loss of generality assume $b \geq 0$. Also as pointed out in Section 2, once the distribution of U is determined, then the distribution of X is determined too. So we only give the solution of U in each theorem.

Theorem 1 Assume

$$E(a_1 e^{U-X} + a_2 e^{2(U-X)}|X) = b \tag{10}$$

holds for real constants a_1, a_2 and b, where $a_1^2 + a_2^2 \neq 0$. Assume additionally that $f_U(u)$ is continuous with the support $[0, T]$, where $0 < T \leq \infty$. Then there are the following cases:
Case (I): $b = 0$

(i) $a_1 + a_2 \neq 0$

(1) $a_1 = 0$, $a_2 \neq 0$ or $a_1 \neq 0$, $a_2 = 0$. Then there does not exist random variable U satisfying (10).

(2) $a_1 \neq 0$ and $a_2 \neq 0$. If $(2a_1 + a_2)(a_1 + a_2) \leq 0$, then there does not exist random variable U satisfying (10); if$(2a_1 + a_2)(a_1 + a_2) > 0$, then $T = \infty$, and U is $\Gamma(p + 1, (a_1 + a_2)/(2a_1 + a_2))$ distributed.

(ii) $a_1 + a_2 = 0$
There does not exist random variable U satisfying (10).

Case (II): $b > 0$

(i) $b \neq a_1 + a_2$. Then $T = \infty$.

(1) $a_1 = 0$ and $a_2 \neq 0$. If $b > a_2$, then U is $\Gamma(p + 1, (b - a_2)/(2b))$ distributed; if $b \leq a_2$, then there does not exist random variable U satisfying (10).

(2) $a_1 \neq 0$ and $a_2 = 0$. If $b > a_1$, then U is $\Gamma(p + 1, (b - a_1)/b)$ distributed; if $b \leq a_1$, then there does not exist random variable U satisfying (10).

(3) $a_1 \neq 0$ and $a_2 \neq 0$. Let the equation

$$(b - a_1 - a_2)x^2 + (3b - 2a_1 - a_2)x + 2b = 0 \tag{11}$$

have roots $-1/r_1$ and $-1/r_2$, where $-1/r_1 \leq -1/r_2$ if both are real numbers. Then $T = \infty$.

(a) If both $-1/r_1$ and $-1/r_2$ are imaginaries, or $-1/r_1 > 0$, then there does not exist random variable U satisfying (10).

(b) If $-1/r_1 < 0$, $-1/r_2 > 0$, then U is $\Gamma(p + 1, r_1)$ distributed.

(c) If $-1/r_2 < 0$, then

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p + 1) r_1^{p+1}} + c_2 \frac{u^p e^{-u/r_2}}{\Gamma(p + 1) r_2^{p+1}}, \ u > 0,$$

where $c_1 + c_2 = 1$.

(d) If $-1/r_1 = -1/r_2 < 0$, then

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p+1) r_1^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_1}}{\Gamma(p+2) r_1^{p+2}},\ u > 0,$$

where $c_1,\ c_2 \geq 0$ and $c_1 + c_2 = 1$.

(ii) $b = a_1 + a_2$

(1) $a_1 a_2 = 0$. Then there does not exist random variable U satisfying (10).

(2) $a_1 a_2 \neq 0$. If $a_1 + 2a_2 > 0$, then $T = \infty$, and U is $\Gamma(p+1, (a_1+2a_2)/(2a_1+2a_2))$ distributed; if $a_1 + 2a_2 \leq 0$, then there does not exist random variable U satisfying (10).

Proof. Observe that it follows immediately for X/U being $\mathcal{B}e(p,1)$ distributed, support(X)= $[0,T] \subset [0,\infty)$(if $T = \infty$ then we write $[0,T)$ instead of $[0,T]$), and $\inf\{u : F_U(u) = 1\} = T$. Note also that since $X \leq U$, a.s., if $T < \infty$, it follows that

$$E(a_1 e^{U-T} + a_2 e^{2(U-T)} | X = T) = E(a_1 + a_2 | X = T) = a_1 + a_2. \tag{12}$$

By letting $g(u) = u^{-p} f_U(u)$, $u > 0$, (10) and (3) imply

$$b \int_x^T g(u)du - \int_x^T a_1 e^{u-x} g(u)du - \int_x^T a_2 e^{2(u-x)} g(u)du = 0,\ 0 < x < u < T. \tag{13}$$

Assume $b = 0$, then (13) becomes

$$\int_x^T a_1 e^{u-x} g(u)du + \int_x^T a_2 e^{2(u-x)} g(u)du = 0,\ 0 < x < T. \tag{14}$$

(i) $a_1 = 0,\ a_2 \neq 0$, or $a_1 \neq 0,\ a_2 = 0$. Taking the derivatives of both sides of (14) with respect to x, we obtain

$$g(x) = 0,\ 0 < x < T. \tag{15}$$

(ii) $a_1 \neq 0$, $a_2 \neq 0$. Taking the derivatives of both sides of (14) with respect to x twice, we obtain

$$(a_1 + a_2) g'(x) + (2a_1 + a_2) g(x) = 0,\ 0 < x < T. \tag{16}$$

Next, assume $b \neq 0$.

(i) $a_1 = 0,\ a_2 \neq 0$ or $a_1 \neq 0,\ a_2 = 0$, taking the derivatives of both sides of (13) with respect to x two times, we obtain

$$(b - a_2) g'(x) + 2b g(x) = 0,\ 0 < x < T, \tag{17}$$

and

$$(b - a_1) g'(x) + b g(x) = 0,\ 0 < x < T, \tag{18}$$

respectively.

(ii) $a_1 \neq 0,\ a_2 \neq 0$. Taking the derivatives of both sides of (13) with respect to x three times, we obtain

$$(b - a_1 - a_2)g''(x) + (3b - 2a_1 - a_2)g'(x) + 2bg(x) = 0,\ 0 < x < T. \tag{19}$$

Case (I): $b = 0$.

(i). $a_1 + a_2 \neq 0$, from (12) we obtain $T = \infty$, otherwise if $T < \infty$, we have $a_1 + a_2 = b = 0$.

(1). Let $a_1 = 0,\ a_2 \neq 0$ or $a_1 \neq 0,\ a_2 = 0$. From (15), we have

$$g(x) = 0,\ x > 0.$$

Hence there does not exist random variable U satisfying (10).

(2). Let $a_1 \neq 0$ and $a_2 \neq 0$. Solving the differential equation (16), yields

$$g(x) = s_1 e^{-(2a_1+a_2)x/(a_1+a_2)},\ x > 0,$$

where s_1 is a constant. If $(2a_1 + a_2)(a_1 + a_2) \leq 0$, then there does not exist random variable U satisfying (10). If $(2a_1 + a_2)(a_1 + a_2) > 0$, then

$$f_U(u) = s_1 u^p e^{-(2a_1+a_2)u/(a_1+a_2)},\ u > 0,$$

and U is $\Gamma(p+1, (a_1 + a_2)/(2a_1 + a_2))$ distributed follows.

(ii). $a_1 + a_2 = 0$. Then (16) becomes $(2a_1 + a_2)g(x) = 0,\ 0 < x < T$, and (note that $a_1^2 + a_2^2 \neq 0$) $g(x) = 0,\ 0 < x < T$ follows. Hence there does not exist random variable U satisfying (10).

Case (II): $b > 0$.

(i). $b \neq a_1 + a_2$, from (12) we obtain $T = \infty$.

(1). Let $a_1 = 0$ and $a_2 \neq 0$. Solving the differential equation (17), yields

$$g(x) = s_1 e^{-2bx/(b-a_2)},\ x > 0. \tag{20}$$

If $b > a_2$, (20) in turn implies

$$f_U(u) = s_1 u^p e^{-2bu/(b-a_2)},\ u > 0.$$

Therefore U is $\Gamma(p+1, (b - a_2)/(2b))$ distributed follows. If $b \leq a_2$, then there does not exist random variable U satisfying (10).

(2). Let $a_1 \neq 0$ and $a_2 = 0$. Solving the differential equation (18), yields

$$g(x) = s_1 e^{-bx/(b-a_1)},\ x > 0. \tag{21}$$

If $b > a_1$, (21) in turn implies U is $\Gamma(p+1, (b - a_1)/b)$ distributed. If $b \leq a_1$, then there does not exist random variable U satisfying (10).

(3). The proof of this case is obvious hence is omitted.

(ii). $b = a_1 + a_2$

(1). $a_1 a_2 = 0$. If $T = \infty$, and $a_1 = 0,\ a_2 \neq 0$, then $b = a_2$, from (17), we have $g(x) = 0,\ x > 0$. Hence there also does not exist random variable U satisfying (10). Similarly, if $a_1 \neq 0,\ a_2 = 0$, then $b = a_1$, from (18), again we obtain there does not exist random variable U satisfying (10).

If $T < \infty$, and $a_1 \neq 0,\ a_2 = 0$, then $b = a_1$ and (10) implies $E(a_1 e^{U-X}|X) = a_1$, or $E(e^{U-X}|X) = 1$, which nevertheless is impossible. Hence there does not exist random variable U satisfying (10). Similarly, if $a_1 = 0,\ a_2 \neq 0$, then there also does not exist random variable U satisfying (10).

(2). $a_1 a_2 \neq 0$. In this case (19) becomes

$$(a_1 + 2a_2)g'(x) + (2a_1 + 2a_2)g(x) = 0,\ 0 < x < T. \tag{22}$$

If $T = \infty$, solving the above differential equation, yields

$$g(x) = s_1 e^{-(2a_1+2a_2)x/(a_1+2a_2)},\ x > 0, \tag{23}$$

where s_1 is a constant. If $a_1+2a_2 > 0$, (23) in turn implies U is $\Gamma(p+1, (a_1+2a_2)/(2a_1+2a_2))$ distributed. If $a_1 + 2a_2 \leq 0$, then there does not exist random variable U satisfying (10).

If $T < \infty$, solving the differential equation (22), yields

$$g(x) = s_1 e^{-(2a_1+2a_2)x/(a_1+2a_2)},\ 0 < x < T, \tag{24}$$

where s_1 is a constant. Substitute (24) into (13) for $T < \infty$, we obtain the left hand side of (13) is equal to $-(1/2)e^{2Ta_2/(a_1+2a_2)}(e^{-T} - e^{-x})^2(a_1 + 2a_2)s_1$, which is equal to zero, hence we obtain $s_1 = 0$.

Then (24) becomes

$$g(x) = 0,\ 0 < x < T.$$

Therefore there does not exist random variable U satisfying (10).

Under different conditions for the constants a_1, a_2 and b, the following corollary gives more complete solutions for the Case(II)-(i)-(3) in the above Theorem. Recall that this is the case that $b > 0$, $b \neq a_1 + a_2$, $a_1 \neq 0$ and $a_2 \neq 0$, also $T = \infty$ as obtained in Theorem 1 in this case. The two roots of (11) are $(-3b + 2a_1 + a_2 - \sqrt{D}\ )/2(b - a_1 - a_2)$ and $(-3b + 2a_1 + a_2 + \sqrt{D}\ )/2(b - a_1 - a_2)$, where $D = b^2 - 4a_1 b + 4a_1^2 + 2a_2 b + 4a_1 a_2 + a_2^2$. As in Theorem 1, the two roots are denoted by $-1/r_1$ and $-1/r_2$, where $-1/r_1 \leq -1/r_2$ if both are real numbers.

Corollary 1 Assume the conditions in Theorem 1 holds. Then we give the solution of U in the following.

(i) $b - a_1 - a_2 > 0$

(1) $D < 0$.
Then there does not exist random variable U satisfying (10).

(2) $D > 0$.

(a) If $3b - 2a_1 - a_2 > 0$, then

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p+1) r_1^{p+1}} + c_2 \frac{u^p e^{-u/r_2}}{\Gamma(p+1) r_2^{p+1}}, \ u > 0, \tag{25}$$

where $0 \le c_2 \le r_2^{p+1}/(r_2^{p+1} - r_1^{p+1})$ and $c_1 = 1 - c_2$.

(b) If $3b - 2a_1 - a_2 \le 0$, then there does not exist random variable U satisfying (10).

(3) $D = 0$.

(a) If $3b - 2a_1 - a_2 > 0$, then

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p+1) r_1^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_1}}{\Gamma(p+2) r_1^{p+2}}, \ u > 0, \tag{26}$$

where $c_1, c_2 \ge 0$ and $c_1 + c_2 = 1$.

(b) If $3b - 2a_1 - a_2 \le 0$, then there does not exist random variable U satisfying (10).

(ii) $b - a_1 - a_2 < 0$

(1) $D < 0$.
Then there does not exist random variable U satisfying (10).

(2) $D > 0$.
Then U is $\Gamma(p+1, r_1)$ distributed.

Proof. (i). $b - a_1 - a_2 > 0$.

If $D < 0$, then both $-1/r_1$ and $-1/r_2$ are imaginaries. Hence there does not exist random variable U satisfying (10).

If $D > 0$, the equation (11) has two real roots. Solving the differential equation (19), yields

$$g(x) = s_1 e^{-x/r_1} + s_2 e^{-x/r_2}, \ x > 0, \tag{27}$$

where s_1 and s_2 are constants. If $3b - 2a_1 - a_2 > 0$, then all the coefficients of the equation (11) are positive, hence $-1/r_1 < -1/r_2 < 0$, and $r_2 > r_1 > 0$. Consequently, by Lemma 1, U has the p.d.f. as given in (25), where $0 \le c_2 \le r_2^{p+1}/(r_2^{p+1} - r_1^{p+1})$. If $3b - 2a_1 - a_2 \le 0$, since $b > 0$, the equation (11) has no negative root. Hence there does not exist random variable U satisfying (10).

If $D = 0$, solving the differential equation (19), yields

$$g(x) = s_1 e^{-x/r_1} + s_2 x e^{-x/r_1}, \ x > 0,$$

where s_1 and s_2 are constants. If $3b - 2a_1 - a_2 > 0$ then $-1/r_1 < 0$. Hence U has the p.d.f. as given in (26), where $c_1,\ c_2 \geq 0$ and $c_1 + c_2 = 1$. Again if $3b - 2a_1 - a_2 \leq 0$, then there does not exist random variable U satisfying (10).

(ii). $b - a_1 - a_2 < 0$.

Again if $D < 0$, then there does not exist random variable U satisfying (10).

If $D > 0$, solving the differential equation (19), yields (27). Now $b - a_1 - a_2 < 0$ and $b > 0$, yields $-1/r_1 < 0$ and $-1/r_2 > 0$. Hence $s_2 = 0$, and U is $\Gamma(p + 1, r_1)$ distributed follows.

This completes the proof of this corollary.

A natural extension is to use

$$E(a_1 e^{U-X} + a_2 e^{2(U-X)} + a_3 e^{3(U-X)}|X) = b$$

to determine the distribution of U. The general case is too tedious, as an example in the following Theorem 3 we only consider the simple case $a_1 = a_2 = a_3 = 1$ and $b > 3$. The other cases will be discussed in Remark 1 after Theorem 3. First we give a lemma.

Lemma 2 For $b > 3$, the following cubic equation

$$(b-3)x^3 + (6b-12)x^2 + (11b-11)x + 6b = 0 \tag{28}$$

has three distinct negative roots.

Proof. First let $m_3 = b - 3,\ m_2 = 6b - 12,\ m_1 = 11b - 11,\ m_0 = 6b$. As $b > 3$, the discriminant of (28) is given by

$$\begin{aligned}
\triangle &= \frac{m_1^2 m_2^2 - 4m_0 m_2^3 - 4m_1^3 m_3 + 18 m_0 m_1 m_2 m_3 - 27 m_0^2 m_3^2}{m_3^4} \\
&= \frac{4(b-3)^4 + 24(b-3)^3 + 192(b-3)^2 + 608(b-3) + 1872}{(b-3)^4} \\
&> 0.
\end{aligned}$$

Hence (28) has three distinct real roots.

Let $f(x) = (b-3)x^3 + (6b-12)x^2 + (11b-11)x + 6b$. By Bolzano's theorem, as $f(-1) = 2 > 0$, $f(-2) = -2 < 0$ and $f(-3) = 6 > 0$, there exist two roots of $f(x) = 0$ in the intervals $(-1,-2)$ and $(-2,-3)$, respectively. Finally, the conclusion follows by noting that the coefficients $b-3$ and $6b$ of the term x^3 and constant term of (28) both are positive.

Denote the three distinct negative roots of (28) by $-1/r_1,\ -1/r_2$ and $-1/r_3$, where $r_1,\ r_2,\ r_3 > 0$.

Theorem 2 Assume

$$E(e^{U-X} + e^{2(U-X)} + e^{3(U-X)}|X) = b \tag{29}$$

holds for constant $b > 3$. Assume additionally that $f_U(u)$ is continuous with the support $[0, T]$, where $0 < T \leq \infty$. Then

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p+1) r_1^{p+1}} + c_2 \frac{u^p e^{-u/r_2}}{\Gamma(p+1) r_2^{p+1}} + c_3 \frac{u^p e^{-u/r_3}}{\Gamma(p+1) r_3^{p+1}}, \ u > 0, \tag{30}$$

where $c_1 + c_2 + c_3 = 1$, such that $f_U(u) > 0$, $\forall u > 0$.

Proof. Again if $T < \infty$, then

$$E(e^{U-T} + e^{2(U-T)} + e^{3(U-T)} | X = T) = 3.$$

By assumption $b > 3$, by (29) we obtain $T = \infty$.

By letting $g(u) = u^{-p} f_U(u)$, $u > 0$, (3) and (29) imply

$$b \int_x^\infty g(u) du - \int_x^\infty e^{u-x} g(u) du - \int_x^\infty e^{2(u-x)} g(u) du - \int_x^\infty e^{3(u-x)} g(u) du = 0, \ x > 0. \tag{31}$$

Taking the derivatives of both sides of (31) with respect to x four times, we obtain

$$(b-3) g'''(x) + (6b - 12) g''(x) + (11b - 11) g'(x) + 6b g(x) = 0, \ x > 0. \tag{32}$$

Solving the differential equation (32), yields

$$g(x) = s_1 e^{-x/r_1} + s_2 e^{-x/r_2} + s_3 e^{-x/r_3}, \ x > 0,$$

where s_1, s_2, s_3 are constants, and (30) follows.

It is desired to use

$$E(\sum_{i=1}^n a_i e^{i(U-X)} | X) = b$$

to determine the distributions of U. As in the case $n = 3$, we only consider the simple case $a_1 = \cdots = a_n = 1$ and $b > n$. Note that $\sum_{j=1}^k a_j$ is defined to be 0 if $k < 1$.

Theorem 3 Assume

$$E(\sum_{i=1}^n e^{i(U-X)} | X) = b \tag{33}$$

holds for some integer $n \geq 1$, and constant $b > n$. Assume additionally that $f_U(u)$ is continuous with the support $[0, T]$, where $0 < T \leq \infty$. If the equation

$$(b-n) x^n + \binom{n+1}{2} (b - (n-1)) x^{n-1} + \sum_{j=1}^{n-1} f_{i_j} (b - (n-1-j)) x^{n-1-j} = 0, \tag{34}$$

where

$$f_{i_j} = \sum_{i_j=j}^{n-1} \sum_{i_{j-1}=i_j}^{n-1} \cdots \sum_{i_2=i_3}^{n-1} \sum_{i_1=i_2}^{n-1} (\binom{n+1-i_1}{2}(n+1-i_1)(n+2-i_2)\cdots(n+j-i_j)),$$

has negative roots $-1/r_1, \cdots, -1/r_k$, with multiplications $m_1, \cdots, m_k$, respectively, where $m_1, \cdots, m_k \geq 1$, then

$$f_U(u) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \frac{u^{p+j-1} e^{-u/r_i}}{\Gamma(p+j) r_i^{p+j}}\ ,\ u > 0, \tag{35}$$

where $\sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} = 1$, such that $f_U(u) > 0$, $\forall u > 0$.

Proof. Again if $T < \infty$, then

$$E(\sum_{i=1}^{n} e^{i(U-T)} | X = T) = n. \tag{36}$$

By assumption $b > n$, by (33) we obtain $T = \infty$.

By letting $g(u) = u^{-p} f_U(u)$, $u > 0$, (3) and (36) imply

$$b \int_x^\infty g(u)du - \sum_{i=1}^{n} \int_x^\infty e^{i(u-x)} g(u)du = 0,\ x > 0. \tag{37}$$

Taking the derivatives on both sides of (37) with respect to x $(n+1)$ times, we obtain

$$(b-n)g^{(n)}(x) + \binom{n+1}{2}(b-(n-1))g^{(n-1)}(x) + \sum_{j=1}^{n-1} f_{i_j}(b-(n-1-j))g^{(n-1-j)}(x) = 0, \tag{38}$$

where $g^{(0)}(x) = g(x)$. Solving the differential equation (38), yields

$$g(x) = s_1 e^{t_1 x} + s_2 e^{t_2 x} + \cdots + s_n e^{t_n x},\ x > 0, \tag{39}$$

where $s_1, \cdots, s_n$ are constants, and $t_1, \cdots, t_n$ are the roots of the equation (34).

Now without loss of generality, we assume all the roots of (34) have no multiplicities. As $f_U(u) = u^p g(u)$, $u > 0$, is a p.d.f., where g given in (39), is a linear combination of exponential functions, we have $\lim_{u\to\infty} f_U(u) = 0$, otherwise f_U cannot be a p.d.f. Hence the coefficient s_i which corresponds to positive t_i must be zero. Consequently

$$f_U(u) = s_1^{'} u^p e^{-u/r_1} + s_2^{'} u^p e^{-u/r_2} + \cdots + s_k^{'} u^p e^{-u/r_k},$$

where $-1/r_1, \cdots, -1/r_k$ are the distinct negative roots of (34), and $s_i^{'}$ is the constant in (39) which corresponds to the root $-1/r_i$ of (34), $i = 1, \cdots, k$. Therefore we obtain U has the p.d.f. as given in (35).

Remark 1 For $n = 1,\ 2$, and $b > n$, (33) corresponds to Case(II)-(i)-(2) and Case(II)-(i)-(3)-(c) in Theorem 1, respectively. It can be seen easily that

$$(b-1)x + b = 0 \tag{40}$$

and

$$(b-2)x^2 + (3b-3)x + 2b = 0 \tag{41}$$

has one negative root and two distinct negative roots, respectively. For $n = 3$, (34) reduces to (28), which has three distinct negative roots by Lemma 2.

We conjecture that the equation (34) also has n distinct negative roots for every $n \geq 4$. Let $t_1,\ t_2, \cdots, t_n$ be the roots of the equation (34). For $n = 4,\ 5,\ 6,\ 7$ and some different b's, we give the numerical roots of (34) in Table 1. As expected all the roots are distinct negative number which coincides with our conjecture. This and some other related problems will be studied in the future.

Table 1. Some numerical roots of (34).

(a) $n = 4$

b	$t_1,\ t_2,\ t_3,\ t_4$
5	-1.156, -2.293, -3.456, -13.094
10	-1.090, -2.182, -3.306, -5.089
45	-1.022, -2.044, -3.070, -4.108
61	-1.016, -2.033, -3.051, -4.076
200	-1.005, -2.010, -3.015, -4.021
1000	-1.001, -2.002, -3.003, -4.004

(b) $n = 5$

b	$t_1,\ t_2,\ t_3,\ t_4,\ t_5$
6	-1.132, -2.243, -3.358, -4.502, -18.766
10	-1.087, -2.171, -3.265, -4.398, -7.078
35	-1.028, -2.056, -3.087, -4.126, -5.204
61	-1.016, -2.032, -3.050, -4.070, -5.100
100	-1.010, -2.020, -3.030, -4.042, -5.056
1000	-1.001, -2.002, -3.003, -4.004, -5.005

(c) $n = 6$

b	$t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6$
7	-1.114, -2.209, -3.302, -4.404, -5.535, -25.435
10	-1.086, -2.164, -3.245, -4.339, -5.471, -9.945
55	-1.018, -2.035, -3.054, -4.075, -5.101, -6.146
100	-1.010, -2.020, -3.030, -4.041, -5.053, -6.070
200	-1.005, -2.010, -3.015, -4.020, -5.026, -6.032
1000	-1.001, -2.002, -3.003, -4.004, -5.005, -6.006

(d) $n = 7$

b	$t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6,\ t_7$
8	-1.101, -2.185, -3.264, -4.347, -5.440, -6.561, -33.103
10	-1.085, -2.158, -3.231, -4.309, -5.401, -6.526, -14.624
35	-1.027, -2.054, -3.082, -4.113, -5.151, -6.205, -7.367
51	-1.019, -2.038, -3.057, -4.079, -5.103, -6.136, -7.204
100	-1.010, -2.020, -3.030, -4.040, -5.052, -6.065, -7.085
1000	-1.001, -2.002, -3.003, -4.004, -5.005, -6.006, -7.007

In Huang and Chang (2005), they also used the condition $E((U - X)^n|X) = b$ to characterize the distribution of U. For our present situation, it is easy to see that the solution of U of $E(e^{n(U-X)}|X) = b$, can be obtained immediately from the solution of $E(e^{U-X}|X) = b$. We omit the details here.

Theorem 4 Assume

$$E(sin(U - X)|X) = b \tag{42}$$

holds for constant $b \neq 0$. Assume additionally that $f_U(u)$ is continuous with the support $[0, T]$, where $0 < T \leq \infty$. Let the equation

$$bx^2 + x + b = 0 \tag{43}$$

have roots $-1/r_1 = (-1 - \sqrt{D_1})/2b$ and $-1/r_2 = (-1 + \sqrt{D_1})/2b$, where $D_1 = 1 - 4b^2$. Then there are the following possible cases:

(i) $b < 0$
Then there does not exist random variable U satisfying (42).

(ii) $b > 0$

(1) Let $D_1 < 0$. Then there does not exist random variable U satisfying (42).

(2) Let $D_1 > 0$. Then $T = \infty$,

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p+1) r_1^{p+1}} + c_2 \frac{u^p e^{-u/r_2}}{\Gamma(p+1) r_2^{p+1}}, \ u > 0, \tag{44}$$

where $0 \leq c_2 \leq r_2^{p+1}/(r_2^{p+1} - r_1^{p+1})$ and $c_2 = 1 - c_1$.

(3) Let $D_1 = 0$. Then $T = \infty$,

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p+1) r_1^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_1}}{\Gamma(p+2) r_1^{p+2}}, \ u > 0, \tag{45}$$

where $c_1, \ c_2 \geq 0$ and $c_1 + c_2 = 1$.

Proof. Again if $T < \infty$, then

$$E(sin(U - T)|X = T) = 0.$$

By assumption $b \neq 0$, by (42) we obtain $T = \infty$.

By letting $g(u) = u^{-p} f_U(u)$, $u > 0$, (3) and (42) imply

$$b \int_x^\infty g(u)du - \int_x^\infty sin(u - x) g(u) du = 0, \ x > 0. \tag{46}$$

Taking the derivatives of both sides of (46) with respect to x three times, we obtain

$$bg''(x) + g'(x) + bg(x) = 0, \ x > 0. \tag{47}$$

Solving the differential equation (47), yields

$$g(x) = s_1 e^{-x/r_1} + s_2 e^{-x/r_2}, \ x > 0, \tag{48}$$

where $s_1, \ s_2$ are constants and $-1/r_1, \ -1/r_2$ are the root of the equation (43).

(i). $b < 0$.

If $D_1 < 0$, then $-1/r_1$ and $-1/r_2$ are imaginaries. Hence there does not exist random variable U satisfying (42). Next consider $D_1 \geq 0$. From (43) and $b < 0$, we obtain $-1/r_1 > 0$ and $-1/r_2 > 0$. Hence there does not exist random variable U satisfying (42).

(ii). $b > 0$.

(1). Let $D_1 < 0$. Again there does not exist random variable U satisfying (42).

(2). Let $D_1 > 0$. From $b > 0$, we obtain all the coefficients of the equation (43) are positive, consequently, $-1/r_1 < -1/r_2 < 0$. (48) in turn implies

$$f_U(u) = s_1 u^p e^{-u/r_1} + s_2 u^p e^{-u/r_2}, \ u > 0.$$

Therefore we obtain U has the p.d.f. as given in (44), where $0 \leq c_2 \leq r_2^{p+1}/(r_2^{p+1} - r_1^{p+1})$.

(3). Let $D_1 = 0$. From $b > 0$ and (43), we have $-1/r_1 = -1/r_2 < 0$. Hence

$$f_U(u) = s_1 u^p e^{-u/r_1} + s_2 u^{p+1} r^{-u/r_1}, \ u > 0.$$

Therefore we obtain U has the p.d.f. as given in (45), where $c_1,\ c_2 \geq 0$ and $c_1 + c_2 = 1$.

It is expected that there is also a characterizating result based $E(cos(U - X)|X) = b$. As it is similar, we omit the statement and proof for this result. Finally we have also tried to use $h(U, X) = a_1 sin(U - X) + a_2 cos(U - X)$ to determine the distribution of (U, X). But the discussion is too cumbersome, hence is omitted.

4. Conclusion

In this work, we characterized the distribution of (U, X) by $E(h(U, X)|X) = b$, where h is allowed to be exponential functions or trigonometric functions of $U - X$. It is expected that there are some other functions of $h(U, X)$ can be used to characterize the distribution of (U, X).

For example, in Theorem 1, let $a_1 = 1$ and $a_2 = 0$, then (10) becomes $E(e^{U-X}|X) = b$, or

$$E(e^U|X) = be^X. \tag{49}$$

This is a special form of $E(e^U|X) = ae^X + b$. It can be shown the distribution of (U, X) can be determined under the assumption $E(e^U|X) = ae^X + b$. We omit the details here.

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小傳

劉嘉樺於民國71年6月13日生於高雄。民國92年6月畢業於私立義守大學應用數學系、民國93年9月進入國立高雄大學統計學研究所就讀。