Stat Papers
DOI 10.1007/s00362-011-0421-2

REGULAR ARTICLE

Identification of power distribution mixtures through regression of exponentials

Wen-Jang Huang · Nan-Cheng Su

Received: 12 October 2008 / Revised: 28 November 2011

Abstract Given two independent non-degenerate positive random variables X and Y, Lukacs (Ann Math Stat 26:319–324, 1955) proved that $X/(X+Y)$ and $X+Y$ are independent if and only if X and Y are gamma distributed with the same scale parameter. In this work, under the assumption X/U and U are independent, and X/U has a $\mathcal{B}e(p, q)$ distribution, we characterize the distribution of (U, X) by the condition $E(h(U-X)|X) = b$, where h is allowed to be a linear combination of exponential functions. Since the assumption for X and U above is equivalent to $X|U$ being $\mathcal{B}e(p, 1)$ scaled by U, hence our results can be viewed as identification of a power distribution mixture.

Keywords Beta distribution · Characterization · Constant regression · Conditional expectation · Gamma distribution · Mixture distributions

Mathematics Subject Classification (2000) Primary 60E05 · Secondary 62E10

1 Introduction

It is known that if X and Y are independent gamma random variables (rvs) with the same scale parameter, i.e. X has a $\Gamma(p, r)$ distribution, Y has a $\Gamma(q, r)$ distribution, for some constants $p, q, r > 0$, then the two rvs

W.-J. Huang (✉)
Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 81148, Taiwan
e-mail: huangwj@nuk.edu.tw

N.-C. Su
Department of Statistics, National Cheng-Kung University, Tainan 70101, Taiwan
e-mail: sunanchen@gmail.com

Published online: 21 December 2011

$$X + Y \quad \text{and} \quad \frac{X}{X + Y}$$

are mutually independent and have $\Gamma(p+q, r)$ and $\mathcal{B}e(p, q)$ distributions, respectively. Here the notation $\Gamma(p, r)$, $p, r > 0$, and $\mathcal{B}e(p, q)$, $p, q > 0$, denote the gamma distribution and beta distribution having the probability density functions (pdf)

$$f_1(x) = \frac{x^{p-1}e^{-x/r}}{\Gamma(p)r^p}, \quad x > 0,$$

and

$$f_2(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1}, \quad 0 < x < 1,$$

respectively, where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \quad t > 0.$$

Lukacs (1955) showed that the above property can be used to characterize the gamma distributions in the following sense. If X and Y are independent non-degenerate positive rvs and $X + Y$ and $X/(X + Y)$ are mutually independent, then X and Y must have gamma distributions with the same scale parameter, but possibly with different values of the shape parameter.

By setting $U = X + Y$ and $W = X/(X + Y)$ in Lukacs type characterization, we obtain another form of characterization using the independence of U and W, and independence of UW and $U(1 - W)$. Note that $X = UW$, both X and U have gamma distributions, and W has beta distribution in this case.

Under the condition X and Y are independent, Bolger and Harkness (1965), Hall and Simons (1969), Wesolowski (1989, 1990), Li et al. (1994), Huang and Su (1997), Bobecka and Wesolowski (2002), Chou and Huang (2003), Huang and Chou (2004) and many others characterized the distribution of X and Y by weakening the independence of $X + Y$ and $X/(X + Y)$ to the so-called constant regression.

Instead of weakening the independence condition of $X/(X+Y)$ and $X+Y$, weakening the independence of X and Y, and replacing the independence of $X/(X + Y)$ and $X + Y$ by the stronger assumption: X/U and U are independent and X/U is $\mathcal{B}e(p, q)$ distributed, Gupta and Wesolowski (1997, 2001), Huang and Wong (1998), Huang and Liu (2006), and Huang and Chang (2007) characterized the distribution of U by using $E(h(U, X)|X) = b$, where $h(U, X)$ is some function of (U, X) and b is a constant. In particular, Huang and Chang (2007) proved if $q = 1$, and for some integer $n \geq 1$, $E(\sum_{i=1}^n a_i(U - X)^i|X) = b$, where $a_1, \ldots, a_n, b$, are real constants such that $a_1^2 + \cdots + a_n^2 \neq 0$ and $b \neq 0$, or for some real number $n > 0$, $E((U - X)^n|X) = b$, where $b > 0$ is a constant, then the distribution of (U, X) can be determined. Recently, other related works of Lukacs characterization were done by Bobecka and Wesolowski (2008) and Meszaros (2010).

In this work, under the assumption X/U and U are independent and X/U is $\mathcal{B}e(p,\ 1)$ distributed, we characterize the distribution of $(U,\ X)$ by $E(h(U-X)|X) = b$, where h is a linear combinations of exponential functions. Since the assumption for X and U above is equivalent to $X|U$ being $\mathcal{B}e(p,\ 1)$ scaled by U, hence our results can be viewed as identification of a power distribution mixture.

2 Preliminaries

Let $(X,\ Y)$ have the pdf

$$f_{X,Y}(x,\ y) = \sum_{i=1}^{k} c_i \frac{x^{p-1}e^{-x/r_i}}{\Gamma(p)r_i^p} \frac{y^{q-1}e^{-y/r_i}}{\Gamma(q)r_i^q}, \quad x,\ y > 0, \tag{1}$$

where $k \geq 1$, $p,\ q > 0$, $r_1, \ldots, r_k > 0$, $c_1, \ldots, c_k > 0$, $\sum_{i=1}^{k} c_i = 1$. The distribution of $(X,\ Y)$ is the mixture of k distributions $F_1(x,\ y), \ldots, F_k(x,\ y)$, where $F_i(x,\ y)$, $i = 1, \ldots, k$, is the joint distribution function (df) of two independent rvs with $\Gamma(p,\ r_i)$, and $\Gamma(q,\ r_i)$ distributions, respectively. Obviously when $(X,\ Y)$ has the pdf given in (1), then both the marginal distributions of X and Y are also mixed gamma distributions. Let $U = X + Y$, and $W = X/(X + Y)$. Then it is easy to see that the pdf of $(U,\ W)$ is

$$f_{U,W}(u,\ w) = \left(\sum_{i=1}^{k} c_i \frac{u^{p+q-1}e^{-u/r_i}}{\Gamma(p+q)r_i^{p+q}}\right) \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} w^{p-1}(1-w)^{q-1},$$
$$0 < u < \infty, \quad 0 < w < 1.$$

Hence for the mixed case, U and W are still independent, the distribution of U is the mixture of k distributions $\Gamma(p+q,\ r_1), \ldots, \Gamma(p+q,\ r_k)$, and W has a $\mathcal{B}e(p,\ q)$ distribution. This is an example for $X/(X+Y)$ and $X+Y$ being independent, and $X/(X+Y)$ has a beta distribution, yet X and Y are not independent and neither of the marginal distribution of X and Y is gamma.

Conversely let X and U be two rvs. Assume X/U and U are independent, and X/U is $\mathcal{B}e(p,\ q)$ distributed. Then $(X,\ U)$ has the pdf

$$f_{X,U}(x,\ u) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}u^{1-p-q}(u-x)^{q-1} f_U(u), \quad 0 < x < u < T \leq \infty, \tag{2}$$

where $f_U(u)$, $0 < u < T$, is the pdf of U, $T = \inf\{u : F_U(u) = 1\}$, and $F_U(u)$, $u \in \mathbb{R}$, is the df of U. From (2), the marginal pdf of X, and the conditional pdf of U given X can be determined while knowing f_U. That is

$$f_X(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1} \int_x^T u^{1-p-q}(u-x)^{q-1} f_U(u)du, \quad 0 < x < T,$$

and

$$f_{U|X}(u|x) = \frac{u^{1-p-q}(u-x)^{q-1} f_U(u)}{\int_x^T u^{1-p-q}(u-x)^{q-1} f_U(u)du}, \quad 0 < x < u < T. \tag{3}$$

For example,

(i) if U has a $\Gamma(p+1, r)$ distribution, then X has a $\Gamma(p, r)$ distribution;
(ii) if

$$f_U(u) = \sum_{i=1}^{k}\sum_{j=1}^{m_i} c_{ij} \frac{u^{p+j-1}e^{-u/r_i}}{\Gamma(p+j)r_i^{p+j}}, \quad 0 < u < \infty,$$

where $\sum_{i=1}^{k}\sum_{j=1}^{m_i} c_{ij} = 1$, such that $f_U(u) \geq 0$, $0 < u < \infty$, then

$$f_X(x) = \sum_{i=1}^{k}\sum_{j=1}^{m_i} c_{ij} \frac{x^{p+j-2}e^{-x/r_i}}{\Gamma(p+j-1)r_i^{p+j-1}}, \quad 0 < x < \infty.$$

The above two distributions of U play important roles in the next section.

3 Main results

Throughout this section, assume X/U and U are independent and X/U is $\mathcal{B}e(p, 1)$ distributed, $p > 0$. As pointed out in Sect. 2, once the distribution of U is determined, then the distribution of X is determined too. So we only present the solution of U in each of the following theorem. Theorem 1 below can be proved along the lines of the more general Theorem 2. For the sake of completeness, we state without proving the theorem.

Theorem 1 *Assume*

$$E(e^{\alpha(U-X)}|X) = b \tag{4}$$

holds for real constants $\alpha \neq 0$ *and* $b > 0$. *Assume additionally that* $f_U(u)$ *is continuous with support* $[0, T]$, *where* $0 < T \leq \infty$. *Then* $T = \infty$, $\alpha(b-1) > 0$ *and* U *is* $\Gamma(p+1, (b-1)/(\alpha b))$ *distributed.*

In order to prove Theorem 2, we need the following Lemma, which gives necessary and sufficient conditions for some linear combinations of gamma distributed pdfs to be a pdf.

Lemma 1 (i) *Let* $p > 0$, $0 < r < s$ *and* $c_1 + c_2 = 1$. *Then*

$$f_3(u) = c_1 \frac{u^{p-1}e^{-u/r}}{\Gamma(p)r^p} + c_2 \frac{u^{p-1}e^{-u/s}}{\Gamma(p)s^p}, \quad 0 < u < \infty, \tag{5}$$

is a pdf if and only if

$$0 \leq c_2 \leq \frac{s^p}{s^p - r^p}. \tag{6}$$

(ii) *Let* $p > q > 0,\ r \geq s > 0$ *and* $c_1 + c_2 = 1$. *Then*

$$f_4(u) = c_1 \frac{u^{p-1} e^{-u/r}}{\Gamma(p) r^p} + c_2 \frac{u^{q-1} e^{-u/s}}{\Gamma(q) s^q}, \quad 0 < u < \infty,$$

is a pdf if and only if $0 \leq c_2 \leq 1$.

Proof (i) First we show the sufficiency. That $\int_0^\infty f_3(u)du = 1$ is obvious. We now prove $f_3(u) \geq 0,\ 0 < u < \infty$. From (6), we obtain

$$\frac{c_2}{s^p} \geq \frac{c_2 - 1}{r^p}.$$

Consequently,

$$\frac{c_2}{s^p} \geq \frac{(c_2 - 1)e^{-(1/r - 1/s)u}}{r^p}, \quad 0 < u < \infty, \tag{7}$$

which is equivalent to

$$(1 - c_2)\frac{u^{p-1} e^{-u/r}}{\Gamma(p) r^p} + c_2 \frac{u^{p-1} e^{-u/s}}{\Gamma(p) s^p} \geq 0, \quad 0 < u < \infty,$$

as desired.

Next we show the necessity. Being a pdf, $f_3(u) \geq 0,\ 0 < u < \infty$, hence (7) holds. If $c_2 < 0$, then (7) in turn implies

$$\frac{(1 - c_2)e^{-(1/r - 1/s)u}}{r^p} > -\frac{c_2}{s^p} \geq 0, \quad 0 < u < \infty. \tag{8}$$

As $s > r$ and $1 - c_2 > 0$, the left side of (8) tends to 0 as $u \to \infty$. The contradiction implies $c_2 \geq 0$. On the other hand, assume c_2 can be greater than $s^p/(s^p - r^p)$. It turns out that there exists some $h > 0$, such that (5) can be a pdf when $c_2 = (s^p + h)/(s^p - r^p)$. Substituting this c_2 into (7), yields

$$r^p s^p + r^p h \geq (r^p s^p + s^p h)e^{-(1/r - 1/s)u}, \quad 0 < u < \infty. \tag{9}$$

As $u \to 0$, the right side of (9) tends to $r^p s^p + s^p h$, which is greater than the left side of (9). The contradiction implies $c_2 \leq s^p/(s^p - r^p)$. This completes the proof of (i).

(ii) The sufficiency is obvious, we only need to prove the necessity. Being a pdf, $f_4(u) \geq 0,\ 0 < u < \infty$, hence

$$(1 - c_2)\Gamma(q) s^q u^{p-q} e^{-(1/r - 1/s)u} + c_2 \Gamma(p) r^p \geq 0, \quad 0 < u < \infty. \tag{10}$$

As $p > q$, $r \geq s$, if $c_2 > 1$, the left side of (10) tends to $-\infty$ as $u \to \infty$; if $c_2 < 0$, the left side of (10) tends to $c_2\Gamma(p)r^p < 0$ as $u \to 0$. Consequently, $0 \leq c_2 \leq 1$. This completes the proof of (ii). □

An extension of Theorem 1 is to characterize the distribution of U by using

$$E(a_1 e^{\alpha_1(U-X)} + a_2 e^{\alpha_2(U-X)}|X) = b, \tag{11}$$

where a_1, a_2, α_1, α_2 and b are real constants. Note that if $a_1 > 0$, $\alpha_1 \neq 0$, $a_2 = 0$ and $b > 0$, then (11) reduces to (4) with the constant b in (4) being replaced by b/a_1. Therefore, in the following Theorem 2, the previous case will not be considered.

Theorem 2 *Assume* (11) *holds for* $\alpha_1\alpha_2 a_1 a_2 \neq 0$ *and* $b \geq 0$. *Assume additionally that* $f_U(u)$ *is continuous with support* $[0, T]$, *where* $0 < T \leq \infty$. *Then only the following cases are possible*:

(i) $b = 0$.
Then $T = \infty$ *and* U *is* $\Gamma(p+1, (a_1+a_2)/(\alpha_1 a_2 + \alpha_2 a_1))$ *distributed*, *where* $a_1 a_2 < 0$, $(\alpha_1 - \alpha_2)(a_1 + a_2)a_2 > 0$ *and* $(\alpha_1 a_2 + \alpha_2 a_1)(a_1 + a_2) > 0$;

(ii) $b > 0$ *and* $a_1 + a_2 = b$.
Then $T = \infty$ *and* U *is* $\Gamma(p+1, (\alpha_1 a_1 + \alpha_2 a_2)/(\alpha_1\alpha_2(a_1+a_2)))$ *distributed*, *where* $(\alpha_1 a_1 + \alpha_2 a_2)\alpha_1\alpha_2 > 0$;

(iii) $b > 0$ *and* $a_1 + a_2 \neq b$.
Let the equation

$$(b - a_1 - a_2)x^2 + ((\alpha_1 + \alpha_2)b - \alpha_2 a_1 - \alpha_1 a_2)x + \alpha_1\alpha_2 b = 0$$

have real roots $-1/r$ *and* $-1/s$, *where* $-1/r \leq -1/s$. *Then* $T = \infty$ *and*

(1) *if* $-1/r < 0$ *and* $1/s > 0$, *then* U *is* $\Gamma(p+1, r)$ *distributed*;

(2) *if* $-1/r < -1/s < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r}}{\Gamma(p+1)r^{p+1}} + c_2 \frac{u^p e^{-u/s}}{\Gamma(p+1)s^{p+1}}, \quad 0 < u < \infty,$$

where $0 \leq c_2 \leq s^p/(s^p - r^p)$ *and* $c_1 + c_2 = 1$;

(3) *if* $-1/r = -1/s < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r}}{\Gamma(p+1)r^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r}}{\Gamma(p+2)r^{p+2}}, \quad 0 < u < \infty,$$

where $c_1, c_2 \geq 0$ *and* $c_1 + c_2 = 1$.

Proof Observe that it follows immediately for X/U being $\mathcal{B}e(p, 1)$ distributed, support $(X) = [0, T] \subset [0, \infty)$ (if $T = \infty$, then we write $[0, T)$ instead of $[0, T]$), and $\inf\{u : F_U(u) = 1\} = T$. Note also that since $X \leq U$, a.s., if $T < \infty$, it follows that

$$E(a_1 e^{\alpha_1(U-T)} + a_2 e^{\alpha_2(U-T)}|X = T) = a_1 + a_2. \tag{12}$$

By letting $g(u) = u^{-p} f_U(u)$, $u > 0$, (3) and (11) imply

$$\int_x^T a_1 e^{\alpha_1(u-x)} g(u)du + \int_x^T a_2 e^{\alpha_2(u-x)} g(u)du = b \int_x^T g(u)du, \quad 0 < x < T. \tag{13}$$

(i) $b = 0$. From (13), it yields $a_1 a_2 < 0$. Taking the derivatives of both sides of (13) with respect to x, we obtain

$$(\alpha_1 - \alpha_2)a_2 \int_x^T e^{\alpha_2(u-x)} g(u)du = (a_1 + a_2)g(x), \quad 0 < x < T. \tag{14}$$

Obviously, $(\alpha_1 - \alpha_2)(a_1 + a_2)a_2 > 0$. Suppose $T < \infty$. From (11) and (12) we obtain the contradiction $a_1 + a_2 = b = 0$. Hence $T = \infty$. Again taking the derivatives of both sides of (14) with respect to x, we obtain

$$(a_1 + a_2)g'(x) + (\alpha_1 a_2 + \alpha_2 a_1)g(x) = 0, \quad 0 < x < \infty. \tag{15}$$

Obviously, $\alpha_1 a_2 + \alpha_2 a_1 \neq 0$ follows. Now solving (15) yields

$$g(x) = ke^{-(\alpha_1 a_2 + \alpha_2 a_1)x/(a_1 + a_2)}, \quad 0 < x < \infty,$$

where $k > 0$ is a constant, and then

$$f_U(u) = ku^p e^{-(\alpha_1 a_2 + \alpha_2 a_1)u/(a_1 + a_2)}, \quad 0 < u < \infty.$$

Consequently, the assertion (i) can be obtained immediately.

(ii) $b > 0$ and $a_1 + a_2 = b$. After taking the derivatives of both sides of (13) with respect to x twice, we obtain

$$\alpha_1 \alpha_2 \,(a_1 + a_2) \int_x^T g(u)du = (\alpha_1 a_1 + \alpha_2 a_2)\, g(x), \quad 0 < x < T. \tag{16}$$

It yields that $(\alpha_1 a_1 + \alpha_2 a_2)\alpha_1 \alpha_2 > 0$. Again taking the derivatives of both sides of (16) with respect to x, it arrives at

$$(\alpha_1 a_1 + \alpha_2 a_2)g'(x) + \alpha_1 \alpha_2 (a_1 + a_2)g(x) = 0, \quad 0 < x < T,$$

which yields the solution

$$g(x) = ke^{-\alpha_1 \alpha_2 (a_1 + a_2)x/(\alpha_1 a_1 + \alpha_2 a_2)}, \quad 0 < x < T, \tag{17}$$

where $k > 0$ is a constant. Substituting (17) into (16) for $T < \infty$, we have

$$k(\alpha_1 a_1 + \alpha_2 a_2)e^{-\alpha_1\alpha_2(a_1+a_2)T/(\alpha_1 a_1+\alpha_2 a_2)} = 0, \quad 0 < x < T,$$

and, consequently, $k = 0$. Hence T cannot be finite. Therefore, (17) in turn implies

$$f_U(u) = ku^p e^{-\alpha_1\alpha_2(a_1+a_2)u/(\alpha_1 a_1+\alpha_2 a_2)}, \quad 0 < u < \infty.$$

This proves the assertion (ii).

(iii) $b > 0$ and $a_1 + a_2 \neq b$. Suppose $T < \infty$. Again from (11) and (12), we obtain the contradiction $a_1 + a_2 = b$. Hence $T = \infty$. After taking the derivatives of both sides of (13) with respect to x three times, we obtain the second order linear homogeneous differential equation

$$(b-a_1-a_2)g''(x)+(b\alpha_1+b\alpha_2-\alpha_1 a_2-\alpha_2 a_1)g'(x)+\alpha_1\alpha_2 b g(x)=0, \quad 0<x<\infty.$$

The remainder of the proof of assertion (iii) is straightforward hence is omitted. □

A natural extension is to use

$$E(a_1e^{\alpha_1(U-X)} + a_2e^{\alpha_2(U-X)} + a_3e^{\alpha_3(U-X)}|X) = b, \tag{18}$$

where a_1, a_2, a_3, α_1, α_2, α_3 and b are real constants, to determine the distribution of U.

Theorem 3 *Assume* (18) *holds for* $\alpha_1\alpha_2\alpha_3 a_1 a_2 a_3 \neq 0$ *and* $b \geq 0$. *Assume additionally that* $f_U(u)$ *is continuous with support* $[0, T]$, *where* $0 < T \leq \infty$. *Then only the following cases are possible*:

(i) $b = 0$.
Let the equation

$$\begin{aligned}(a_1 + a_2 + a_3)x^2 &+ (\alpha_1 a_2 + \alpha_2 a_1 + \alpha_1 a_3 + \alpha_3 a_1 + \alpha_2 a_3 + \alpha_3 a_2)x \\ &+ \alpha_1\alpha_2 a_3 + \alpha_1\alpha_3 a_2 + \alpha_2\alpha_3 a_1 = 0\end{aligned}$$

have real roots $-1/r_1$ *and* $-1/s_1$, *where* $-1/r_1 \leq -1/s_1$. *Then* $T = \infty$ *and*

(1-1) *if* $-1/r_1 < 0$ *and* $1/s_1 > 0$,*then* U *is* $\Gamma(p + 1, r_1)$ *distributed*;

(1-2) *if* $-1/r_1 < -1/s_1 < 0$, *then*

$$f_U(u) = c_1\frac{u^p e^{-u/r_1}}{\Gamma(p+1)r_1^{p+1}} + c_2\frac{u^p e^{-u/s_1}}{\Gamma(p+1)s_1^{p+1}}, \quad 0 < u < \infty,$$

where $0 \leq c_2 \leq s_1^p/(s_1^p - r_1^p)$ *and* $c_1 + c_2 = 1$;

(1-3) *if* $-1/r_1 = -1/s_1 < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_1}}{\Gamma(p+1) r_1^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_1}}{\Gamma(p+2) r_1^{p+2}}, \quad 0 < u < \infty,$$

where $c_1, c_2 \geq 0$ *and* $c_1 + c_2 = 1$.

(ii) $b > 0$ *and* $a_1 + a_2 + a_3 = b$.
For $\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 \neq 0$, *let the equation*

$$(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) x^2 + (\alpha_1 \alpha_2 a_1 + \alpha_1 \alpha_2 a_2 + \alpha_1 \alpha_3 a_1 + \alpha_1 \alpha_3 a_3 + \alpha_2 \alpha_3 a_2 + \alpha_2 \alpha_3 a_3) x + \alpha_1 \alpha_2 \alpha_3 (a_1 + a_2 + a_3) = 0$$

have real roots $-1/r_2$ *and* $-1/s_2$, *where* $-1/r_2 \leq -1/s_2$. *Then* $T = \infty$ *and*

(2-1) *if* $-1/r_2 < 0$ *and* $1/s_2 > 0$, *then* U *is* $\Gamma(p+1, r_2)$ *distributed*;
(2-2) *if* $-1/r_2 < -1/s_2 < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_2}}{\Gamma(p+1) r_2^{p+1}} + c_2 \frac{u^p e^{-u/s_2}}{\Gamma(p+1) s_2^{p+1}}, \quad 0 < u < \infty,$$

where $0 \leq c_2 \leq s_2^p/(s_2^p - r_2^p)$ *and* $c_1 + c_2 = 1$;
(2-3) *if* $-1/r_2 = -1/s_2 < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_2}}{\Gamma(p+1) r_2^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_2}}{\Gamma(p+2) r_2^{p+2}}, \quad 0 < u < \infty,$$

where $c_1, c_2 \geq 0$ *and* $c_1 + c_2 = 1$.

For $\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0$, U *is* $\Gamma(p+1, \beta)$ *distributed, where* $\beta = (\alpha_1 \alpha_2 a_1 + \alpha_1 \alpha_2 a_2 + \alpha_1 \alpha_3 a_1 + \alpha_1 \alpha_3 a_3 + \alpha_2 \alpha_3 a_2 + \alpha_2 \alpha_3 a_3)/\alpha_1 \alpha_2 \alpha_3 (a_1 + a_2 + a_3) > 0$.

(iii) $b > 0$ *and* $a_1 + a_2 + a_3 \neq b$.
Let the equation

$$(b - a_1 - a_2 - a_3) x^3 - (\alpha_1 a_2 + \alpha_1 a_3 + \alpha_2 a_1 + \alpha_2 a_3 + \alpha_3 a_1 + \alpha_3 a_2 - (\alpha_1 + \alpha_2 + \alpha_3) b) x^2 + ((\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) b - \alpha_1 \alpha_2 a_3 - \alpha_1 a_2 \alpha_3 - a_1 \alpha_2 \alpha_3) x + \alpha_1 \alpha_2 \alpha_3 b = 0$$

have roots $-1/r_3$, $-1/s_3$ *and* $-1/t_3$, *where* $-1/r_3 \leq -1/s_3 \leq -1/t_3$ *if all are real numbers. Then* $T = \infty$ *and*

(3-1) *if* $-1/r_3 < 0$ *and* $-1/t_3 \geq -1/s_3 > 0$, *or* $-1/t_3$ *and* $-1/s_3$ *are nonreal, then* U *is* $\Gamma(p+1, r_3)$;
(3-2) *if* $-1/r_3 < -1/s_3 < 0$ *and* $-1/t_3 > 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_3}}{\Gamma(p+1) r_3^{p+1}} + c_2 \frac{u^p e^{-u/s_3}}{\Gamma(p+1) s_3^{p+1}}, \quad 0 < u < \infty,$$

where $0 \leq c_2 \leq s_3^p/(s_3^p - r_3^p)$ *and* $c_1 + c_2 = 1$;

(3-3) *if* $-1/r_3 = -1/s_3 < 0$ *and* $-1/t_3 > 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_3}}{\Gamma(p+1) r_3^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_3}}{\Gamma(p+2) r_3^{p+2}}, \quad 0 < u < \infty,$$

where $c_1,\ c_2 \geq 0$ *and* $c_1 + c_2 = 1$;

(3-4) *if* $-1/r_3 < -1/s_3 < -1/t_3 < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_3}}{\Gamma(p+1) r_3^{p+1}} + c_2 \frac{u^p e^{-u/s_3}}{\Gamma(p+1) s_3^{p+1}} + c_3 \frac{u^p e^{-u/t_3}}{\Gamma(p+1) t_3^{p+1}}, \quad 0 < u < \infty,$$

where $c_1 + c_2 + c_3 = 1$, *such that* $f_U(u) > 0,\ 0 < u < \infty$;

(3-5) *if* $-1/r_3 = -1/s_3 < -1/t_3 < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_3}}{\Gamma(p+1) r_3^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_3}}{\Gamma(p+2) r_3^{p+2}} + c_3 \frac{u^p e^{-u/t_3}}{\Gamma(p+1) t_3^{p+1}}, \quad 0 < u < \infty,$$

where $c_1 + c_2 + c_3 = 1$, *such that* $f_U(u) > 0,\ 0 < u < \infty$;

(3-6) *if* $-1/r_3 < -1/s_3 = -1/t_3 < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_3}}{\Gamma(p+1) r_3^{p+1}} + c_2 \frac{u^p e^{-u/s_3}}{\Gamma(p+1) s_3^{p+1}} + c_3 \frac{u^{p+1} e^{-u/s_3}}{\Gamma(p+2) s_3^{p+2}}, \quad 0 < u < \infty,$$

where $c_1 + c_2 + c_3 = 1$, *such that* $f_U(u) > 0,\ 0 < u < \infty$;

(3-7) *if* $-1/r_3 = -1/s_3 = -1/t_3 < 0$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r_3}}{\Gamma(p+1) r_3^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r_3}}{\Gamma(p+2) r_3^{p+2}} + c_3 \frac{u^{p+2} e^{-u/r_3}}{\Gamma(p+3) r_3^{p+3}}, \quad 0 < u < \infty,$$

where $c_1 + c_2 + c_3 = 1$, *such that* $f_U(u) > 0,\ 0 < u < \infty$.

Proof Again if $T < \infty$, then

$$E(a_1 e^{\alpha_1(U-T)} + a_2 e^{\alpha_2(U-T)} + a_3 e^{\alpha_3(U-T)} | X = T) = a_1 + a_2 + a_3. \tag{19}$$

By letting $g(u) = u^{-p} f_U(u)$, $u > 0$, (3) and (18) imply

$$a_1 \int_x^T e^{\alpha_1(u-x)} g(u) du + a_2 \int_x^T e^{\alpha_2(u-x)} g(u) du + a_3 \int_x^T e^{\alpha_3(u-x)} g(u) du$$
$$= b \int_x^T g(u) du, \quad 0 < x < T. \tag{20}$$

(i) $b = 0$. Suppose $T < \infty$. Then (18) and (19) imply $a_1 + a_2 + a_3 = b = 0$. Taking the derivatives of both sides of (20) with respect to x twice yields

$$(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(a_1 + a_2) \int_x^T e^{\alpha_3(u-x)} g(u) du = A_1 g(x), \quad 0 < x < T, \tag{21}$$

where $A_1 = \alpha_1 a_1 + \alpha_2 a_2 - \alpha_3 a_1 - \alpha_3 a_2$. It follows $(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)(a_1 + a_2)A_1 > 0$. Again taking the derivatives of both sides of (21) with respect to x yields

$$A_1 g'(x) + B_1 g(x) = 0, \quad 0 < x < T, \tag{22}$$

where $B_1 = \alpha_1\alpha_2 a_1 + \alpha_1\alpha_2 a_2 - \alpha_1\alpha_3 a_2 - \alpha_2\alpha_3 a_1$. Obviously, $B_1 \neq 0$. Hence

$$g(x) = ke^{-A_1 x/B_1}, \quad 0 < x < T, \tag{23}$$

where $k > 0$ is a constant. Substituting (23) into (21), we find $e^{(\alpha_3 - B_1/A_1)T} = 0,\ 0 < x < T$, which contradicts the assumption $T < \infty$. Hence $T = \infty$. Now taking the derivatives of both sides of (20) with respect to x three times yields

$$\begin{aligned}(a_1 + a_2 + a_3)g''(x) &+ (\alpha_1 a_2 + \alpha_2 a_1 + \alpha_1 a_3 + \alpha_3 a_1 + \alpha_2 a_3 + \alpha_3 a_2)g'(x)\\ &+ (\alpha_1\alpha_2 a_3 + \alpha_1\alpha_3 a_2 + \alpha_2\alpha_3 a_1)g(x) = 0, \quad 0 < x < \infty.\end{aligned}$$

The remainder of the proof of assertion (i) is straightforward hence is omitted.

(ii) $b > 0$ and $a_1 + a_2 + a_3 = b$. Taking the derivatives of both sides of (20) with respect to x twice yields

$$\begin{aligned}&\alpha_1\alpha_2 (a_1 + a_2 + a_3) \int_x^T g(u)du - (\alpha_2 - \alpha_3)(\alpha_1 a_3 - \alpha_3 a_3)\, e^{-\alpha_3 x} \int_x^T e^{\alpha_3 u} g(u)du\\ &= (\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3)\, g(x), \quad 0 < x < T,\end{aligned} \tag{24}$$

and then taking the derivatives of both sides of (24) with respect to x yields

$$C_2 \int_x^T g(u)du = A_2 g'(x) + B_2 g(x), \quad 0 < x < T, \tag{25}$$

where $A_2 = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$, $B_2 = \alpha_1\alpha_2 a_1 + \alpha_1\alpha_2 a_2 + \alpha_1\alpha_3 a_1 + \alpha_1\alpha_3 a_3 + \alpha_2\alpha_3 a_2 + \alpha_2\alpha_3 a_3$ and $C_2 = \alpha_1\alpha_2\alpha_3(a_1 + a_2 + a_3)$. Finally after taking the derivatives of both sides of (25) with respect to x, we obtain

$$A_2 g''(x) + B_2 g'(x) + C_2 g(x) = 0, \quad 0 < x < T, \tag{26}$$

which yields the solution

$$g(x) = \begin{cases} k_1 e^{\frac{-B_2 + \sqrt{B_2^2 - 4A_2C_2}}{2A_2}x} + k_2 e^{\frac{-B_2 - \sqrt{B_2^2 - 4A_2C_2}}{2A_2}x}, & \text{if } A_2 \neq 0,\ B_2^2 - 4A_2C_2 > 0;\\ k_3 e^{-\frac{B_2}{2A_2}x} + k_4 x e^{-\frac{B_2}{2A_2}x}, & \text{if } A_2 \neq 0,\ B_2^2 - 4A_2C_2 = 0;\\ k_5 e^{-\frac{C_2}{B_2}x}, & \text{if } A_2 = 0. \end{cases} \tag{27}$$

where $k_1, \ldots, k_5$ are constants. For $A_2 = 0$, substituting $g(x)$ as in (27) into (25) yields $k_5 B_2 e^{-C_2 T/B_2} = 0,\ 0 < x < T$. This in turn implies that $T = \infty$. Similarly, for each of the other two situations in (27), by substituting $g(x)$ into (24), will also lead to $T = \infty$. Having obtained $T = \infty$, the remainder of the assertions in (ii) follow immediately.

(iii) $b > 0$ and $a_1 + a_2 + a_3 \neq b$. Again if $T < \infty$, then (18) and (19) lead to $a_1 + a_2 + a_3 = b$. This contradiction implies $T = \infty$. Taking the derivatives of both sides of (20) with respect to x four times yields the third order linear homogeneous differential equation

$$\begin{aligned}&(b - a_1 - a_2 - a_3)g'''(x) - (\alpha_1 a_2 + \alpha_1 a_3 + \alpha_2 a_1 + \alpha_2 a_3 + \alpha_3 a_1 + \alpha_3 a_2\\ &\quad - (\alpha_1 + \alpha_2 + \alpha_3)b)g''(x) + ((\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)b - \alpha_1\alpha_2 a_3 - \alpha_1 a_2 \alpha_3\\ &\quad - a_1\alpha_2\alpha_3)g'(x) + \alpha_1\alpha_2\alpha_3 b g(x) = 0, \quad 0 < x < \infty.\end{aligned}$$

The remainder of the proof of assertion (iii) is straightforward hence is omitted. □

Remark 1 It is known that every cubic equation of the form

$$Ax^3 + Bx^2 + Cx + D = 0, \tag{28}$$

where $A \neq 0$, B, C, D are real constants, has at least one real root. Denote the discriminant of (28) by

$$\Delta = 18ABCD - 4B^3 D + B^2 C^2 - 4AC^3 - 27A^2 D^2. \tag{29}$$

Then (29) can be used to distinguish the nature of the roots that (28) has three distinct real roots if $\Delta > 0$; a multiple root and all its roots are real if $\Delta = 0$; one real root and two nonreal complex conjugate roots if $\Delta < 0$.

Remark 2 As in Lemma 1, it is desired to investigate necessary and sufficient conditions for linear combinations of three gamma distributed pdfs which appear in Theorem 3 to be a pdf. Similar to Theorem 2, in Case (iii) of Theorem 3, the coefficients c_1, c_2 and c_3 do not need to satisfy $0 \le c_1, c_2, c_3 \le 1$. We give example for each situation below.

(i) Let $c_1 = -3/5$, $c_2 = 3/2$, $c_3 = 1/10$, $p = 0$, $r_3 = 1$, $s_3 = 2$, $t_3 = 3$ in assertion (3-4), then

$$f_U(u) = -\frac{3}{5}e^{-u} + \frac{3}{2}\left(\frac{1}{2}e^{-\frac{u}{2}}\right) + \frac{1}{10}\left(\frac{1}{3}e^{-\frac{u}{3}}\right) > 0, \quad 0 < u < \infty; \tag{30}$$

(ii) Let $c_1 = -1/20$, $c_2 = 9/10$, $c_3 = 3/20$, $p = 0$, $r_3 = s_3 = 1$, $t_3 = 3$ in assertion (3-5), then

$$f_U(u) = -\frac{1}{20}e^{-u} + \frac{9}{10}ue^{-u} + \frac{3}{20}\left(\frac{1}{3}e^{-\frac{u}{3}}\right) > 0, \quad 0 < u < \infty; \tag{31}$$

(iii) Let $c_1 = -1/20$, $c_2 = 9/10$, $c_3 = 3/20$, $p = 0$, $r_3 = 1$, $s_3 = t_3 = 3$ in assertion (3-6), then

$$f_U(u) = -\frac{1}{20}e^{-u} + \frac{9}{10}\left(\frac{1}{3}e^{-\frac{u}{3}}\right) + \frac{3}{20}\left(\frac{1}{9}ue^{-\frac{u}{3}}\right) > 0, \quad 0 < u < \infty; \tag{32}$$

(iv) Let $c_1 = 3/10$, $c_2 = -1$, $c_3 = 17/10$, $p = 0$, $r_3 = s_3 = t_3 = 1$ in assertion (3-7), then

$$f_U(u) = \frac{3}{10}e^{-u} - ue^{-u} + \frac{17}{10}\left(\frac{1}{2}u^2e^{-u}\right) > 0, \quad 0 < u < \infty. \tag{33}$$

However, we do not have a lemma which is parallel to Lemma 1 for linear combinations of three gamma distributed pdfs to be a pdf.

Inspired by the previous theorems, it is natural to ask whether

$$E\left(\sum_{i=1}^{n} a_i e^{i(U-X)}|X\right) = b$$

can be used to determine the distributions of U. The general case is too cumbersome, for the special case $a_1 = \cdots = a_n = 1$ and $b > n$, a characterization can be obtained. We omit the statement of this result.

In Huang and Chang (2007), they also used the condition $E((U-X)^n|X) = b$ to characterize the distribution of U. For our present situation, it is easy to see that the solution of U of $E(e^{n(U-X)}|X) = b$, can be obtained immediately from the solution of $E(e^{U-X}|X) = b$. We omit the details here. In Theorem 2, along the lines of the present proof, it can be shown the result still holds, if α_1, α_2, a_1, a_2 are allowed to be complex numbers such that $\alpha_1\alpha_2$, a_1a_2, $\alpha_1+\alpha_2$, a_1+a_2 and $\alpha_2a_1+\alpha_1a_2$ are real numbers (b is still a real number). Then by letting $\alpha_1 = i$, $\alpha_2 = -i$, $a_1 = -i/2$ and $a_2 = i/2$, we have the following consequence immediately.

Corollary 1 *Assume*

$$E(\sin(U-X)|X) = b \tag{34}$$

holds for constant $b > 0$. *Assume additionally that* $f_U(u)$ *is continuous with the support* $[0, T]$, *where* $0 < T \leq \infty$. *Let the equation*

$$bx^2 + x + b = 0 \tag{35}$$

have roots $-1/r = (-1-\sqrt{1-4b^2})/2b$ *and* $-1/s = (-1+\sqrt{1-4b^2})/2b$. *Then* $T = \infty$ *and there are only the following possible cases*:

(i) *if* $0 < b < 1/2$, *then*

$$f_U(u) = c_1\frac{u^pe^{-u/r}}{\Gamma(p+1)r^{p+1}} + c_2\frac{u^pe^{-u/s}}{\Gamma(p+1)s^{p+1}}, \quad 0 < u < \infty, \tag{36}$$

where $0 \leq c_2 \leq s^{p+1}/(s^{p+1}-r^{p+1})$ *and* $c_1 + c_2 = 1$.

(ii) *if* $b = 1/2$, *then*

$$f_U(u) = c_1 \frac{u^p e^{-u/r}}{\Gamma(p+1) r^{p+1}} + c_2 \frac{u^{p+1} e^{-u/r}}{\Gamma(p+2) r^{p+2}}, \quad 0 < u < \infty,$$

where $c_1,\ c_2 \geq 0$ *and* $c_1 + c_2 = 1$.

We omit the statement of a similar characterizing result based on $E(\cos(U - X)|X) = b$.

4 Conclusion

In this work, we characterized the distribution of $(U,\ X)$ by $E(h(U - X)|X) = b$, where h is allowed to be an exponential or trigonometric function of $U - X$. It is expected that there are some other functions of $h(U,\ X)$ can be used to characterize the distribution of $(U,\ X)$.

For example, in Theorem 1, let $\alpha = 1$, then (4) becomes $E(e^{U-X}|X) = b$, or $E(e^U|X) = be^X$. This is a special form of $E(e^U|X) = ae^X + b$. It can be shown the distribution of $(U,\ X)$ can be determined under the assumption $E(e^U|X) = ae^X + b$. We omit the details here.

Acknowledgements The authors would like to thank the referee for many constructive and useful suggestions which enable them to improve the original version. Support for this research by Wen-Jang Huang was provided in part by the National Science Council of the Republic of China, Grant No. NSC 96-2118-M-390-001-MY2 and NSC 98-2118-M-390-001-MY2. Support for this research by Nan-Cheng Su was provided in part by the National Science Council of the Republic of China, Grant No. NSC 99-2118-M-006-005.

References

Bobecka K, Wesolowski J (2002) Three dual regression schemes for the Lukacs theorem. Metrika 56:43–54
Bobecka K, Wesolowski J (2008) Bivariate Lukacs type regression characterizations. In: New developments in applied statistics research, Chap. 5. NOVA Science Publishers, Inc., New York, pp 55–63
Bolger EM, Harkness WL (1965) A characterization of some distributions by conditional moments. Ann Math Stat 36:703–705
Chou CW, Huang WJ (2003) Characterizations of the gamma distribution via conditional moments. Sankhyā 65:271–283
Gupta AK, Wesolowski J (1997) Uniform mixtures via posterior means. Ann Inst Stat Math 49:171–180
Gupta AK, Wesolowski J (2001) Regressional identifiability and identification for beta mixtures. Stat Decis 19:71–82
Hall WJ, Simons G (1969) On characterizations of the gamma distribution. Sankhyā A 31:385–390
Huang WJ, Chang SH (2007) On some characterizations of the mixture of gamma distributions. J Stat Plan Inference 137:2964–2974
Huang WJ, Chou CW (2004) Characterizations of the gamma distribution via conditional expectations. Technical Report, Department of Applied Mathematics, National Kaohsiung University
Huang WJ, Liu, C-H (2006) Some characterization results based on certain conditional expectations. Technical Report, Department of Applied Mathematics, National University of Kaohsiung
Huang WJ, Su JC (1997) On a study of renewal process connected with certain conditional moments. Sankhyā A 59:28–41
Huang WJ, Wong HL (1998) On a study of beta and geometric mixtures by conditional moments. Technical Report, Department of Applied Mathematics, National Sun Yat-Sen University

Li SH, Huang WJ, Huang MNL (1994) Characterizations of the Poisson process as a renewal process via two conditional moments. Ann Inst Stat Math 46:351–360

Lukacs E (1955) A characterization of the gamma distribution. Ann Math Stat 26:319–324

Meszaros F (2010) A functional equation and its application to the characterization of gamma distributions. Aequ Math 79:53–59

Wesolowski J (1989) A characterization of the gamma process by conditional moments. Metrika 36:299–309

Wesolowski J (1990) A constant regression characterization of the gamma law. Adv Appl Probab 22:488–490