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A Study of Some Different Concepts of Symmetry on the Real Line

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Recently, different concepts of symmetry on R^+ such as R-symmetry, log-symmetry, and double symmetry are studied. Analogous concepts and their properties of these symmetries on R will be studied in this work. Based on skewing representation and previous studies, characterizations of double symmetry on R will be given. Among others, some interesting examples of the so-called I-symmetry, that is the analogue of log-symmetry on R, will also be presented.

Keywords Characterization; Double symmetry; I-symmetry; Log-symmetry; R-symmetry; Skewing representation.

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1. Introduction

A random variable (r.v.) X is said to be symmetric about a constant μ, if $X-\mu$ and $\mu-X$ have the same distribution, denote it by $X-\mu \stackrel{d}{=} \mu-X$. If $\mu=0$, we simply say X is symmetric. Recently, different concepts of symmetry on R^+ are introduced and investigated. Mudholkar and Wang (2007) gave the definition of R-symmetric distribution on R^+. According to their definition, a positive r.v. X with probability density function (pdf) f_X is said to be R-symmetric about the R-center θ, where $\theta>0$, if $f_X(\theta x)=f_X(\theta/x)$, $x>0$. By using the Cauchy-Schlömilch transformation, Baker (2008) provided an efficient way to construct R-symmetric distributions on R^+, a lot of examples through this transformation were then given. Later, Chaubey et al. (2010) offered a similar theorem to clarify the correspondence between the ordinary symmetric distributions and the R-symmetric distributions on R^+. For R-symmetric and unimodal r.v.'s on R^+, Mudholkar and Wang (2007) and Chaubey et al. (2010) gave representations of their pdf's. Mudholkar and

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Wang (2010) showed that the product-convolution of two R-symmetric unimodal distributions on R^+ is still R-symmetric unimodal on R^+.

Earlier, Seshadri (1965) studied another nonordinary symmetry. He characterized those non negative r.v.'s X's such that $X \stackrel{d}{=} 1/X$. Since $\log X \stackrel{d}{=} -\log X$, this was referred as log-symmetry by Jones (2008). When X is a non negative r.v., Jones (2008) studied R-symmetry and log-symmetry about δ, $\delta > 0$. The latter is $X/\delta \stackrel{d}{=} \delta/X$, which is equivalent to the ordinary symmetry about $\log \delta$ of the r.v. $\log X$. Jones (2008) also pointed out that when X has a pdf f, $X/\delta \stackrel{d}{=} \delta/X$ is equivalent to $x^2 f_X(\delta x) = f_X(\delta/x)$, $x > 0$. For non negative r.v.'s link between R-symmetry and log-symmetry was studied by Chaubey et al. (2010).

Jones and Arnold (2008) studied the r.v.'s on R^+ which are both R-symmetric and log-symmetric, the so-called double symmetry. An example of doubly symmetric distribution is lognormal. They also characterized the class of absolutely continuous distributions on R^+ that are doubly symmetric. It turns out to be a proper subset of absolutely continuous distributions on R^+ which are moment-equivalent to the lognormal distribution.

In this work, we will investigate natural analog of the concepts of R-symmetry, log-symmetry, and double symmetry on R. More precisely, we call the analog of log-symmetry on R as I-symmetry. Here, "I" stands for "inverse." Throughout this work, unless it is stated, every r.v. is assumed to follow an absolutely continuous distribution. Also for an r.v., say X, let f_X denote the pdf of X.

First, we give definitions of those symmetries mentioned above.

Definition 1.1. An r.v. X on R is said to be R-symmetric about the R-center θ, where $\theta > 0$, if

$$f_X(\theta x) = f_X\left(\frac{\theta}{x}\right), \quad x \in R\backslash\{0\}, \tag{1}$$

or equivalently, if

$$f_X(x) = f_X\left(\frac{\theta^2}{x}\right), \quad x \in R\backslash\{0\}.$$

It can be shown easily from (1), if X is R-symmetric on R, then $f_X(0) = 0$.

Definition 1.2. An r.v. X on R is said to be I-symmetric about δ, where $\delta > 0$, if

$$\frac{X}{\delta} \stackrel{d}{=} \frac{\delta}{X},$$

or equivalently, if

$$x^2 f_X(\delta x) = f_X\left(\frac{\delta}{x}\right), \quad x \in R\backslash\{0\}. \tag{2}$$

Definition 1.3. An r.v. X on R is said to be doubly symmetric about (θ, δ), where $\theta, \delta > 0$, if X is both R-symmetric about θ and I-symmetric about δ.

Through this extension of symmetric concepts from R^+ to R, some new distributional properties of interest are established. In Sec. 2, based on a mixture representation, we investigate the relationship between double symmetry on R^+ and double symmetry on R. In Secs. 3, 4, and 5, we give some elementary propositions of R-symmetry, I-symmetry, and double symmetry, respectively. Next, in Sec. 6, we characterize the doubly symmetric distributions on R, and in Sec. 7, we give some further study about R-symmetry and I-symmetry. It turns out the results are much related to the skewing representation of a pdf Hence, these results will shed some insight into skew distributions.

2. Preliminary Results

Let X be an r.v. on R. Obviously, f_X can have the following mixture representation:

$$f_X(x) = af_1(x)I_{\{x>0\}} + (1-a)f_2(-x)I_{\{x\leq 0\}}, \tag{3}$$

where I is the indicator function, and

$$a = P(X > 0) = \int_0^\infty f_X(x)dx. \tag{4}$$

Then both $f_1(x) = f_X(x)/a$ and $f_2(x) = f_X(-x)/(1-a)$ are pdf's on R^+. Note that f_1 is defined to be 0 if $a = 0$, and f_2 is defined to be 0 if $a = 1$. It can be seen if $a = 0$, then X is on R^-; if $a = 1$, then X is on R^+. Based on the above representation, we have the following simple lemma.

Lemma 2.1. *Let X be an r.v. on R with* $0 < a < 1$, *where a is defined in* (4). *Then X is doubly symmetric about* (θ, δ) *if and only if both* f_1 *and* f_2 *in* (3) *are doubly symmetric about* (θ, δ).

By Lemma 2.1, we have the following consequence.

Corollary 2.1. *Let X be an r.v. on R with* $0 < a < 1$, *where a is defined in* (4). *Then X is R-symmetric about* θ *if and only if both* f_1 *and* f_2 *are R-symmetric about* θ; *X is I-symmetric about* δ *if and only if both* f_1 *and* f_2 *are log-symmetric about* δ.

3. R-Symmetry on *R*

Mudholkar and Wang (2007) investigated the properties of R-symmetry on R^+. It can be shown easily for R-symmetric on R that many similar properties still hold. As an example, for independent nonnegative r.v.'s X and Y which are R-symmetric about θ_1 and θ_2, respectively, Mudholkar and Wang (2007) proved that XY is R-symmetric about $\theta_1\theta_2$. This property also holds for R-symmetry on R. Yet if X is R-symmetric, $1/X$ may not be R-symmetric about any center. Consequently, if the independent r.v.'s X and Y are both R-symmetric, it may happen that X/Y is not R-symmetric about any center. The following is an example.

Example 3.1. Let X and Y be i.i.d. with the common distribution of the root-reciprocal of $IG(1, \lambda)$ (IG stands for inverse Gaussian). That is,

$$f_X(x) = f_Y(x) = \sqrt{\frac{2\lambda}{\pi}} \exp\left(-\frac{\lambda}{2}\left(\frac{1}{x} - x\right)^2\right), \quad x > 0.$$

Then as pointed out by Mudholkar and Wang (2007), both X and Y are R-symmetric in R about 1. The pdf's of the r.v.'s $U = 1/X$, and $V = X/Y$ are given by

$$f_U(u) = \frac{\sqrt{2\lambda}}{\sqrt{\pi u^2}} \exp\left(-\frac{\lambda}{2}\left(\frac{1}{u} - u\right)^2\right), \quad u > 0,$$

and

$$f_V(v) = \frac{\lambda e^{2\lambda}}{\pi v} \int_0^\infty \exp\left(-\frac{\lambda}{2}\left(\frac{1}{v} + v\right)\left(\frac{1}{y} + y\right)\right) dy, \quad v > 0,$$

respectively. Now it can be shown easily that neither of U and V is R-symmetric.

Although an R-symmetric pdf f_X on R^+ may not be unimodal, under certain conditions f_X can be unimodal, see, e.g., Sec. 2.1 of Baker (2008) and Remark 3.2 of Chaubey et al. (2010). Yet if f_X is R-symmetric on R, then certainly it cannot be unimodal. This can be seen by noting $f_X(0) = 0$. Mudholkar and Wang (2007) proved that if X is R-symmetric about θ in R^+, and X is unimodal, then θ is the mode. Similarly, it can be shown if X is R-symmetric about θ in R, and both f_1 and f_2 in (3) are unimodal, then θ is their common mode. Hence, after reflecting of f_2 through the origin, f_X is bimodal. On the other hand, Mudholkar and Wang (2010) showed that on R^+ the product-convolution of two R-symmetric unimodal distributions is still R-symmetric unimodal. Along the lines of Mudholkar and Wang (2010) and using the representation in (3), we have the following analogous result for R-symmetry on R.

Proposition 3.1. *Let the independent r.v.'s X and Y be R-symmetric bimodal with the R-centers θ_1 and θ_2, respectively. Then XY is R-symmetric and bimodal with the R-center $\theta_1\theta_2$.*

4. I-Symmetry

We now give some simple properties of I-symmetry. The proofs are similar to those of the situation of log-symmetry on R^+, hence are omitted.

Proposition 4.1. *Let the r.v. X on R be I-symmetric about δ. Then $P(-\delta < X \leq \delta) = 1/2$. Also for every constant $a > 0$, aX is I-symmetric about $a\delta$.*

Proposition 4.2. *Let the independent r.v.'s X and Y on R be I-symmetric about δ_1 and δ_2, respectively. Then XY is I-symmetric about $\delta_1\delta_2$.*

In contrast to R-symmetry, for I-symmetry we have the following result.

Proposition 4.3. *Let the independent r.v.'s X and Y on R be I-symmetric about δ_1 and δ_2, respectively. Then X/Y is I-symmetric about δ_1/δ_2. In particular, $1/X$ is I-symmetric about $1/\delta_1$.*

5. Double Symmetry

In this section some simple properties of double symmetry will be given. Again the proofs are all omitted.

Proposition 5.1. *Let the r.v. X on R be doubly symmetric about (θ, δ). Then for any constant $a > 0$, aX is doubly symmetric about $(a\theta, a\delta)$.*

Proposition 5.2. *Let the independent r.v.'s X and Y on R be doubly symmetric about (θ_2, δ_2) and (θ_2, δ_2), respectively. Then XY is doubly symmetric about $(\theta_1\theta_2, \delta_1\delta_2)$.*

The following proposition points out that the ratio of two independent doubly symmetric r.v.'s is still doubly symmetric.

Proposition 5.3. *Let the independent r.v.'s X and Y on R be doubly symmetric about (θ_1, δ_1) and (θ_2, δ_2), respectively. Then X/Y is doubly symmetric about $(\theta_1\theta_2/\delta_2^2, \delta_1/\delta_2)$. In particular, $1/X$ is doubly symmetric about $(\theta_1/\delta_1^2, 1/\delta_1)$.*

6. Main Results

Based on the study of univariate skew normal distribution, see, e.g., Azzalini (1985), for a pdf f_X, it can also be represented as

$$f_X(x) = 2f(x)G(x), \quad x \in R, \tag{5}$$

where

$$f(x) = \frac{1}{2}(f_X(x) + f_X(-x)) \tag{6}$$

is a symmetric pdf, and

$$G(x) = \frac{f_X(x)}{f_X(x) + f_X(-x)} = \begin{cases} af_1(x)/(af_1(x) + (1-a)f_2(x)), & x > 0, \\ (1-a)f_2(-x)/(af_1(-x) + (1-a)f_2(-x)), & x \le 0, \end{cases} \tag{7}$$

is a skewing function, that is $G(x) \geq 0$ and $G(x) + G(-x) = 1$, $x \in R$. In this section, we will characterize double symmetry through skewing representation.

As the skew representation plays an important role in characterization of double symmetry, we are interested in knowing is it possible that f is not doubly symmetric, yet f_X is doubly symmetric? The next lemma will answer this question.

Lemma 6.1. *Let the r.v. X on R be doubly symmetric about (θ, δ). Then the f in (5) is also doubly symmetric about (θ, δ).*

Jones and Arnold (2008) characterized the class of absolutely continuous distributions on R^+ which are doubly symmetric. By using their result and the skewing representation of a distribution as in (5), we have the following characterization of the double symmetry on R.

Theorem 6.1. *Let the r.v. X on R be doubly symmetric about* (θ, δ). *Let* $k = \delta/\theta$. *Also, let* f_X *be represented as in* (5). *Then* f *has the form*

$$f(x) \propto \sum_{i=-\infty}^{\infty} \theta^{-2i} k^{2i(i+1)} x^{2i} |x| \omega(\theta^{-2} k^{4(i-1)} x^2) I(\theta k^{-2i} < |x| \leq \theta k^{2-2i}), \quad x \in R\backslash\{0\}, \quad (8)$$

where ω *is a non negative function on* $(k^{-4}, 1]$ *and chosen to satisfy*

$$\psi(u) = \psi\left(\frac{1}{k^4 u}\right), \quad k^{-4} < u \leq 1, \quad (9)$$

where

$$\psi(u) \equiv u\omega(u), \quad (10)$$

and G *is chosen to satisfy*

$$G(\theta x) = G\left(\frac{\theta}{x}\right), \quad \text{and} \quad G(\delta x) = G\left(\frac{\delta}{x}\right) \quad x \in R\backslash\{0\}. \quad (11)$$

Proof. First from Lemma 6.1, we obtain $X_1 = |X|$ is doubly symmetric. Now by Jones and Arnold (2008),

$$f_{X_1}(x) \propto \sum_{i=-\infty}^{\infty} \theta^{-2i} k^{2i(i+1)} x^{2i+1} \omega(\theta^{-2} k^{4(i-1)} x^2) I(\theta k^{-2i} < x \leq \theta k^{2-2i}), \quad x > 0,$$

where the non negative function ω defined on $(k^{-4}, 1]$ satisfying (9) and (10). Note that $f(x) = f_{X_1}(|x|)/2$, $x \in R$, hence (8) is obtained immediately.

Next due to the doubly symmetric property of X, we have (1) and (2). Then by the representation of (5), (1), and (2) in turn imply

$$2f(\theta x)G(\theta x) = 2f\left(\frac{\theta}{x}\right)G\left(\frac{\theta}{x}\right), \quad (12)$$

and

$$2x^2 f(\delta x)G(\delta x) = 2f\left(\frac{\delta}{x}\right)G\left(\frac{\delta}{x}\right), \quad (13)$$

respectively. Furthermore, from Lemma 6.1, we obtain $f(\theta x) = f(\theta/x)$ and $x^2 f(\delta x) = f(\delta/x)$. These together with (12) and (13) imply (11) immediately. This completes the proof. □

The "if" part of the next theorem is obvious. By Lemma 6.1 and Theorem 6.1, the "only if" part follows.

Theorem 6.2. *Let the pdf of the r.v. X be written as in* (5). *Then X is doubly symmetric about* (θ, δ) *if and only if*

(i) *f is doubly symmetric about* (θ, δ), *and*
(ii) *G satisfies* (11).

We give an illustration of Theorem 6.2.

Example 6.1. Let the pdf of the r.v. X be written as in (5), where

$$f(x) = \frac{1}{2\sqrt{2\pi}\sigma|x|}\exp\left(-\frac{(\log|x|-\mu)^2}{2\sigma^2}\right),$$

and

$$G(x) = \frac{1}{2} + \frac{\varepsilon}{2}\text{sgn}(x)\cos\left(\frac{2\pi(\log|x|-\mu)}{\sigma^2}\right), \quad |\varepsilon| \leq 1.$$

Note that $X_1 = |X|$ has Log-$\mathcal{N}(\mu, \sigma^2)$ distribution, which is doubly symmetric about $(e^{\mu-\sigma^2}, e^{\mu})$. From Lemma 2.1, f is doubly symmetric about $(\theta, \delta) = (e^{\mu-\sigma^2}, e^{\mu})$. Also, it can be checked easily that $G(e^{\mu-\sigma^2}x) = G(e^{\mu-\sigma^2}/x)$ and $G(e^{\mu}x) = G(e^{\mu}/x)$. Hence the conditions for G in (11) are satisfied. Therefore, X is doubly symmetric about $(e^{\mu-\sigma^2}, e^{\mu})$.

7. Further Study of R-Symmetry and I-Symmetry

7.1. *Generation of R-Symmetric Distributions*

Baker (2008) showed that any pdf on R^+ may be transformed to an R-symmetric pdf on R^+ by the Cauchy-Schlömilch transformation. Later, Chaubey et al. (2010) gave a similar transformation. More precisely they offered a simple method to generate an R-symmetric pdf on R^+ by a symmetric pdf. Along the lines of Chaubey et al. (2010), the following corresponding result for R-symmetry on R can be obtained immediately.

Theorem 7.1. *Let h be a symmetric pdf Then*

$$g(x) = \frac{2}{\theta}h\left(\frac{x}{\theta} - \frac{\theta}{x}\right)G\left(\frac{x}{\theta}\right), \quad x \in R\backslash\{0\}, \tag{14}$$

where $\theta > 0$, *and G is a skewing function which satisfies*

$$G(x) = G(1/x), \quad x \in R\backslash\{0\}, \tag{15}$$

is an R-symmetric pdf with the R-center θ.

We now give a simple class of examples of skewing function which satisfies (15). Note that the term $G(x/\theta)$ in (14) can be replaced by $G(x)$, then instead of (15), the skewing function G must satisfy $G(\theta x) = G(\theta/x)$, $x \in R\backslash\{0\}$. Also, if a skewing function G satisfies (15), then $G_1(x) = G(x/\theta)$, $x \in R\backslash\{0\}$, is a skewing function satisfying $G_1(\theta x) = G_1(\theta/x)$, $x \in R\backslash\{0\}$.

Example 7.1. The function defined below is a skewing function satisfying (15),

$$G(x) = \begin{cases} \frac{1}{2}(1 + h(x)), & 0 \le |x| \le 1, \\ \frac{1}{2}\left(1 + h\left(\frac{1}{x}\right)\right), & |x| > 1, \end{cases} \tag{16}$$

where $|h(x)| \le 1$, $|x| \le 1$, is an odd function. When $h(x) = cx^n$, where $|c| \le 1$ and n is 0 or odd number, then

$$G(x) = \begin{cases} \frac{1}{2}(1 + cx^n), & 0 \le |x| \le 1, \\ \frac{1}{2}(1 + cx^{-n}), & |x| > 1. \end{cases} \tag{17}$$

In particular, if $c = 0$, then $G(x) = 1/2$, $x \in R$.

7.2. *Characterization of I-Symmetry*

First, we give a characterization by Seshadri (1965) of the distributions with support on R^+ which are log-symmetric about 1.

Lemma 7.1. *Let X be an r.v. on R^+. Then X is log-symmetric about* 1 *if and only if*

$$f_X(x) = \frac{1}{x} g(\log x), \quad x > 0, \tag{18}$$

where g is a symmetric pdf.

The next lemma is an extension of the above lemma, which concerns r.v.'s on R.

Lemma 7.2. *Let X be an r.v. on R. Then X is I-symmetric about* 1 *if and only if*

$$f_X(x) = \frac{1}{|x|} g(\log |x|) G(x), \quad x \in R \backslash \{0\}, \tag{19}$$

where g is a symmetric pdf and G is a skewing function which satisfies (15).

Proof. First we prove the "if" part. Suppose (19) holds. Let $Z = 1/X$. Then,

$$f_Z(z) = f_X\left(\frac{1}{z}\right)\frac{1}{z^2} = \frac{|z|}{z^2} g\left(\log\left|\frac{1}{z}\right|\right) G\left(\frac{1}{z}\right) = \frac{1}{|z|} g(\log |z|) G(z) = f_X(z), \quad z \in R \backslash \{0\},$$

where the third equality is by the symmetry of g and (15). This proves the "if" part.

Next, assume X is I-symmetric about 1. According to Corollary 2.1, both f_1 and f_2 in (3) are log-symmetric about 1. From Lemma 7.1, $f_1(x) = g_1(\log x)/x$ and $f_2(x) = g_2(\log x)/x$, where g_1 and g_2 are symmetric pdf's. By (5), (6), and (7), it follows

$$f(x) = \frac{1}{2|x|}(ag_1(\log |x|) + (1 - a)g_2(\log |x|)) = \frac{1}{2|x|} g(\log |x|), \quad x \in R \backslash \{0\}, \tag{20}$$

and

$$G(x) = \begin{cases} ag_1(\log x)/(ag_1(\log x) + (1-a)g_2(\log x)), & x > 0, \\ (1-a)g_2(\log(-x))/(ag_1(\log(-x)) + (1-a)g_2(\log(-x))), & x < 0, \end{cases} \quad (21)$$

where $g(x) = ag_1(x) + (1-a)g_2(x)$, $x \in R$, which is a mixture pdf of g_1 and g_2. Due to the fact that both g_1 and g_2 are symmetric, g, and hence f, is also symmetric. Finally, that the function G in (21) satisfies (15) is obvious. This completes the proof. □

Remark 7.1. If the G in (19) is $G(x) = 0$, $x \leq 0$, and $G(x) = 1$, $x > 0$, then $X > 0$ and f_X is reduced to (18).

Consider an r.v. X which is I-symmetric about δ. Then by Proposition 4.1 and Lemma 7.2, the consequence given below follows.

Theorem 7.2. *Let X be an r.v. on R. Then X is I-symmetric about δ, if and only if*

$$f_X(x) = \frac{1}{|x|} g\left(\log \frac{|x|}{\delta}\right) G\left(\frac{x}{\delta}\right), \quad x \in R \backslash \{0\}, \quad (22)$$

where g is a symmetric pdf and G is a skewing function which satisfies (15).

The following are examples to illustrate Theorem 7.2.

Example 7.2. Let X be $\mathscr{C}(0, 1)$ distributed with pdf $f_X(x) = (\pi(1 + x^2))^{-1}$, $x \in R$. Obviously $X \stackrel{d}{=} 1/X$. By choosing

$$g(x) = \frac{2e^x}{\pi(1 + e^{2x})}, \quad x \in R,$$

and $G(x) = 1/2$, $x \in R$, then

$$f_X(x) = \frac{1}{\pi(1 + x^2)} = \frac{1}{|x|} g(\log |x|) G(x), \quad x \in R \backslash \{0\}.$$

On the other hand, if

$$f_X(x) = \frac{2}{\pi(1 + x^2)} G(x), \quad x \in R,$$

where G is a skewing function which satisfies (15), then for this X, it is still I-symmetric about 1.

Example 7.3. Let $g(x) = \frac{1}{2} e^{-|x|}$, $x \in R$, the pdf of a Laplace distribution, and $G(x) = I_{\{x \geq 0\}}$, $x \in R$, a skewing function which satisfies (15). Then,

$$f_X(x) = \frac{1}{|x|} g(\log x) G(x) = \begin{cases} \dfrac{1}{2}, & 0 < x < 1, \\ \dfrac{1}{2x^2}, & x \geq 1. \end{cases}$$

This is the pdf of U/V, where U and V are i.i.d. $\mathcal{U}(0, 1)$ r.v.'s. Note that if $X \stackrel{d}{=} U/V$, where U and V are i.i.d. r.v.'s, then X is I-symmetric about 1.

Although the ratio of two i.i.d. r.v.'s has a distribution that is I-symmetric about 1, for an r.v. Z which is log-symmetric about 1, we will show below that there may not exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$. As $\log Z$ is symmetric, if there is a symmetric r.v. which is not distributed as the difference of two i.i.d. r.v.'s, then this offers an example that a log-symmetric r.v. may not be distributed as the ratio of two i.i.d. r.v.'s.

Throughout the rest of this section, let $\psi_Z(t)$, $t \in R$, denote the characteristic function (ch.f.) of the r.v. Z. First by noting the ch.f. of the difference of two i.i.d. r.v.'s is real and nonnegative, we have the following lemma.

Lemma 7.3. *Let the r.v. Z on R^+ be log-symmetric about* 1. *Also, let $Z_1 = \log Z$. If there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$, then $\psi_{Z_1}(t) \geq 0$, $\forall t \in R$.*

For a symmetric r.v. Z_1, let $Z = e^{Z_1}$. Then Z is log-symmetric about 1. According to Lemma 7.3, if $\psi_{Z_1}(t) < 0$ for some $t \in R$, then there do not exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$. The following example was given by Seshadri (1965).

Example 7.4. Let the pdf of the r.v. Z_1 mentioned above be

$$f_{Z_1}(z) = \frac{1}{\sqrt{2\pi}} z^2 e^{-z^2/2}, \quad z \in R.$$

Then,

$$\psi_{Z_1}(t) = \sqrt{\frac{2}{\pi}}(1 - t^2)e^{-t^2/2}, \quad t \in R.$$

As $\psi_{Z_1}(t) < 0$ when $|t| > 1$, there do not exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$ follows.

Next, we give a sufficient condition for a log-symmetric r.v. on R^+ which can be represented as X/Y, where X and Y are i.i.d. r.v.'s.

Theorem 7.3. *Let the r.v. Z on R^+ be log-symmetric about* 1, *and let $Z_1 = \log Z$. If $\sqrt{\psi_{Z_1}}$ is a ch.f., then there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$.*

Proof. That Z is log-symmetric about 1 implies ψ_{Z_1} is a real and even function. Hence $\sqrt{\psi_{Z_1}}$ is also even. Let i.i.d. r.v.'s X_1 and Y_1 have ch.f. $\sqrt{\psi_{Z_1}}$. Then $\psi_{X_1 - Y_1}(t) = \psi_{Z_1}(t)$, $t \in R$. Consequently, $Z_1 \stackrel{d}{=} X_1 - Y_1$. The rest of the proof follows easily. □

Theorem 7.3 has the following immediate consequence.

Corollary 7.1. *Let the r.v. Z on R^+ be log-symmetric about* 1. *Also, let $Z_1 = \log Z$. If ψ_{Z_1} is infinitely divisible, then there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$.*

It is known that the function $\psi(t)$, $t \in R$, is a Pólya type ch.f. if

$$\psi(0) = 1, \quad \psi(t) \geq 0, \quad \psi(t) = \psi(-t), \quad t \in R, \tag{23}$$

where ψ is decreasing and continuous convex on R^+; see, e.g., Chung (2001). The following is another consequence.

Corollary 7.2. *Let the r.v. Z on R^+ be log-symmetric about* 1. *Also, let $Z_1 = \log Z$. If ψ_{Z_1} is a Pólya type ch.f. satisfying*

$$\psi_{Z_1}(t) > 0, \quad \psi''_{Z_1}(t)\psi_{Z_1}(t) - \frac{1}{2}(\psi'_{Z_1}(t))^2 \geq 0, \quad t > 0, \tag{24}$$

then there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$.

Proof. Firstly, we show that $\sqrt{\psi_{Z_1}}$ is also a Pólya type ch.f. That $\sqrt{\psi_{Z_1}}$ satisfies (23) is obvious. Also, since ψ_{Z_1} is decreasing and continuous on R^+, so is $\sqrt{\psi_{Z_1}}$. Now

$$\left(\sqrt{\psi_{Z_1}(t)}\right)'' = \frac{\psi''_{Z_1}(t)\psi_{Z_1}(t) - (\psi'_{Z_1}(t))^2/2}{2\psi_{Z_1}(t)\sqrt{\psi_{Z_1}(t)}} \geq 0 \tag{25}$$

by (24). Consequently, $\sqrt{\psi_{Z_1}}$ is convex on R^+. Therefore, $\sqrt{\psi_{Z_1}}$ is a Pólya type ch.f. The proof then follows by Theorem 7.3. □

It is known that both $\mathscr{C}(0, 1)$ and $\mathscr{N}(0, 1)$ distributions are infinitely divisible. We now present some examples to illustrate Corollary 7.1.

Example 7.5. Let the pdf of the r.v. Z be

$$f_Z(z) = \frac{1}{\pi z(1 + (\log z)^2)}, \quad z > 0.$$

Then, Z is log symmetric about 1, and Z_1 is $\mathscr{C}(0, 1)$ distributed, where $Z_1 = \log Z$. Since $\mathscr{C}(0, 1)$ distribution is infinitely divisible, according to Corollary 7.1, there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$. As can be seen, if the common pdf of X and Y is

$$f_X(x) = \frac{1/2}{\pi x(1/4 + (\log x)^2)}, \quad x > 0,$$

then this can be served as an example.

Example 7.6. Let the r.v. Z be Log-$\mathscr{N}(0, 1)$ distributed. Also let $Z_1 = \log Z$. Then Z is log-symmetric about 1, and Z_1 is $\mathscr{N}(0, 1)$ distributed, which is infinitely divisible. Hence, there exist two i.i.d. r.v.'s X and Y such that $Z \stackrel{d}{=} X/Y$. The r.v.'s X and Y with Log-$\mathscr{N}(0, 1/2)$ being their common distribution is an example.

Example 7.7. Let the r.v. Z_1 have the Pólya type ch.f. $\psi_{Z_1}(t) = (1 + |t|)^{-1}$, $t \in R$. Also let $Z = e^{Z_1}$. Then Z is log-symmetric about 1. Since

$$\psi''_{Z_1}(t)\psi_{Z_1}(t) - \frac{1}{2}(\psi'_{Z_1}(t))^2 = \frac{3}{2(1+t)^4} \geq 0, \quad t \in R^+,$$

according to Corollary 4, there exist two i.i.d. r.v.'s X and Y such that $Z \overset{d}{=} X/Y$.

In Examples 7.4 and 7.5, that the common distribution of X and Y is related to that of Z by a scale factor of 2 on the logged scale can be seen from the fact that the ch.f. of $\log X$ is the square root of the ch.f. of $\log Z$, the ch.f.'s of $\mathscr{C}(0, 1)$ and $\mathscr{N}(0, 1)$ have the forms $\exp\{-|s|\}$ and $\exp\{is\}$, respectively, whose square root have similar forms $\exp\{-|s|/2\}$ and $\exp\{is/2\}$, respectively, which corresponding to the ch.f.'s of $\mathscr{C}(0, 1/2)$ and $\mathscr{N}(0, 1/2)$ distributions, respectively. Not only more general Cauchy and normal distributions, among those common distributions, the family of gamma distributions also have such kind of property.

7.3. *I-Symmetry Arising from Trigonometric Formulas*

Let $Z = X/Y$. Although Z is I-symmetric about 1 if X and Y are i.i.d., as mentioned before, the converse may not be true. That the joint pdf of X, Y satisfies

$$f_{X,Y}(x, y) = f_{X,Y}(y, x), \quad x, y \in R, \tag{26}$$

is sufficient to imply Z is I-symmetric about 1. See also the following example by Jones (1999).

Example 7.8. Let (X, Y) have the polar representation

$$X = R\cos\Theta \quad \text{and} \quad Y = R\sin\Theta, \tag{27}$$

where Θ is $\mathscr{U}(0, 2\pi)$ distributed, and R is a positive r.v. independent of Θ. Then (X, Y) has a spherically symmetric distribution with pdf

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sqrt{x^2 + y^2}} f_R(\sqrt{x^2 + y^2}), \quad x, y \in R, \tag{28}$$

which satisfies (26). Hence, $\tan\Theta$ $(= Y/X)$ is I-symmetric about 1. In fact, $\tan\Theta$ is $\mathscr{C}(0, 1)$ distributed.

Example 7.8 shows that there exists an I-symmetric distribution about 1 arising from trigonometric functions. Jones (1999) also pointed out if the Θ in (27) is $\mathscr{U}(a, b)$ distributed, where $b - a = m\pi$, m is a positive integer, then $\tan\Theta$ has a $\mathscr{C}(0, 1)$ distribution. It follows immediately that for S being an r.v. independent of Θ, where Θ is $\mathscr{U}(-\pi/2, \pi/2)$ distributed, then $\tan(n\Theta + S)$ is also $\mathscr{C}(0, 1)$ distributed, where n is a positive integer. Furthermore, Jones (1999) gave some multiple angle and angle sum formulas for tangent functions, which remain $\mathscr{C}(0, 1)$ distributed. For example, the double angle formula for tangent function yields $(\tan\Theta - 1/\tan\Theta)/2$ is $\mathscr{C}(0, 1)$ distributed. Also, the multiple angle and angle sum formulas for sine and cosine functions yield some functions of X and Y have the same distribution as X and some functions of X and Y have a $\mathscr{C}(0, 1)$ distribution. For example, $2XY/\sqrt{X^2+Y^2} \stackrel{d}{=} X$ and $2XY/(Y^2 - X^2)$ is $\mathscr{C}(0, 1)$ distributed (see Jones, 1999).

Inspired by Jones (1999), we present some related results in the following. Let

$$f_U(u) = \frac{2}{\pi} G(\tan u), \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{29}$$

where G is a skewing function satisfying (15). Let $T = \tan U$. Then the pdf of T is

$$f_T(t) = \frac{2}{\pi(1+t^2)} G(t), \quad t \in R, \tag{30}$$

which is I-symmetric about 1 (see Example 7.2). The following theorem points out that some of the results presented by Jones (1999) still hold for the r.v.'s U and T given above.

Theorem 7.4. *Let U have the pdf given in* (29), *and* $T = \tan U$. *Then*:

(i) $T \stackrel{d}{=} 1/T$;
(ii) $\tan(2U)$ *is* $\mathscr{C}(0, 1)$ *distributed*;
(iii) $(T - 1/T)/2$ *is* $\mathscr{C}(0, 1)$ *distributed*;
(iv) $\tan(2U + S)$ *is* $\mathscr{C}(0, 1)$ *distributed, where S is an r.v. independent with U*;
(v) $2XY/(Y^2 - X^2)$ *is* $\mathscr{C}(0, 1)$ *distributed, where* $X = R\cos U$, $Y = R \sin U$, *and R is a positive r.v. independent with U*;
(vi) *let* $V = \sin(4U)$, *then*

$$f_V(v) = \frac{1}{\pi\sqrt{1-v^2}}, \quad |v| < 1.$$

The proof of the above theorem is standard hence is omitted.

Remark 7.2. If $G(x) = 1/2$, $x \in R$, that is U is $\mathscr{U}(-\pi/2, \pi/2)$ distributed, then $2U$ in (ii) and (iv) can be replaced by U, and $4U$ in (vi) can be replaced by U.

Remark 7.3. Let $X = R\cos U$, $Y = R\sin U$, where U has the pdf given in (29), and R is a positive r.v. independent of U. Then

$$f_{X,Y}(x, y) = \frac{2}{\pi\sqrt{x^2+y^2}} f_R(\sqrt{x^2+y^2}) G(y/x).$$

Thus, (X, Y) can be viewed as having a generalized spherically symmetric distribution.

8. Conclusion

As mentioned by Mudholkar and Wang (2010), data are usually non negative, right-skewed, and unimodal. Hence, unimodal distributions provide realistic models for data in practice. I-symmetrically distributed r.v.'s on R can be unimodal.

$\mathscr{C}(0, 1)$ distribution is an example. Yet r.v.'s which are R-symmetric and hence doubly-symmetric on R cannot be unimodal. As there are also many examples in applications with bimodal phenomena, the bimodal distributions provided in this work may then be considered as an alternative to fit the data with bimodal R-symmetric property.

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