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Characterizations of Distributions Based on Moments of Residual Life

WEN-JANG HUANG[1] AND NAN-CHENG SU[2]

[1]Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung, Taiwan
[2]Department of Statistics, National Cheng-Kung University, Tainan, Taiwan

Let T be a random variable having an absolutely continuous distribution function. It is known that linearity of $E(T \mid T > t)$ can be used to characterize distributions such as exponential, power and Pareto distribution. In this work, we will extend the above results. More precisely, we characterize the distribution of T by using certain relationships of conditional moments of T. Our results can also be used to obtain new characterization of distributions based on adjacent order statistics or record values.

Keywords Characterization; Conditional moment; Exponential distribution; Order statistics; Residual lifetime; Record values.

Mathematics Subject Classification Primary 60E05; Secondary 62E10.

1. Introduction

For a non negative random variable (r.v.) T, during the past decades, there are many studies on characterizing exponential distribution under some weaker conditions than lack-of-memory property:

$$P(T > s + t \mid T > t) = P(T > s), \quad 0 \leq s, \ \ t < \infty.$$

For example, each of the following assumptions based on moments of residual life:

$$E((T - t)^k \mid T > t) = c_1, \quad 0 \leq t < \infty,$$
$$Var(T - t \mid T > t) = c_2, \quad 0 \leq t < \infty,$$

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Address correspondence to Nan-Cheng Su, Department of Statistics, National Cheng-Kung University, Tainan 70101, Taiwan; E-mail: sunanchen@gmail.com

and

$$Var(T - t \mid T > t) = (E(T - t \mid T > t))^2, \quad 0 \le t < \infty, \tag{1}$$

where $k \geq 1$ is a fixed integer and $c_1, c_2 > 0$ are constants, can be used to characterize T to be exponentially distributed; see Dallas (1979), Azlarov and Sultanava (1972), and Nagaraja (1975), respectively. Recently, Gupta and Kirmani (2004) gave a uniqueness theorem showing that, under some mild conditions, the function $Var(T - t \mid T > t)$, $t \geq 0$, can be used to determine the distribution of T.

Related results can be found in the book of Rao and Shanbhag (1994) and articles, such as Navarro et al. (1998), Gupta and Kirmani (2000), Su and Huang (2000), Gupta (2006), Nair and Sudheesh (2006, 2010), and references therein. Among them, Su and Huang (2000) showed that the cumulative distribution function (cdf) F of T is determined by giving the function

$$v(t) = E(g(T) \mid T > t), \quad a < t < b,$$

where g is a continuous function and (a, b) is the support of F. Note that if T has a probability density function (pdf) f, then $F(t), a < t < b$, $\gamma(t), a < t < b$, or $E(T \mid T > t), a < t < b$, are equivalent, in the sense that given one of them, the other two can be determined, where $\gamma(t)$, the hazard function, is defined as

$$\gamma(t) = \frac{f(t)}{1 - F(t)}, \quad a < t < b.$$

Gupta and Kirmani (2004) considered the case $a = 0$ and $b = \infty$ and used the ratio of $\gamma(t)$ and $E(T \mid T > t)$ to characterize the distribution of T. Nair and Sudheesh (2006) showed that, for a continuous function g and a differentiable function h, the pdf f satisfies the differential equation

$$\frac{f'(t)}{f(t)} = -\frac{h'(t)}{h(t)} + \frac{\mu - g(t)}{\sigma h(t)}, \quad a < t < b, \tag{2}$$

where μ and σ are the mean and the standard deviation of $g(T)$, if and only if

$$E(g(T) \mid T > t) = \mu + \sigma h(t)\gamma(t), \quad a < t < b. \tag{3}$$

Later, for $a = 0$, $b = \infty$, $g'(t) \neq 0$, $0 < t < \infty$, and T being positive, Nair and Sudheesh (2010) first showed that

$$\begin{aligned} Var(g(T) \mid T > t) = {} & (\mu - E(g(T) \mid T > t))(E(g(T) \mid T > t) - g(t)) \\ & + \sigma E(g'(T)h(T) \mid T > t), \quad 0 < t < \infty, \end{aligned} \tag{4}$$

if and only if (3) holds, then we obtained the characterization of distributions. For example, when $g(t) = t$ and $h(t) = \mu t/\sigma$, where $\mu = E(T)$ and $\sigma = \sqrt{Var(T)}$, (4) becomes

$$E((T - t)^2 + (t - 2\mu)(T - t) \mid T > t) = \mu t, \quad 0 < t < \infty, \tag{5}$$

and

$$\frac{f'(t)}{f(t)} = -\frac{1}{\mu}, \quad 0 < t < \infty,$$

follows. Consequently the relationship (5) characterizes the exponential distribution with cdf $F(t) = 1 - e^{-t/\mu}$, $0 < t < \infty$.

In Sec. 2, we first generalize Nagaraja (1975) by using the weaker condition

$$E((T-t)^2 \mid T > t) = \eta(E(T-t \mid T > t))^2, \quad a < t < b, \tag{6}$$

where η is a constant, or equivalently,

$$Var(T-t \mid T > t) = (\eta - 1)(E(T-t \mid T > t))^2, \quad a < t < b. \tag{7}$$

It can be seen that when $\eta = 2$, $a = 0$, and $b = \infty$, (7) reduces to (1). We identify all distributions allowing the condition (6). These distributions can also be characterized by the linearity of $E(T \mid T > t)$, $a < t < b$. Motivated by this, we characterize the distribution of T by using certain relationships of conditional moments of T. It turns out that some common distributions, such as exponential, uniform and Pareto, can be characterized by using our theorems. Note that the pdf f satisfying (2) is not assumed in this work.

In Sec. 3, we present some applications of our results in the characterization of distributions by using order statistics and record values.

2. Characterizations Based on Moments of Left Truncated r.v.

From now on, let T be an r.v. with an absolutely continuous cdf F and a pdf f. Also assume that F has support (a, b), where $-\infty \le a < b \le \infty$. First we give an extension of Nagaraja (1975), where F is determined by using $Var(T-t \mid T > t)$ proportional to $(E(T-t \mid T > t))^2$, $\forall a < t < b$.

Theorem 2.1. *Assume that f is continuous and*

$$E((T-t)^2 \mid T > t) = \eta(E(T-t \mid T > t))^2, \quad a < t < b, \tag{8}$$

where η is a constant. Then only the following three cases are possible:

(i) *$\eta = 2$, $a > -\infty$, $b = \infty$ and $F(t) = 1 - e^{-\lambda(t-a)}$, $a < t < \infty$, where $\lambda > 0$ is a constant, namely T has an exponential distribution with location parameter;*
(ii) *$1 < \eta < 2$, $-\infty < a < b < \infty$ and $F(t) = 1 - ((b-t)/(b-a))^{2(\eta-1)/(2-\eta)}$, $a < t < b$, in particular, if $\eta = 4/3$, then T has a $U(a, b)$ distribution;*
(iii) *$\eta > 2$, $a > -\infty$, $b = \infty$ and $F(t) = 1 - ((a+\delta)/(t+\delta))^{2(\eta-1)/(\eta-2)}$, $a < t < \infty$, where $\delta > -a$ is a constant, namely T has a Pareto distribution.*

Proof. As T is non degenerate, from (8), it follows that $\eta > 1$, and

$$(1 - F(t)) \int_t^b (x-t)^2 f(x)dx = \eta \left(\int_t^b (x-t) f(x)dx \right)^2, \quad a < t < b. \tag{9}$$

Taking the derivatives of both sides of (9) with respect to t yields

$$f(t)\int_t^b (x-t)^2 f(x)dx = 2(\eta-1)(1-F(t))\int_t^b (x-t)f(x)dx, \quad a<t<b. \tag{10}$$

In view of (9) and (10), we obtain (note that $\int_t^b (x-t)^2 f(x)dx \neq 0$)

$$f(t)\int_t^b (x-t)f(x)dx = 2(1-1/\eta)(1-F(t))^2, \quad a<t<b, \tag{11}$$

this in turn implies that f is differentiable. Now by taking the derivatives of both sides of (11) with respect to t yields

$$f'(t)\int_t^b (x-t)\, f(x)dx = (-3+4/\eta)\, f(t)\,(1-F(t))\,, \quad a<t<b. \tag{12}$$

(11) and (12) together imply

$$\frac{f'(t)}{f(t)} = \frac{4-3\eta}{2(\eta-1)}\frac{f(t)}{1-F(t)}, \quad a<t<b.$$

Consequently,

$$f(t) = \lambda(1-F(t))^{(3\eta-4)/(2\eta-2)}, \quad a<t<b,$$

where $\lambda>0$ is a constant. From this we have, for $a<t<b$,

$$F(t) = \begin{cases} 1-c_1e^{-\lambda t}, & \eta=2, \\ 1-(c_2t+c_3)^r, & \eta\neq 2, \end{cases} \tag{13}$$

where $c_1>0$ is a constant, $r=(2\eta-2)/(2-\eta)$, $c_2=-\lambda/r$ and c_3 is a constant.

First, consider the case $\eta=2$. Then (13) yields $a>-\infty$, $b=\infty$ and $c=e^{\lambda a}$. This proves assertion (i).

Next, consider the case $\eta\neq 2$. Assume that $1<\eta<2$. This in turn implies that $r>0$ and $c_2<0$. As $F(t)$ is a cdf, from (13), it yields that $-\infty<a<b<\infty$, $c_2=1/(a-b)$ and $c_3=b/(b-a)$. The last case to be considered is $\eta>2$. Then $r<0$ and $c_2>0$ follows. From (13), it can be seen that this can only happen if $a=(1-c_3)/c_2>-\infty$ and $b=\infty$. The remaining assertions of (ii) and (iii) can be obtained immediately.

Remark 2.1. Let $k\geq 1$ be an integer. If $F(t)=1-e^{-\lambda(t-a)}$, $a<t<\infty$, $\lambda>0$, then

$$E((T-t)^k \mid T>t) = \frac{\Gamma(k+1)}{\lambda^k}, \quad a<t<\infty; \tag{14}$$

if $F(t)=1-((b-t)/(b-a))^r$, $a<t<b$, $r>0$, then

$$E((T-t)^k \mid T>t) = \frac{\Gamma(k+1)\Gamma(r+1)}{\Gamma(k+r+1)}(b-t)^k, \quad a<t<b; \tag{15}$$

if $F(t) = 1 - ((a+\delta)/(t+\delta))^r$, $a < t < \infty$, $\delta > -a$, $r > k$, then

$$E((T-t)^k \mid T > t) = \frac{\Gamma(k+1)\Gamma(r-k)}{\Gamma(r)}(t+\delta)^k, \quad a < t < \infty. \tag{16}$$

The next theorem can be compared to Gupta (2006), where the monotonic behavior of $Var(T-t \mid T > t)$ was studied through the function $E((T-t)^2 \mid T > t)/E(T-t \mid T > t)$.

Theorem 2.2. *Assume that f is continuous and for some integer $k \geq 1$,*

$$E((T-t)^k \mid T > t) = (\eta t + \theta)E((T-t)^{k-1} \mid T > t), \quad a < t < b, \tag{17}$$

where η and θ are constants. Then only the following three cases are possible:

(i) $\eta = 0$, $\theta > 0$, $a > -\infty$, $b = \infty$ *and* $F(t) = 1 - e^{-(k/\theta)(t-a)}$, $a < t < \infty$;
(ii) $-1 < \eta < 0$, $a > -\infty$, $b = -\theta/\eta$ *and* $F(t) = 1 - ((b-t)/(b-a))^{-k(1+1/\eta)}$, $a < t < b$;
(iii) $\eta > 0$, $a > -\delta$, $b = \infty$ *and* $F(t) = 1 - ((a+\delta)/(t+\delta))^{k(1+1/\eta)}$, $a < t < \infty$, *where* $\delta = \theta/\eta$.

Proof. From (17), it can be obtained that

$$\eta t + \theta > 0, \quad a < t < b, \tag{18}$$

and

$$\int_t^b (x-t)^k f(x)dx = (\eta t + \theta)\int_t^b (x-t)^{k-1} f(x)dx, \quad a < t < b. \tag{19}$$

Taking the kth derivatives of both sides of (19) with respect to t and after some manipulations, we obtain

$$\frac{f(t)}{1-F(t)} = \frac{k(\eta+1)}{\eta t + \theta}, \quad a < t < b, \tag{20}$$

In view of (18) and (20), it follows that $\eta > -1$.

First, consider the case $\eta = 0$. Then $\theta > 0$ follows. Solving (20) yields $F(t) = 1 - c_1 e^{-(k/\theta)t}$, $a < t < b$, where $c_1 > 0$ is a constant. As $F(t)$ is a cdf, it turns out that $a > -\infty$, $b = \infty$ and $c_1 = e^{ka/\theta}$. The proof of assertion (i) is completed.

Next, consider the case $\eta \neq 0$. The solutions of (20) is

$$F(t) = 1 - c_2(\eta t + \theta)^r, \quad a < t < b, \tag{21}$$

where $r = -k(1 + 1/\eta)$ and $c_2 > 0$ is a constant. Assume that $-1 < \eta < 0$. This gives $r > 0$. Again as $F(t)$ is a cdf, from (21), it can be seen that $-\infty < a < b < \infty$, $c_2 = (\eta a + \theta)^{-r}$ and $\theta = -\eta b$. On the other hand, assume that $\eta > 0$. Then $r < -k$ follows. This, together with (18) and (21), shows that $a > -\infty$, $b = \infty$ and $c_2 = (\eta a + \theta)^{-r}$. The assertions (ii) and (iii) now follow immediately.

In the above theorem, again we have characterizations of the exponential distribution in (i), uniform distribution in (ii), where $\eta = -k/(k+1)$, and Pareto distribution in (iii).

Before proving the next two theorems, we need the following lemma which can be found in Boyce and DiPrima (1997).

Lemma 2.1. *Consider the Euler equation*:

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0, \tag{22}$$

where t belongs to an interval not containing the origin, and α and β are some fixed real numbers. Then

$$y(t) = \begin{cases} c_1|t|^{(1-\alpha+\sqrt{(1-\alpha)^2-4\beta})/2} + c_2|t|^{(1-\alpha-\sqrt{(1-\alpha)^2-4\beta})/2}, & \text{if } (1-\alpha)^2 > 4\beta, \\ (c_3 + c_4 \log|t|)|t|^{(1-\alpha)/2}, & \text{if } (1-\alpha)^2 = 4\beta, \end{cases}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

For $a = 0$ and $b = \infty$, Dallas (1979) used the condition $E((T-t)^k \mid T > t) = c$, $t \geq 0$, where $k \geq 1$ is a fixed integer and c is a constant, to characterize T to be exponentially distributed, and Galambos and Hagwood (1992) gave a uniqueness theorem showing that the function $E((T-t)^2 \mid T > t)$, $t \geq 0$ can determine the distribution of T. Inspired by this and in view of Remark 2.1, we give the following theorem.

Theorem 2.3. *Assume that f is differentiable and for some integer $k \geq 2$,*

$$E((T-t)^k \mid T > t) = (\eta t + \theta)^2 E((T-t)^{k-2} | T > t), \quad a < t < b, \tag{23}$$

where η and θ are constants. Then only the following three cases are possible:

(i) $\eta = 0$, $\theta \neq 0$, $a > -\infty$, $b = \infty$ *and* $F(t) = 1 - e^{-\sqrt{k(k-1)/\theta^2}(t-a)}$, $a < t < \infty$;
(ii) $0 < \eta^2 < 1$, $a > -\infty$, $b = -\theta/\eta$ *and* $F(t) = 1 - ((b-t)/(b-a))^{(\sqrt{1+4k(k-1)/\eta^2} - 2k + 1)/2}$, $a < t < b$;
(iii) $\eta^2 > 0$, $a > -\delta$, $b = \infty$ *and* $F(t) = 1 - ((a+\delta)/(t+\delta))^{(2k-1+\sqrt{1+4k(k-1)/\eta^2})/2}$, $a < t < \infty$, *where* $\delta = \theta/\eta$.

Proof. First consider $\eta = 0$. From (23), it follows that $\theta \neq 0$ and

$$\int_t^b (x-t)^k f(x)dx = \theta^2 \int_t^b (x-t)^{k-2} f(x)dx, \quad a < t < b. \tag{24}$$

Taking the kth derivatives of both sides of (24) with respect to t and after some manipulations, we obtain the second order linear homogeneous differential equation

$$F''(t) + \frac{k(k-1)}{\theta^2}(1 - F(t)) = 0, \quad a < t < b,$$

which yields the solution

$$F(t) = 1 + c_1 e^{\lambda t} + c_2 e^{-\lambda t}, \quad a < t < b, \tag{25}$$

where $\lambda = \sqrt{k(k-1)/\theta^2}$. As $F(t)$ is a cdf, (25) yields

$$\text{(1-1)}\quad a > -\infty, \quad b = \infty \quad \text{and} \quad F(t) = 1 - e^{-\lambda(t-a)}, \ a < t < \infty;$$

$$\text{(1-2)}\quad -\infty < a < b < \infty \quad \text{and} \quad F(t) = 1 + e^{\lambda(t+a)}/(e^{2b\lambda} - e^{2a\lambda}) + e^{-\lambda(t-a)}/(e^{2(a-b)\lambda} - 1), \quad a < t < b.$$

It can be readily checked that the equality in (23) with $\eta = 0$ is achieved only for case (1-1). This proves assertion (i).

Next, consider $\eta \neq 0$. First note that (23) can be rewritten as

$$E((T_1 - t_1)^k \mid T_1 > t_1) = \eta^2 t_1^2 E((T_1 - t_1)^{k-2} \mid T_1 > t_1), \quad a_1 < t_1 < b_1,$$

where $T_1 \equiv T + \theta/\eta$ having a cdf $F_1(x) \equiv F(x - \theta/\eta)$, $a_1 \equiv a + \theta/\eta$ and $b_1 \equiv b + \theta/\eta$. Hence, it is sufficient to prove the situation $\theta = 0$. Now from (23) with $\theta = 0$, it follows that $ab \geq 0$ and

$$\int_t^b (x-t)^k f(x)dx = \eta^2 t^2 \int_t^b (x-t)^{k-2} f(x)dx, \quad a < t < b. \tag{26}$$

Similarly, taking the kth derivatives of both sides of (26) with respect to t, we arrive at

$$t^2 \overline{F}''(t) + 2kt\overline{F}'(t) + k(k-1)\left(1 - 1/\eta^2\right)\overline{F}(t) = 0, \quad a < t < b, \tag{27}$$

where $\overline{F}(t) = 1 - F(t)$. By Lemma 2.1, the solution of (27) is

$$F(t) = 1 - c_3|t|^{-\mu-\sigma} - c_4|t|^{\sigma-\mu}, \quad a < t < b, \tag{28}$$

where $\mu = k - 1/2 > 0$, $\sigma = \sqrt{1 + 4k(k-1)/\eta^2}/2 > 0$ and c_3, c_4 are constants. Note that $\mu > \sigma$ if $\eta^2 > 1$; $\mu = \sigma$ if $\eta^2 = 1$; $\mu < \sigma$ if $0 < \eta^2 < 1$. Since $ab \geq 0$, $\lim_{t\downarrow a} F(t) = 0$ and $\lim_{t\uparrow b} F(t) = 1$, (28) leads to the following eight cases:

$$\text{(2-1)}\quad 0 < \eta^2 < 1, \quad a > -\infty, \ b = 0 \quad \text{and} \quad F(t) = 1 - (t/a)^{\sigma-\mu}, \ a < t < 0;$$

$$\text{(2-2)}\quad 0 < \eta^2 \leq 1, \quad a > 0, \ b = \infty \quad \text{and} \quad F(t) = 1 - (a/t)^{\mu+\sigma}, \ a < t < \infty;$$

$$\text{(2-3)}\quad \eta^2 > 1, \quad a > 0, \ b = \infty \quad \text{and} \quad F(t) = 1 - c_3 t^{-\mu-\sigma} - (a^{\mu-\sigma} - c_3 a^{-2\sigma})t^{\sigma-\mu}, \ a < t < \infty;$$

$$\text{(2-4)}\quad \eta^2 \neq 1, \quad a > 0, \ b < \infty \quad \text{and}$$

$$F(t) = 1 - (a^{\mu+\sigma}/(b^{2\sigma} - a^{2\sigma}))(b^{2\sigma}t^{-\mu-\sigma} - t^{\sigma-\mu}), \ a < t < b;$$

$$\text{(2-5)}\quad \eta^2 = 1, \quad a > 0, \ b < \infty \quad \text{and}$$

$$F(t) = 1 - (t^{-2\mu} - b^{-2\mu})/(a^{-2\mu} - b^{-2\mu}), \ a < t < b;$$

$$\text{(2-6)}\quad \eta^2 \neq 1, \quad a > -\infty, \ b < 0 \quad \text{and}$$

$$F(t) = 1 - (|a|^{\mu+\sigma}/(|a|^{2\sigma} - |b|^{2\sigma}))(|t|^{\sigma-\mu} - |b|^{2\sigma}|t|^{-\mu-\sigma}), \ a < t < b;$$

$$\text{(2-7)}\quad \eta^2 = 1, \quad a > -\infty, \ b < 0 \quad \text{and}$$

$$F(t) = 1 - (|b|^{-2\mu} - |t|^{-2\mu})/(|b|^{-2\mu} - |a|^{-2\mu}), \;\; a < t < b;$$

(2-8) $\eta^2 = 1, \quad a = -\infty, \;\; b < 0 \quad$ and $F(t) = (b/t)^{2\mu}, \;\; -\infty < t < b.$

Since $F(t)$ is a cdf and satisfies (23) with $\theta = 0$, it can be seen that cases (2-4)–(2-8) are excluded, and there remains cases (2-1), (2-2), and case (2-3) only for $c_3 = a^{\mu+\sigma}$. Note that when $c_3 = a^{\mu+\sigma}$, $F(t)$ in case (2-3) is exactly the same as that in case (2-2). Consequently, the assertions (ii) and (iii) follow immediately. The proof is finished.

Note that (8) can be rewritten as

$$E(T^2 + 2(\eta-1)tT - (\eta-1)t^2 \mid T > t) = \eta E^2(T \mid T > t), \quad a < t < b.$$

Inspired by this, we are interested in knowing which distributions can be characterized by using the following more general form

$$E(T^2 + (q_1 t + q_2)T + q_3 t^2 + q_4 t + q_5 \mid T > t) = q_6 E^2(T \mid T > t), \quad a < t < b, \quad (29)$$

where $q_1, \cdots, q_6$ are constants. In particular, if $q_1 = q_2 = q_3 = q_4 = 0$, $q_5 > 0$ and $q_6 = 1$, then (29) is equivalent to $Var(T \mid T > t) = q_5$, $a < t < b$; if $q_1 = -\eta - 2$, $q_2 = -\theta$, $q_3 = \eta + 1$, $q_4 = \theta$ and $q_5 = q_6 = 0$, then (29) reduces to (17) with $k = 2$; if $q_1 = -2$, $q_2 = q_6 = 0$, $q_3 = 1 - \eta^2$, $q_4 = 2\eta\theta$ and $q_5 = -\theta^2$, then (29) reduces to (23) with $k = 2$. The following theorem corresponds to the situation $q_1 = q_3 = q_4 = 0$, $q_2 = 2\delta(1-\eta)$, $q_5 = (1-\eta)\delta^2$ and $q_6 = \eta$.

Theorem 2.4. *Assume that f is differentiable and*

$$E((T+\delta)^2 \mid T > t) = \eta(E(T+\delta \mid T > t))^2, \quad a < t < b, \quad (30)$$

where η and δ are constants. Then $\eta > 1$ and only the following two cases are possible:

(i) $a > -\infty$, $b = -\delta$ *and* $F(t) = 1 - ((b-t)/(b-a))^{\sqrt{\eta/(\eta-1)}-1}$, $a < t < b$;
(ii) $a > -\delta$, $b = \infty$ *and* $F(t) = 1 - ((a+\delta)/(t+\delta))^{\sqrt{\eta/(\eta-1)}+1}$, $a < t < \infty$.

Proof. As in the proof of Theorem 2.3, it is sufficient to consider the case $\delta = 0$. From (30), it follows that $\eta > 1$ and

$$(1 - F(t)) \int_t^b x^2 f(x) dx = \eta \left(\int_t^b x f(x) dx \right)^2, \quad a < t < b. \quad (31)$$

Taking the derivatives of both sides of (31) with respect to t yields that, for $a < t < b$,

$$\int_t^b x^2 f(x) dx = 2\eta t \int_t^b x f(x) dx - t^2(1 - F(t)). \quad (32)$$

If $ab < 0$, by letting $t = 0$, it leads to a contradiction that the left-hand side of (32) is greater than 0 and the right hand side of (32) is equal to 0. Hence, $ab \geq 0$.

By taking the second derivatives of both sides of (32) with respect to t and after some manipulations, we obtain

$$t^2\overline{F}''(t) + 3t\overline{F}'(t) - \frac{1}{\eta - 1}\overline{F}(t) = 0, \quad a < t < b, \tag{33}$$

where $\overline{F}(t) = 1 - F(t)$. By Lemma 2.1, the solution of (33) is

$$F(t) = 1 + c_1|t|^{\mu-1} + c_2|t|^{-\mu-1}, \quad a < t < b, \tag{34}$$

where $\mu = \sqrt{\eta/(\eta - 1)} > 1$ and c_1, c_2 are constants. Using (34), $ab \geq 0$, and the fact that $F(t)$ is a cdf, we obtain the following solutions:

(3-1) $a > -\infty, \quad b = 0$ and $F(t) = 1 - (t/a)^{\mu-1}, \ a < t < 0$;

(3-2) $a > 0, \ b = \infty$ and $F(t) = 1 - (a/t)^{\mu+1}, \ a < t < \infty$;

(3-3) $a > 0, \quad b < \infty$ and $F(t) = 1 - (a^{\mu+1}/(b^{2\mu} - a^{2\mu}))(b^{2\mu}t^{-\mu-1} - t^{\mu-1})$, $a < t < b$;

(3-4) $a > -\infty, \quad b < 0$ and $F(t) = 1 - (|a|^{\mu+1}/(|a|^{2\mu} - |b|^{2\mu}))$ $(|t|^{\mu-1} - |b|^{2\mu}|t|^{-\mu-1}), \ a < t < b$.

Again for each $F(t)$ in cases (3-1)–(3-4), $E(T^k \mid T > t)$ can be obtained easily, and then we conclude that the equality in (30) with $\delta = 0$ holds only for $F(t)$ as given in cases (3-1) and (3-2), which proves our result for the case $\delta = 0$. This completes the proof.

Again when $\eta = 4/3$, the above theorem provides a characterization of the $\mathcal{U}(a, b)$ distribution.

Remark 2.2. Let $k \geq 1$ be an integer. If $F(t) = 1 - ((b - t)/(b - a))^r$, $a < t < b$, $r > 0$, then

$$E((b - T)^k \mid T > t) = \frac{r}{r + k}(b - t)^k, \quad a < t < b;$$

if $F(t) = 1 - ((a + \delta)/(t + \delta))^r$, $a < t < \infty$, $\delta > -a$, $r > k$, then

$$E((T + \delta)^k \mid T > t) = \frac{r}{r - k}(t + \delta)^k, \quad a < t < \infty.$$

3. Applications to Characterizations Based on Adjacent Order Statistics and Record Values

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed r.v's with a common absolutely continuous cdf G. Also, assume that G has support (a, b), where $-\infty \leq a < b \leq \infty$. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics based on $\{X_1, X_2, \ldots, X_n\}$. Let the sequences of record times $U(n)$ and record values Y_n be defined as follows: $U(1) = 1$, $U(n + 1) = \min\{i > U(n) : X_i > X_{U(n)}\}$ and $Y_n = X_{U(n)}$, $n \geq 1$. The properties and characterizations related to order statistics and record values have been widely studied and some excellent reviews can be found in books such as Arnold et al. (1998, 2008), Nevzorov (2001), and David and Nagaraja (2003). Also, it has been pointed out by Deheuvels (1984), Gupta (1984), Nagaraja (1988), Huang and Su (1999), and Huang et al. (2007), etc., there are many parallel characterizations for order statistics and record values. In this section, we will illustrate how our previous results can be applied to order statistics and record values.

It is noted that, for $a < t < b$, $1 \le i < n$, and $j \ge 1$, when $F = 1 - (1-G)^{n-i}$, the conditional distribution of T, given $T > t$, is identical with that of $X_{i+1:n}$ given $X_{i:n} = t$. In particular, when $F = G$, the conditional distribution of T, given $T > t$, is also identical with that of Y_{j+1} given $Y_j = t$. This in turn implies that every characterization result in Sec. 2 has a parallel version based on the conditional moments of $X_{i+1:n}$ given $X_{i:n} = t$, as well as that of Y_{j+1} given $Y_j = t$. For example, Theorem 2.2 yields characterizations of G by using

$$E((X_{i+1:n} - X_{i:n})^k \mid X_{i:n} = t) = (\eta t + \theta) E((X_{i+1:n} - X_{i:n})^{k-1} \mid X_{i:n} = t), \quad a < t < b,$$

or

$$E((Y_{j+1} - Y_j)^k \mid Y_j = t) = (\eta t + \theta) E((Y_{j+1} - Y_j)^{k-1} \mid Y_j = t), \quad a < t < b,$$

where $k \ge 1$ is an integer, and η, θ are constants. For $k = 1$, the above two characterizations can be found in Dembińska and Wesołowski (1998) and Nagaraja (1977), respectively.

On the other hand, for $a < t < b$, $1 \le i < n$, and $j \ge 1$, the conditional distribution of T, given $T < t$, is identical to that of $X_{i:n}$ given $X_{i+1:n} = t$, if $F = G^i$, and that of Y_j given $Y_{j+1} = t$, if

$$\frac{F(x)}{F(t)} = \frac{(-\log(1 - G(x)))^j}{(-\log(1 - G(t)))^j}, \quad a < x < t. \tag{35}$$

Similarly, the characterizations of F based on conditional moments of T given $T < t$ can deduce the characterizations of G based on the conditional moments of $X_{i:n}$ given $X_{i+1:n} = t$, or Y_j given $Y_{j+1} = t$. Note that, for $k \ge 1$, $a < t < b$, and δ is a constant, let $u = -t$ and $U = -T$, then

$$E((T - t)^k \mid T > t) = E((u - U)^k \mid U < u),$$

and

$$E((T + \delta)^k \mid T > t) = (-1)^k E((U - \delta)^k \mid U < u).$$

This leads to the conclusion that every characterization result in Sec. 2 has a corresponding form based on conditional moments of T given $T < t$. Consequently, the corresponding characterizations of G based on the conditional moments of backward order statistics or record values follows. The followings are two examples of applications of Theorem 2.2.

Theorem 3.1. *Assume that G' is continuous and for some integers $1 \le i \le n-1$ and $k \ge 1$,*

$$E((X_{i+1:n} - X_{i:n})^k \mid X_{i+1:n} = t) = (\eta t + \theta) E((X_{i+1:n} - X_{i:n})^{k-1} \mid X_{i+1:n} = t), \quad a < t < b, \tag{36}$$

where η and θ are constants. Then only the following three cases are possible:

(i) $\eta = 0$, $\theta > 0$, $a = -\infty$, $b < \infty$ *and* $G(x) = e^{-(k/(i\theta))(b-x)}$, $-\infty < x < b$;
(ii) $0 < \eta < 1$, $a = \theta/\eta$, $b < \infty$ *and* $G(x) = ((x-a)/(b-a))^{k(1-\eta)/(i\eta)}$, $a < x < b$;
(iii) $\eta < 0$, $a = -\infty$, $b < \delta$ *and* $G(x) = ((\delta - b)/(\delta - x))^{k(\eta-1)/(i\eta)}$, $-\infty < x < b$, *where* $\delta = -\theta/\eta$.

In the above theorem, we have a characterization of $\mathcal{U}(a, b)$ distribution in (ii), where $\eta = k/(k+i)$. Note that Theorem 3.1 covers the result reported by Ferguson (1967), where it was proved that G can be determined by (36) with $k = 1$.

The next theorem extends Nagaraja (1988), where G was characterized by using linearity of $E(Y_{j+1} - Y_j \mid Y_{j+1})$. It provides a characterization of the exponential distribution in (ii), where $\eta = k/(k+j)$.

Theorem 3.2. *Assume that G' is continuous and for some integers $j, k \ge 1$,*

$$E((Y_{j+1} - Y_j)^k \mid Y_{j+1} = t) = (\eta t + \theta) E((Y_{j+1} - Y_j)^{k-1} \mid Y_{j+1} = t), \quad a < t < b,$$

where η and θ are constants. Then only the following three cases are possible:

(i) $\eta = 0$, $\theta > 0$, $a = -\infty$, $b = \infty$ *and* $G(x) = 1 - \exp\left\{-c_1 e^{(k/(\theta j))x}\right\}$, $-\infty < x < \infty$, *where* $c_1 > 0$ *is a constant*;
(ii) $0 < \eta < 1$, $a = \theta/\eta$, $b = \infty$ *and* $G(x) = 1 - \exp\left\{-c_2 (x-a)^{k(1-\eta)/(j\eta)}\right\}$, $a < x < \infty$, *where* $c_2 > 0$ *is a constant*;
(iii) $\eta < 0$, $a = -\infty$, $b = -\theta/\eta$ *and* $G(x) = 1 - \exp\{-c_3 (b-x)^{k(1-\eta)/(j\eta)}\}$, $-\infty < x < b$, *where* $c_3 > 0$ *is a constant.*

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