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Generalized skew-Cauchy distribution ☆

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Abstract

In this work we investigate the generalized skew-symmetric distributions. Suppose Y is an absolutely continuous random variable symmetric about 0 with probability density function f and cumulative distribution function F. If a random variable X satisfies $X^2 \overset{d}{=} Y^2$, then X is said to have a generalized skew distribution of F (or f). The generalized skew-Cauchy (GSC) distribution are considered and special examples of GSC distribution are presented. Some of these examples are generated from generalized skew-normal or generalized skew-t distributions.

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1. Introduction

The univariate skew-normal distribution has been studied by many authors, see e.g. Azzalini (1985, 1986), Henze (1986), Chiogna (1998) and Gupta et al. (2004b). Following Azzalini (1985), a random variable X is said to have a skew-normal distribution with parameter λ, denoted by $X \sim \mathcal{SN}(\lambda)$, if the probability density function (p.d.f.) is given by

$$f_X(x) = 2\phi(x)\Phi(\lambda x), \quad \lambda, x \in \mathbb{R}, \tag{1}$$

where ϕ and Φ are the p.d.f. and cumulative distribution function (c.d.f.) of the standard normal distribution, respectively.

By letting the p.d.f. of the random variable X be

$$f_X(x) = 2\phi(x)\Phi\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right), \quad \lambda_1, x \in \mathbb{R}, \ \lambda_2 \geqslant 0, \tag{2}$$

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Arellano-Valle et al. (2004) defined a so-called skew-generalized normal distribution, they denoted this distribution by $\mathscr{SGN}(\lambda_1, \lambda_2)$.

The multivariate skew-normal distribution has also been considered by Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Gupta and Kollo (2003), and Gupta et al. (2004a). Here a p-dimensional random vector $\boldsymbol{X}$ is said to have a multivariate skew-normal distribution, denoted by $\boldsymbol{X} \sim \mathscr{SN}_p(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, if it is continuous and its p.d.f. is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = 2\phi_p(\boldsymbol{x}; \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}'\boldsymbol{x}), \tag{3}$$

where $\boldsymbol{\Omega} > 0$, $\boldsymbol{\alpha} \in \mathbb{R}^p$, $\phi_p(\boldsymbol{x}; \boldsymbol{\Omega})$ is the p.d.f. of $\mathscr{N}_p(\boldsymbol{0}, \boldsymbol{\Omega})$ distribution (the p-dimensional normal distribution with zero mean vector and correlation matrix $\boldsymbol{\Omega}$). Quadratic forms of skew-normal random vectors have been studied by Azzalini (1985), Azzalini and Dalla Valle (1996), Azzalini and Capitanio (1999), Loperfido (2001), Genton et al. (2001), and Gupta and Huang (2002). Based on Gupta and Huang (2002), some parallel results for the class of multivariate skew normal-symmetric distributions have also been obtained by Huang and Chen (2006).

If the p.d.f. of a random variable X has the form

$$f_X(x) = 2f(x)G(x), \quad x \in \mathbb{R}, \tag{4}$$

where f is a p.d.f. of a random variable symmetric about 0, and G is a Lebesque measurable function satisfying $0 \leqslant G(x) \leqslant 1$ and $G(x) + G(-x) = 1$ a.e. on $\mathbb{R}$, then X is said to have the so-called skew-symmetric distribution. Gupta et al. (2002) studied the models in which f is taken to be the p.d.f. from one of the following distributions: normal, Student's t, Cauchy, Laplace, logistic, and uniform distribution, and G is a distribution function such that G' is symmetric about 0. Nadarajah and Kotz (2003) considered the models that f is taken to be a normal p.d.f. with zero mean, while G is taken to come from one of the above continuous symmetric distributions. Multivariate skew-symmetric distributions have also been studied by Gupta and Chang (2003) and Wang et al. (2004a). The multivariate skew-Cauchy distribution and multivariate skew t-distribution are studied by Arnold and Beaver (2000), and Gupta (2003), respectively.

It is known that the square of each of the $\mathscr{N}(0,1)$, $\mathscr{SN}(\lambda)$ and $\mathscr{SGN}(\lambda_1, \lambda_2)$ distribution is χ_1^2 distributed. Based on this observation, in this paper, first we introduce the generalized skew-symmetric model in Section 2. Then in Section 3, we introduce the generalized skew-Cauchy (GSC) distribution. In Section 4, some examples as well as their p.d.f.s of GSC distribution generated by the ratio of two independent generalized skew normally distributed random variables will be given. Finally, in Section 5, several of the possible shapes of the p.d.f. of a main example in Section 4 under various choices of parameters will be illustrated.

2. Generalized skew distributions

First we give a definition.

Definition 1. Suppose Y is an absolutely continuous random variable symmetric about 0 with p.d.f. f and c.d.f. F. Assume random variable X satisfies

$$X^2 \overset{\text{d}}{=} Y^2. \tag{5}$$

Then X is said to have a generalized skew distribution of F (or f).

In the above definition, if Y has a common distribution, such as $\mathscr{N}(0,1)$ distribution, then X is said to have a generalized skew-$\mathscr{N}(0,1)$ distribution. The p.d.f. of a generalized skew distribution can be obtained by using the following lemma.

Lemma 1 (*Huang et al., 2005*). *Let n be a positive integer, and $h(t)$, $t \in A$, a continuous p.d.f. Also assume $A \subset [0, \infty)$, when n is even. Then X^n has h as its p.d.f., if and only if the p.d.f. of X is*

$$f_X(x) = \begin{cases} nx^{n-1}h(x^n), & n \text{ is odd}, \\ n|x|^{n-1}h(x^n)G(x), & n \text{ is even}, \end{cases} \tag{6}$$

where $x \in B = \{x | x \in \mathbb{R}, x^n \in A\}$, *and* $G(x)$ *is a Lebesgue measurable function which satisfies* $0 \leqslant G(x) \leqslant 1$ *and* $G(x) + G(-x) = 1$ *a.e.*, $\forall x \in B$.

According to the above lemma, the following theorem is obtained immediately.

Theorem 1. *Assume the random variable Y is defined as in Definition* 1, *and X has a generalized skew distribution of f. Then the p.d.f. of X is*

$$f_X(x) = 2f(x)G(x), \quad x \in \mathbb{R}, \tag{7}$$

or equivalently,

$$f_X(x) = f(x)(1 + H(x)), \quad x \in \mathbb{R}, \tag{8}$$

where G, the skew function, is a Lebesque measurable function satisfying

$$0 \leqslant G(x) \leqslant 1 \quad \text{and} \quad G(x) + G(-x) = 1 \ \text{a.e. on } \mathbb{R}, \tag{9}$$

and $H(x) = 2G(x) - 1$, *satisfying*

$$-1 \leqslant H(x) \leqslant 1 \quad \text{and} \quad H(-x) = -H(x) \ \text{a.e. on } \mathbb{R}. \tag{10}$$

Proof. Let $h(t)$ be the p.d.f. of X^2. Then $h(t) = t^{-1/2} f(t^{1/2})$, $t > 0$. By Lemma 1,

$$f_X(x) = 2|x| \frac{1}{|x|} f(|x|)G(x) = 2f(x)G(x), \quad x \in \mathbb{R},$$

as required, where $f(|x|) = f(x)$ is by the fact that f is symmetric about 0. The rest of the proof is obvious hence is omitted. □

There are infinitely many functions satisfy (9). For example G is the distribution function corresponding to a symmetric random variable (in particular G can be taken as F), $G(x) = (1 + \sin x)/2$ (hence G is not necessary to be increasing), $G(x) \equiv \frac{1}{2}$ (in this case $f_X(x) = f(x), x \in \mathbb{R}$), etc. The p.d.f. given in (7) has the same form as in (4). In fact, Arnold and Lin (2004) have used f_X in (7) with $f = \phi$ to define the generalized skew-$\mathscr{N}(0,1)$ distribution. $\mathscr{SN}(\lambda)$, $\mathscr{SGN}(\lambda_1, \lambda_2)$, the skew-normal symmetric models of Nadarajah and Kotz (2003) all belong to the class of generalized skew-$\mathscr{N}(0,1)$ distribution.

Let (Y_1, Y_2) be $\mathscr{BVN}(0,0,1,1,\rho)$ distributed, $|\rho| \neq 1$. Denote $Y_{(1)} = \min\{Y_1, Y_2\}$ and $Y_{(2)} = \max\{Y_1, Y_2\}$. Loperfido (2002) pointed out that $Y_{(1)} \sim \mathscr{SN}(-\gamma)$ and $Y_{(2)} \sim \mathscr{SN}(\gamma)$, where $\gamma = [(1-\rho)/(1+\rho)]^{1/2}$. For the minimum and maximum of a random sample of size two, we have the following result.

Proposition 1. *Suppose* X_1 *and* X_2 *are two independent and identically distributed random variables with the common absolutely continuous c.d.f. F and p.d.f. f, where f is assumed to be symmetric about* 0. *Let* $X_{(1)} = \min\{X_1, X_2\}$, $X_{(2)} = \max\{X_1, X_2\}$. *Then* $X_{(1)}$ *and* $X_{(2)}$ *are both generalized skew distributions of f with p.d.f.s*

$$f_{X_{(1)}}(y_1) = 2f(y_1)F(-y_1), \quad y_1 \in \mathbb{R}, \tag{11}$$

and

$$f_{X_{(2)}}(y_2) = 2f(y_2)F(y_2), \quad y_2 \in \mathbb{R}, \tag{12}$$

respectively. Also $|X_{(1)}| \overset{d}{=} |X_{(2)}| \overset{d}{=} |X_1|$.

Proof. For independent and identically distributed random variables, the marginal p.d.f.s of $X_{(1)}$ and $X_{(2)}$ can be obtained immediately. By using $1 - F(y_1) = F(-y_1)$, $y_1 \in \mathbb{R}$, it yields (11). The rest of this proposition is obvious. □

It can be seen easily, that in the above proposition, neither the minimum nor the maximum has a generalized skew distribution of f, if the sample size of random variables is greater than two. Also when (X_1, X_2) is $\mathscr{BVN}(0,0,1,1,0)$ distributed, namely X_1 and X_2 are independent $\mathscr{N}(0,1)$ distributed, then Proposition 1 implies $X_{(1)}$ and $X_{(2)}$ are $\mathscr{SN}(-1)$ and $\mathscr{SN}(1)$ distributed, respectively, which coincides with the result by Loperfido (2002).

The next property for generalized skew-$\mathscr{N}(0, \sigma^2)$ distribution is also immediate.

Proposition 2. *Let $X_1, \ldots, X_{n+m}$, $n, m \geqslant 1$, be independent random variables each has a generalized skew-$\mathcal{N}(0, \sigma^2)$ distribution. Then*

$$\frac{\sum_{i=1}^{n} X_i^2/n}{\sum_{i=n+1}^{n+m} X_i^2/m} \sim \mathcal{F}_{n,m},$$

where $\mathcal{F}_{n,m}$ has an $\mathcal{F}$ distribution with n and m degrees of freedom.

Note that it is allowed that the random variables $X_1, \ldots, X_{n+m}$ in the above proposition are not necessary to be identically distributed. The following is an equivalent condition to (5).

Proposition 3 (Wang et al., 2004b). *If $X \sim 2f(x)G(x)$ and $Y \sim 2\tilde{f}(x)\tilde{G}(x)$, where $2f(x)G(x)$ and $2\tilde{f}(x)\tilde{G}(x)$ are two p.d.f.s of generalized skew distributions, then*

$$f(x) = \tilde{f}(x) \Leftrightarrow \tau(X) \stackrel{\mathrm{d}}{=} \tau(Y), \quad \text{for every even function } \tau,$$
$$\Leftrightarrow X^2 \stackrel{\mathrm{d}}{=} Y^2.$$

It should be mentioned here, one even function τ, such that $\tau(X) \stackrel{\mathrm{d}}{=} \tau(Y)$ is enough to imply $X^2 \stackrel{\mathrm{d}}{=} Y^2$. We give a simple proposition below, which can be compared with Proposition 3 of Arellano-Valle et al. (2004).

Proposition 4. *Let X be generalized skew-$\mathcal{N}(0, \sigma^2)$ distributed, Y be $\mathcal{N}(0, \sigma^2)$ distributed, and Z be χ_1^2 distributed, $\sigma > 0$. Then $|X| \stackrel{\mathrm{d}}{=} |Y| \stackrel{\mathrm{d}}{=} \sigma\sqrt{Z} \sim \mathcal{HN}(0, \sigma^2)$, where $\mathcal{HN}(0, \sigma^2)$ denotes the half-normal distribution with parameter σ.*

Although there are some parallel properties between non-skew and skew distributions, there also have many properties hold for non-skew distributions but not for skew distributions. We list some examples below:

Let X_1 and X_2 be independent and identically distributed random variables with $\mathcal{N}(0, \sigma^2)$ being their common distribution. Then

(i) $X_1^2 + X_2^2$ and $X_1/\sqrt{X_1^2 + X_2^2}$ are independent,
(ii) $X_1^2 + X_2^2$ and X_1/X_2 are independent,
(iii) $X_1 - X_2$ and $X_1 + X_2$ are independent.

But none of these properties hold for any other generalized skew-$\mathcal{N}(0, \sigma^2)$ distributions.

3. The GSC models

We now use Definition 1 to define the generalized skew-Cauchy distribution.

X is said to have a generalized skew-$\mathcal{C}(0, \sigma)$ distribution, denoted by $\mathcal{GSC}(\sigma)$, where $\sigma > 0$, if $X^2 \stackrel{\mathrm{d}}{=} Y^2$, where Y has a $\mathcal{C}(0, \sigma)$ distribution. That is X^2 has the p.d.f.

$$h(t) = \frac{\sigma}{\pi[\sqrt{t}(\sigma^2 + t)]}, \quad t \geqslant 0, \quad \sigma > 0. \tag{13}$$

Denote the distribution of X^2 by $\mathcal{C}^2(0, \sigma)$.

By Theorem 1, X has a $\mathcal{GSC}(\sigma)$ distribution, if and only if the p.d.f. of X has either of the following forms:

$$f_X(x) = \frac{2\sigma}{\pi(\sigma^2 + x^2)} G(x), \quad x \in \mathbb{R}, \quad \sigma > 0, \tag{14}$$

or

$$f_X(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)} (1 + H(x)), \quad x \in \mathbb{R}, \quad \sigma > 0, \tag{15}$$

where G and H are Lebesque measurable functions satisfying (9) and (10), respectively. There are some simple properties for the distribution of $\mathcal{GSC}(\sigma)$.

Proposition 5. (i) *The only symmetric* $\mathscr{GSC}(\sigma)$ *distribution is* $\mathscr{C}(0,\sigma)$ *distribution.*

(ii) *Let* $X \sim \mathscr{GSC}(\sigma)$, *and* $r \in \mathbb{R}$. *Then* $E|X|^r$ *exists if and only if* $|r|<1$.

(iii) $X \sim \mathscr{GSC}(\sigma) \Leftrightarrow X^2 \sim \mathscr{C}^2(0,\sigma) \Leftrightarrow \frac{1}{X^2} \sim \mathscr{C}^2(0,\frac{1}{\sigma}) \Leftrightarrow \frac{1}{X} \sim \mathscr{GSC}(\frac{1}{\sigma})$.

Gupta et al. (2002) gave three examples of GSC distribution. The first example is defined in a similar way as the skew-normal distribution defined by Azzalini (1985, 1986). That is the p.d.f. of X is $2f(x)F(\lambda x)$, where $f(\cdot)$ and $F(\cdot)$ are the p.d.f. and c.d.f. of $\mathscr{C}(0,\sigma)$ distribution, respectively. More precisely, the p.d.f. of X is given by

$$f_1(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}\left[1 + \frac{2\arctan(\lambda x/\sigma)}{\pi}\right], \quad \lambda, x \in \mathbb{R}, \quad \sigma > 0. \tag{16}$$

As $\mathscr{C}(0,1)$ distribution is exactly the $\mathscr{T}_1$ distribution, inspired by this, the second example of GSC distribution is based on the skew-$\mathscr{T}_1$ distribution, the latter is defined in a similar way as t distribution.

Example 1. Let $X = U/\sqrt{W}$, where U has a generalized skew-$\mathscr{N}(0,1)$ distribution and W independent of U is χ^2_1 distributed. Then X has a $\mathscr{GSC}(1)$ distribution.

Note that the random variable X given above satisfies $X^2 \stackrel{d}{=} X_1^2$, where $X_1 = U_1/\sqrt{W_1}$, U_1 has a $\mathscr{N}(0,1)$ distribution, and W_1 independent of U_1 is χ^2_1 distributed. That is X_1 is $\mathscr{T}_1$ distributed. Hence $\mathscr{GSC}(1)$ is also a generalized skew-$\mathscr{T}_1$ distribution.

For a special case, let U have a $\mathscr{SN}(\lambda_1)$ distribution. Then the p.d.f. of X is given by

$$f_2(x) = \frac{1}{\pi(1+x^2)}\left[1 + \frac{\lambda_1 x}{\sqrt{1+(1+\lambda_1^2)x^2}}\right], \quad \lambda_1, x \in \mathbb{R}, \tag{17}$$

which is the second example of GSC distribution given by Gupta et al. (2002).

4. More examples of GSC distribution

First we give another GSC example below, which is a slight generalization of the second example given by Gupta et al. (2002).

Example 2. Let U and V be two independent random variables both are generalized skew-$\mathscr{N}(0,\sigma^2)$ distributed. Then $X = U/|V|$ has a $\mathscr{GSC}(1)$ distribution.

In particular, let U be $\mathscr{SN}(\lambda)$ distributed, and V be generalized skew-(0, 1) distributed. Then $X = U/|V|$ has the p.d.f. given in (17).

The reason that the two X's defined in Example 1 and this example are equally distributed is due to Proposition 4.

Suppose that U and V are two independent random variables and both are $\mathscr{N}(0,\sigma^2)$ distributed, $\sigma>0$. It is known that not only $U/|V|$ but also U/V is $\mathscr{C}(0,1)$ distributed. The next example indicates similar result holds for generalized skew-normal distributions. This example nevertheless is a slight generalization of Examples 1 and 2. Note that both $\sqrt{W}$ in Example 1 and $|V|$ in Example 2 are generalized skew-$\mathscr{N}(0,1)$ distributed.

Example 3. Let U and V be two independent random variables both distributed as generalized skew-$\mathscr{N}(0,\sigma^2)$ distribution. Then $X = U/V$ has a $\mathscr{GSC}(1)$ distribution.

The third way of Gupta et al. (2002) to define GSC distribution is by letting $X = U/V$, where U and V are independent random variables both distributed as $\mathscr{SN}(\lambda)$. Obviously X has a $\mathscr{GSC}(1)$ distribution. Although Gupta et al. (2002) failed to obtain the closed form of the p.d.f. of X, the p.d.f. actually can be obtained. The following theorem indicates that the closed form of the p.d.f. of X can be derived, even under a more general setting.

Theorem 2. *Let U and V be independent random variables distributed as $\mathscr{SN}(\lambda_1)$ and $\mathscr{SN}(\lambda_2)$, respectively, $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $X \equiv X(\lambda_1, \lambda_2) = U/V$ has a $\mathscr{GSC}(1)$ distribution with p.d.f.*

$$f_X(x) = \frac{1}{\pi(1+x^2)}\left(1 + \frac{2\lambda_2 \arctan(\lambda_1 x/\sqrt{1+\lambda_2^2+x^2})}{\pi\sqrt{1+\lambda_2^2+x^2}} + \frac{2\lambda_1 x \arctan(\lambda_2/\sqrt{1+(1+\lambda_1^2)x^2})}{\pi\sqrt{1+(1+\lambda_1^2)x^2}}\right),$$
$$x \in \mathbb{R}. \tag{18}$$

Before proving this theorem, we give some preliminary results below about the integrations with regard to the p.d.f. ϕ and c.d.f. Φ of the $\mathscr{N}(0,1)$ distribution. The first lemma can be found in Gupta and Brown (2001).

Lemma 2. *For any $b \in \mathbb{R}$,*

$$\int_0^\infty \phi(t)\Phi(bt)\,\mathrm{d}t = \frac{1}{4} + \frac{1}{2\pi}\arctan(b). \tag{19}$$

Lemma 3. *For $s \geqslant 0$, integer $r \geqslant 1$ and $a_1, \ldots, a_r \in \mathbb{R}, \sum_{i=1}^r a_i^2 \neq 0$,*

$$\int_0^\infty v^s \phi(a_1 v)\cdots\phi(a_r v)\,\mathrm{d}v = \frac{\Gamma((s+1)/2)2^{(s-1)/2}}{(2\pi)^{r/2}(\sum_{i=1}^r a_i^2)^{(s+1)/2}}. \tag{20}$$

Proof. Since $\phi(a_1 v)\cdots\phi(a_r v) = (\sqrt{2\pi})^{-(r-1)}\phi((\sum_{i=1}^r a_i^2)^{1/2} v)$, without loss of generality, it suffices to prove the case $r = 1$. Now by letting $t = a_1^2 v^2$, we have

$$\begin{aligned}\int_0^\infty v^s \phi(a_1 v)\,\mathrm{d}v &= \frac{1}{\sqrt{2\pi}}\int_0^\infty v^s \mathrm{e}^{-a_1^2 v^2/2}\,\mathrm{d}v \\ &= \frac{1}{2\sqrt{2\pi}(a_1^2)^{(s+1)/2}}\int_0^\infty t^{(s-1)/2}\mathrm{e}^{-t/2}\,\mathrm{d}t \\ &= \frac{\Gamma((s+1)/2)2^{(s-1)/2}}{\sqrt{2\pi}(a_1^2)^{(s+1)/2}}\end{aligned}$$

as desired. □

The next lemma is an extension of the above two lemmas.

Lemma 4. *For $s \geqslant 2$, integer $r \geqslant 1$ and $a_1, \ldots, a_r, b \in \mathbb{R}, \sum_{i=1}^r a_i^2 \neq 0$, we have the following recursive formula:*

$$\begin{aligned}\int_0^\infty v^s \phi(a_1 v)\cdots\phi(a_r v)\Phi(bv)\,\mathrm{d}v &= \frac{b\Gamma(s/2)2^{s/2-1}}{(2\pi)^{(r+1)/2}(\sum_{i=1}^r a_i^2)(\sum_{i=1}^r a_i^2 + b^2)^{s/2}} \\ &\quad + \frac{s-1}{\sum_{i=1}^r a_i^2}\int_0^\infty v^{s-2}\phi(a_1 v)\cdots\phi(a_r v)\Phi(bv)\,\mathrm{d}v.\end{aligned} \tag{21}$$

Also

$$\int_0^\infty \phi(a_1 v)\cdots\phi(a_r v)\Phi(bv)\,\mathrm{d}v = \frac{1}{(2\pi)^{(r+1)/2}(\sum_{i=1}^r a_i^2)^{1/2}}\left(\frac{\pi}{2} + \arctan\left(\frac{b}{(\sum_{i=1}^r a_i^2)^{1/2}}\right)\right), \tag{22}$$

and

$$\int_0^\infty v\phi(a_1 v)\cdots\phi(a_r v)\Phi(bv)\,\mathrm{d}v = \frac{1}{2(2\pi)^{r/2}(\sum_{i=1}^r a_i^2)}\left(1 + \frac{b}{(\sum_{i=1}^r a_i^2 + b^2)^{1/2}}\right). \tag{23}$$

Proof. Again it suffices to prove the case $r = 1$. For $s \geqslant 2$, by integration by parts and Lemma 3, it yields

$$\begin{aligned}
\int_0^\infty v^s \phi(a_1 v)\Phi(bv)\,\mathrm{d}v &= \int_0^\infty v^s \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-a_1^2 v^2/2}\Phi(bv)\,\mathrm{d}v \\
&= \frac{-1}{\sqrt{2\pi}a_1^2}\left[v^{s-1}\Phi(bv)\mathrm{e}^{-a_1^2v^2/2}\Big|_0^\infty - \int_0^\infty \mathrm{e}^{-a_1^2v^2/2}\mathrm{d}\big(v^{s-1}\Phi(bv)\big)\right] \\
&= \frac{b}{\sqrt{2\pi}a_1^2}\int_0^\infty v^{s-1}\mathrm{e}^{-a_1^2v^2/2}\phi(bv)\,\mathrm{d}v + \frac{s-1}{\sqrt{2\pi}a_1^2}\int_0^\infty v^{s-2}\mathrm{e}^{-a_1^2v^2/2}\Phi(bv)\,\mathrm{d}v \\
&= \frac{b}{a_1^2}\int_0^\infty v^{s-1}\phi(a_1v)\phi(bv)\,\mathrm{d}v + \frac{s-1}{a_1^2}\int_0^\infty v^{s-2}\phi(a_1v)\Phi(bv)\,\mathrm{d}v \\
&= \frac{b\Gamma(s/2)2^{s/2-1}}{2\pi a_1^2(a_1^2+b^2)^{s/2}} + \frac{s-1}{a_1^2}\int_0^\infty v^{s-2}\phi(a_1v)\Phi(bv)\,\mathrm{d}v.
\end{aligned}$$

This proves (21) for the case $r = 1$.

Next by letting $t = |a_1|v$, from Lemma 2 we have

$$\int_0^\infty \phi(a_1v)\Phi(bv)\,\mathrm{d}v = \frac{1}{|a_1|}\int_0^\infty \phi(t)\Phi\left(\frac{b}{|a_1|}t\right)\mathrm{d}t = \frac{1}{4|a_1|} + \frac{1}{2\pi|a_1|}\arctan\left(\frac{b}{|a_1|}\right),$$

this is exactly (22) for $r = 1$.

Finally, again by integration by parts and Lemma 3,

$$\begin{aligned}
\int_0^\infty v\phi(a_1v)\Phi(bv)\,\mathrm{d}v &= \frac{-1}{\sqrt{2\pi}a_1^2}\left[\Phi(bv)\mathrm{e}^{-a_1^2v^2/2}\Big|_0^\infty - \int_0^\infty \mathrm{e}^{-a_1^2v^2/2}\,\mathrm{d}\Phi(bv)\right] \\
&= \frac{-1}{\sqrt{2\pi}a_1^2}\left[-\frac{1}{2} - b\int_0^\infty \mathrm{e}^{-a_1^2v^2/2}\phi(bv)\,\mathrm{d}v\right] \\
&= \frac{1}{2\sqrt{2\pi}a_1^2} + \frac{b}{a_1^2}\int_0^\infty \phi(av)\phi(bv)\,\mathrm{d}v \\
&= \frac{1}{2\sqrt{2\pi}a_1^2} + \frac{b}{a_1^2}\frac{\Gamma(1/2)2^{-1/2}}{2\pi(a_1^2+b^2)^{1/2}} \\
&= \frac{1}{2\sqrt{2\pi}a_1^2}\left[1 + \frac{b}{(a_1^2+b^2)^{1/2}}\right].
\end{aligned}$$

This completes the proof of this lemma. $\square$

We also have an extended corollary.

Corollary 1. *For integer* $r \geqslant 1$, *and* $a_1, \ldots, a_r, b_1, b_2 \in \mathbb{R}$, $\sum_{i=1}^r a_i^2 \neq 0$,

$$\begin{aligned}
&\int_0^\infty v\phi(a_1v)\cdots\phi(a_rv)\Phi(b_1v)\Phi(b_2v)\,\mathrm{d}v \\
&\quad = \frac{1}{2(2\pi)^{(r+2)/2}(\sum_{i=1}^r a_i^2)}\left[\pi + \frac{b_1(\pi + 2\arctan(b_2/(\sum_{i=1}^r a_i^2 + b_1^2)^{1/2}))}{(\sum_{i=1}^r a_i^2 + b_1^2)^{1/2}}\right. \\
&\qquad \left. + \frac{b_2(\pi + 2\arctan(b_1/(\sum_{i=1}^r a_i^2 + b_2^2)^{1/2}))}{(\sum_{i=1}^r a_i^2 + b_2^2)^{1/2}}\right]. \qquad (24)
\end{aligned}$$

Proof. Again it suffices to prove the case $r = 1$. By integration by parts and Lemma 4, it yields

$$
\begin{aligned}
&\int_0^\infty v\phi(a_1 v)\Phi(b_1 v)\Phi(b_2 v)\,\mathrm{d}v \\
&= \int_0^\infty v\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-a_1^2v^2/2}\Phi(b_1 v)\Phi(b_2 v)\,\mathrm{d}v \\
&= \frac{-1}{\sqrt{2\pi}a_1^2}\left[\Phi(b_1 v)\Phi(b_2 v)\mathrm{e}^{-a_1^2v^2/2}\Big|_0^\infty - \int_0^\infty \mathrm{e}^{-a_1^2v^2/2}\mathrm{d}(\Phi(b_1 v)\Phi(b_2 v))\right] \\
&= \frac{-1}{\sqrt{2\pi}a_1^2}\left[-\frac{1}{4} - \int_0^\infty \mathrm{e}^{-a_1^2v^2/2}[b_1\phi(b_1 v)\Phi(b_2 v) + b_2\phi(b_2 v)\Phi(b_1 v)]\,\mathrm{d}v\right] \\
&= \frac{1}{4\sqrt{2\pi}a_1^2} + \frac{b_1}{a_1^2}\int_0^\infty \phi(a_1 v)\phi(b_1 v)\Phi(b_2 v)\,\mathrm{d}v + \frac{b_2}{a_1^2}\int_0^\infty \phi(a_1 v)\phi(b_2 v)\Phi(b_1 v)\,\mathrm{d}v \\
&= \frac{1}{4\sqrt{2\pi}a_1^2} + \frac{b_1}{a_1^2}\left[\frac{1}{(2\pi)^{3/2}(a_1^2+b_1^2)^{1/2}}\left(\frac{\pi}{2} + \arctan\left(\frac{b_2}{(a_1^2+b_1^2)^{1/2}}\right)\right)\right] \\
&\quad + \frac{b_2}{a_1^2}\left[\frac{1}{(2\pi)^{3/2}(a_1^2+b_2^2)^{1/2}}\left(\frac{\pi}{2} + \arctan\left(\frac{b_1}{(a_1^2+b_2^2)^{1/2}}\right)\right)\right] \\
&= \frac{1}{2(2\pi)^{3/2}a_1^2}\left[\pi + \frac{b_1(\pi + 2\arctan(b_2/(a_1^2+b_1^2)^{1/2}))}{(a_1^2+b_1^2)^{1/2}} + \frac{b_2(\pi + 2\arctan(b_1/(a_1^2+b_2^2)^{1/2}))}{(a_1^2+b_2^2)^{1/2}}\right]. \qquad \square
\end{aligned}
$$

Proof of Theorem 2. That X has a $\mathscr{GSC}(1)$ distribution is obvious. We derive the p.d.f. of X in the following. First the joint p.d.f. of U and V is

$$f_{U,V}(u,v) = 4\phi(u)\phi(v)\Phi(\lambda_1 u)\Phi(\lambda_2 v), \quad u, v \in \mathbb{R}.$$

Hence the p.d.f. of X is

$$
\begin{aligned}
f_X(x) &= 4\int_{-\infty}^\infty |v|\phi(xv)\phi(v)\Phi(\lambda_1 xv)\Phi(\lambda_2 v)\,\mathrm{d}v \\
&= 4\int_0^\infty v\phi(xv)\phi(v)\Phi(\lambda_1 xv)\Phi(\lambda_2 v)\,\mathrm{d}v + 4\int_0^\infty v\phi(xv)\phi(v)\Phi(-\lambda_1 xv)\Phi(-\lambda_2 v)\,\mathrm{d}v.
\end{aligned}
$$

By using Corollary 1, it yields

$$
\begin{aligned}
f_X(x) = 4\cdot\frac{1}{8\pi^2(1+x^2)}&\left[\pi + \frac{\lambda_2(\pi + 2\arctan(\lambda_1 x/\sqrt{1+\lambda_2^2+x^2}))}{\sqrt{1+\lambda_2^2+x^2}}\right. \\
&\left.+\frac{\lambda_1 x(\pi + 2\arctan(\lambda_2/\sqrt{1+(1+\lambda_1^2)x^2}))}{\sqrt{1+(1+\lambda_1^2)x^2}}\right] \\
+ 4\cdot\frac{1}{8\pi^2(1+x^2)}&\left[\pi + \frac{-\lambda_2(\pi + 2\arctan(-\lambda_1 x/\sqrt{1+\lambda_2^2+x^2}))}{\sqrt{1+\lambda_2^2+x^2}}\right. \\
&\left.+\frac{-\lambda_1 x(\pi + 2\arctan(-\lambda_2/\sqrt{1+(1+\lambda_1^2)x^2}))}{\sqrt{1+(1+\lambda_1^2)x^2}}\right]
\end{aligned}
$$

$$= \frac{1}{\pi(1+x^2)}\left(1+\frac{2\lambda_2\,\arctan(\lambda_1 x/\sqrt{1+\lambda_2^2+x^2})}{\pi\sqrt{1+\lambda_2^2+x^2}}+\frac{2\lambda_1 x\,\arctan(\lambda_2/\sqrt{1+(1+\lambda_1^2)x^2})}{\pi\sqrt{1+(1+\lambda_1^2)x^2}}\right),$$

$$\lambda_1, \lambda_2, x \in \mathbb{R},$$

as desired. □

We give another special case of Example 3.

Example 4. Let $X = U/V$, where U is $\mathcal{N}(0,1)$ distributed, V is $\mathcal{SN}(\lambda)$ distributed, and U and V are independent. By noting $\mathcal{SN}(0) \stackrel{d}{=} \mathcal{N}(0,1)$, from (18) we obtain immediately

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

Consequently, X is $\mathscr{C}(0,1)$ distributed and independent of λ. Being $\mathscr{C}(0,1)$ distributed, X and $1/X$ have the same distribution. Hence $X_1 = V/U$ is also $\mathscr{C}(0,1)$ distributed.

The following is an extension of Example 4.

Example 5. Let U be $\mathcal{N}(0,\sigma^2)$ distributed, and V be generalized skew-$\mathcal{N}(0,\sigma^2)$ distributed. Then $X = U/V$ is $\mathscr{C}(0,1)$ distributed.

Proof. First the joint p.d.f. of U and V is

$$f_{U,V}(u,v) = \frac{2}{\sigma^2}\phi\left(\frac{u}{\sigma}\right)\phi\left(\frac{v}{\sigma}\right)G(v), \quad u,v \in \mathbb{R}, \ \sigma > 0,$$

where $G(v)$ is a Lebesque measurable function satisfying condition (9). Hence by letting $t = v/\sigma$, the p.d.f. of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{2}{\sigma^2}\phi\left(\frac{xv}{\sigma}\right)\phi\left(\frac{v}{\sigma}\right)G(v)|v|\,\mathrm{d}v. \\ &= \int_{-\infty}^{\infty} 2\phi(xt)\phi(t)G(\sigma t)|t|\,\mathrm{d}t \\ &= 2\int_0^{\infty} t\phi(xt)\phi(t)G(\sigma t)\,\mathrm{d}t + 2\int_0^{\infty} t\phi(xt)\phi(t)G(-\sigma t)\,\mathrm{d}t \\ &= 2\int_0^{\infty} t\phi(xt)\phi(t)[G(\sigma t)+G(-\sigma t)]\,\mathrm{d}t \\ &= 2\int_0^{\infty} t\phi(xt)\phi(t)\,\mathrm{d}t = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}, \end{aligned}$$

as desired.

Obviously, the result still holds true if U is generalized skew-$\mathcal{N}(0,\sigma^2)$ distributed and V is $\mathcal{N}(0,\sigma^2)$ distributed. □

Finally, limiting distributions of the random variable $X(\lambda_1,\lambda_2)$ defined in Theorem 2 when λ_1, λ_2 tend to $0, \infty$ or $-\infty$ can be obtained easily. The details are omitted.

5. Some figures of the p.d.f. of the GSC distribution

In this section, several of the possible shapes of the p.d.f. of the random variable $X(\lambda_1,\lambda_2)$ in Theorem 2 under various choices of (λ_1,λ_2) are illustrated. From Fig. 1, it seems the p.d.f. of the $X(\lambda_1,\lambda_2)$ distribution may have one side heavier tail and one side thinner tail than the $\mathscr{C}(0,1)$ distribution. However, it can be seen easily that for any $\lambda_1,\lambda_2 \in \mathbb{R}$, the ratio of the p.d.f. of $X(\lambda_1,\lambda_2)$ to the p.d.f. of $\mathscr{C}(0,1)$ distribution tends to 1 as $x \to \infty$ or $-\infty$.

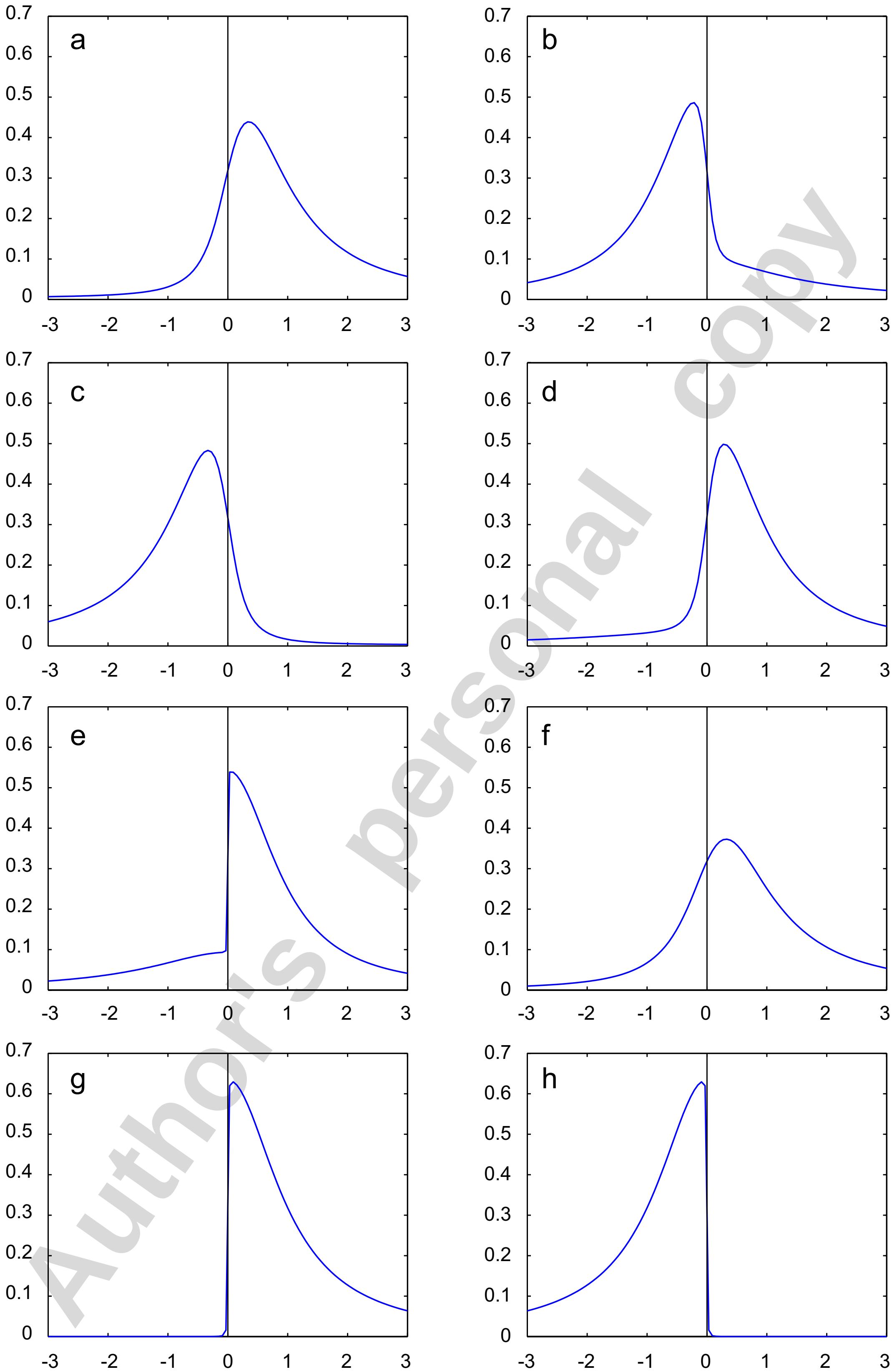

Fig. 1. Probability density function of $X(\lambda_1, \lambda_2)$ for several values of (λ_1, λ_2): (a) $(\lambda_1, \lambda_2) = (2, 5)$, (b) $(\lambda_1, \lambda_2) = (-6, 1)$, (c) $(\lambda_1, \lambda_2) = (3, -7)$, (d) $(\lambda_1, \lambda_2) = (-4, -2)$, (e) $(\lambda_1, \lambda_2) = (100, 1)$, (f) $(\lambda_1, \lambda_2) = (1, 100)$, (g) $(\lambda_1, \lambda_2) = (100, 100)$, and (h) $(\lambda_1, \lambda_2) = (100, -100)$.

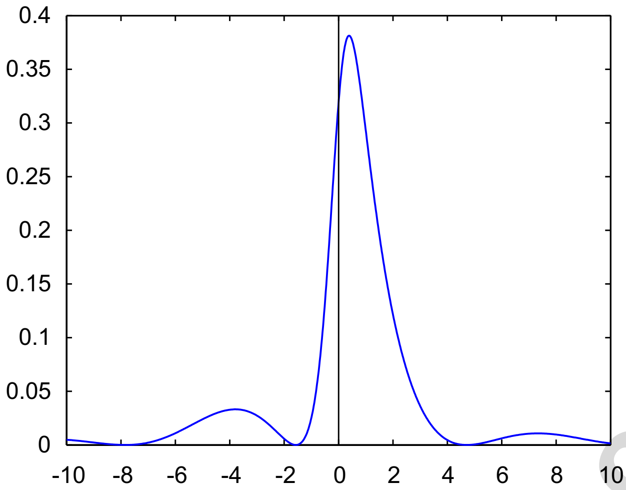

Fig. 2. Probability density function of $f(x) = (1 + \sin x)/(\pi(1 + x^2))$, $x \in \mathbb{R}$.

In general, GSC distribution may not have the same tail heaviness as the $\mathscr{C}(0, 1)$ distribution also may not be unimodal. As an example let $\sigma = 1$ and $H(x) = \sin x$ in (15), Fig. 2 depicts this p.d.f. curve. Yet our conjecture is for any $\lambda_1, \lambda_2 \in \mathbb{R}$, the p.d.f. curve of $X(\lambda_1, \lambda_2)$ is unimodal. This and some other related problems will be studied in the future.

References

Arellano-Valle, R.B., Gómez, H.W., Quintana, F.A., 2004. A new class of skew-normal distributions. Commun. Statist. Theory Meth. 33, 1975–1991.

Arnold, B.C., Beaver, R.J., 2000. The skew-Cauchy distribution. Statist. Probab. Lett. 49, 285–290.

Arnold, B.C., Lin, G.-D., 2004. Characterizations of the skew-normal and generalized chi distributions. Sankhyā 66, 593–606.

Azzalini, A., 1985. A class of distributions which includes the normal ones. Scand. J. Statist. 12, 171–178.

Azzalini, A., 1986. Further results on a class of distributions which includes the normal ones. Statistica 46, 199–208.

Azzalini, A., Capitanio, A., 1999. Statistical applications of the multivariate skew normal distribution. J. Roy. Statist. Soc. B 61, 579–602.

Azzalini, A., Dalla Valle, A., 1996. The multivariate skew-normal distribution. Biometrika 83, 715–726.

Chiogna, M., 1998. Some results on the scalar skew-normal distribution. J. Ital. Statist. Soc. 7, 1–13.

Genton, M.G., He, L., Liu, X., 2001. Moments of skew-normal random vectors and their quadratic forms. Statist. Probab. Lett. 51, 319–325.

Gupta, A.K., 2003. Multivariate skew *t*-distribution. Statistics 37, 359–363.

Gupta, A.K., Chang, F.-C., 2003. Multivariate skew-symmetric distributions. Appl. Math. Lett. 16, 643–646.

Gupta, A.K., Chang, F.-C., Huang, W.-J., 2002. Some skew-symmetric models. Random Operators Stochastic Equation 10, 133–140.

Gupta, A.K., Huang, W.-J., 2002. Quadratic forms in skew normal variates. J. Math. Anal. Appl. 273, 558–564.

Gupta, A.K., Kollo, T., 2003. Density expansions based on the multivariate skew normal distribution. Sankhyā 65, 821–835.

Gupta, A.K., González-Farías, G., Domínguez-Molina, J.A., 2004a. A multivariate skew normal distribution. J. Multivariate Analysis 89, 181–190.

Gupta, A.K., Nguyen, T.T., Sanqui, J.A.T., 2004b. Characterization of the skew-normal distribution. Ann. Inst. Statist. Math. 56, 351–360.

Gupta, R.C., Brown, N., 2001. Reliability studies of the skew-normal distribution and its application to a strength-stress model. Commun. Statist. Theory Meth. 30, 2427–2445.

Henze, N., 1986. A probabilistic representation of the 'skew-normal' distribution. Scand. J. Statist. 13, 271–275.

Huang, W.-J., Chen, Y.-H., 2006. Quadratic forms of multivariate skew normal-symmetric distributions. Statist. Probab. Lett. 76, 871–879.

Huang, W.-J., Huang, Y.-N., Lai, C.-L., 2005. Characterizations of distributions based on certain powers of random variables. J. Chinese Statist. Assoc. 43, 423–432.

Loperfido, N., 2001. Quadratic forms of skew-normal random vectors. Statist. Probab. Lett. 54, 381–387.

Loperfido, N., 2002. Statistical implications of selectively reported inferential results. Statist. Probab. Lett. 56, 13–22.

Nadarajah, S., Kotz, S., 2003. Skewed distributions generated by the normal kernel. Statist. Probab. Lett. 65, 269–277.

Wang, J., Boyer, J., Genton, M.G., 2004a. A skew-symmetric representation of multivariate distributions. Statist. Sinica 14, 1259–1270.

Wang, J., Boyer, J., Genton, M.G., 2004b. A note on an equivalence between chi-square and generalized skew-normal distributions. Statist. Probab. Lett. 66, 395–398.