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On some characterizations of the mixture of gamma distributions☆

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Abstract

Following Gupta and Wesolowski [1997. Uniform mixtures via posterior means. Ann. Inst. Statist. Math. 49, 171–180], in this work, under the condition X/U and U are independent, X/U has a $\mathscr{B}e(p,q)$ distribution, and given X the conditional expectation of a certain function of (U,X) is constant, we characterize the distribution of (U,X). This problem is related to Lukacs type characterization, where both X and Y have to be gamma distributed with the same scale parameter, if both X and Y, and $X/(X+Y)$ and $X+Y$ are independent. Among others, we prove if $q=1$, and for some integer $n\geqslant 1$, $E(\sum_{i=1}^{n} a_i(U-X)^i|X)=b$, where $a_1,\ldots,a_n,b$, are real constants such that $a_1^2+\cdots+a_n^2\neq 0$ and $b\neq 0$, or for some real number $n>0$, $E((U-X)^n|X)=b$, where $b>0$ is a constant, then the distribution of (U,X) can be determined.

MSC: Primary 60E05; secondary 62E10

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1. Introduction

It is known that if X and Y are independent gamma random variables with the same scale parameter, i.e. X has a $\Gamma(p,r)$ distribution, Y has a $\Gamma(q,r)$ distribution, for some constants $p,q,r>0$, then the two random variables

$$X+Y \quad \text{and} \quad \frac{X}{X+Y}$$

are mutually independent and have $\Gamma(p+q,r)$ and $\mathscr{B}e(p,q)$ distributions, respectively. Here the notation $\Gamma(p,r)$, $p,r>0$, and $\mathscr{B}e(p,q)$, $p,q>0$, denotes the gamma distribution and beta distribution having the probability density functions (p.d.f.)

$$f_1(x)=\frac{x^{p-1}e^{-x/r}}{\Gamma(p)r^p},\quad x>0, \tag{1}$$

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and

$$f_2(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} = \frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1}, \quad 0<x<1, \tag{2}$$

respectively, where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx, \quad t>0,$$

and $B(\cdot, \cdot)$ is the beta function defined by

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q>0.$$

Lukacs (1955) showed that the above property can be used to characterize the gamma distributions in the following sense. If X and Y are independent non-degenerate positive random variables and $X+Y$ and $X/(X+Y)$ are mutually independent, then X and Y must have gamma distributions with the same scale parameter, but possibly with different values of the shape parameter.

By setting $U=X+Y$ and $W=X/(X+Y)$ in Lukacs type characterization, we get another form of characterization using the independence of U and W, and independence of UW and $U(1-W)$. Note that $X=UW$, X, U have gamma distributions, and W has beta distribution in this case.

As usual denote identically distributed by "$\overset{d}{=}$". Some related characterizations of the gamma distribution were done by Huang and Chen (1989) using

$$\sum_{i=1}^{M} Y_i \overset{d}{=} \sum_{i=1}^{K} U_i Y_i,$$

where $K>M$, $U_i, i=1,\ldots,K$, are independent and identically distributed (i.i.d.) from the common distribution $\mathscr{Be}(r,1)$, $r=M/(K-M)$, and $Y_i, i=1,\ldots,K$, are i.i.d. non-negative random variables. Furthermore, Huang and Chen (1991) proved that under the condition $Z\overset{d}{=}U_1X$, the distribution of Z can uniquely determine the distribution of X. Other related works were done by Yeo and Milne (1991), Alzaid and Al-Osh (1991), Pakes (1992a, b, 1994, 1997), and Pakes and Khattree (1992).

Let X, U and W be random variables where W is independent of U and has support on [0, 1]. As mentioned by Alzaid and Al-Osh (1991), the formula that $X=UW$ is of paramount importance in many fields. For example, in economic modeling, U may represent the actual income of an individual and X stands for his reported income.

In addition to the results mentioned above, there are many further investigations. Given two independent and non-degenerate positive random variables X and Y, Bolger and Harkness (1965), Hall and Simons (1969), Wesolowski (1989, 1990) and Li et al. (1994), Huang and Su (1997), Bobecka and Wesolowski (2002), Chou and Huang (2003), Huang and Chou (2004) and many others weaken the independent condition to constant regression. A good survey on some aspects of regression characterizations can be found in Rao and Shanbhag (1994, Chapter 9).

Instead of weakening the independence condition of $X/(X+Y)$ and $X+Y$ to constant regressions, conditions which are weaker than the independence of X and Y, and replacing the independence condition of $X/(X+Y)$ and $X+Y$ by the stronger assumption: X/U and U are independent and X/U is $\mathscr{U}(0,1)$ distributed, Gupta and Wesolowski (1997) characterized the distribution of U by using the so-called uniform mixtures via posterior means.

Inspired by Gupta and Wesolowski (1997), under the condition X/U and U are independent and X/U is $\mathscr{Be}(p,1)$ distributed, where $p>0$ is a constant, Huang and Wong (1998) characterized the distribution of U by using one of the following conditions:

1. $E(U|X)=aX+b$;
2. $E(U^2|X)=X^2+2bX+2b^2$;
3. $E((U-X)^2|X)=b$;
4. $\mathrm{Var}(U|X)=b$;

where a and b are some constants. Furthermore, in Gupta and Wesolowski (2001), by allowing X/U to have a general $\mathscr{B}e(p,q)$ distribution, they proved the distribution of U can be determined by giving $E(U|X)=aX+b$, where a and b are constants. Note that U may not be gamma distributed in these cases.

In this work, under the condition X/U and U are independent and X/U has a $\mathscr{B}e(p,q)$ distribution, by giving $E(h(U,X)|X)=b$, where h is some function of (U,X) and b is a constant, we characterize the distribution of (U,X).

2. Preliminaries

Firstly, we introduce some more notation which will be used in this work. Denote by

(i) $\mathscr{B}_{\mathrm{I}}(p,q,r)$ the first kind beta distribution defined by the p.d.f.

$$f(x)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,\frac{x^{p-1}(r-x)^{q-1}}{r^{p+q-1}},\quad 0<x<r, \tag{3}$$

where p,q,r are positive constants.

(ii) $\mathscr{B}_{\mathrm{II}}(p,q,r)$ the second kind beta distribution defined by the p.d.f.

$$f(x)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,\frac{r^{q}x^{p-1}}{(r+x)^{p+q}},\quad 0<x<\infty, \tag{4}$$

where p,q,r are positive constants.

Note that if the random variable W is $\mathscr{B}e(p,q)$ distributed, then for every $r>0$, $S_1=rW$ is $\mathscr{B}_{\mathrm{I}}(p,q,r)$ distributed, and $S_2=rW/(1-W)$ is $\mathscr{B}_{\mathrm{II}}(p,q,r)$ distributed.

Let γ_p and γ_q be independent with $\Gamma(p,1)$ and $\Gamma(q,1)$ distributions, respectively, and R, positive and non-degenerate, be independent of (γ_p,γ_q). Next, let $(X,Y)=(\gamma_p,\gamma_q)R$. Then $W=X/(X+Y)=\gamma_p/(\gamma_p+\gamma_q)$ is independent of $U=X+Y=(\gamma_p+\gamma_q)R$. This is an example for $X/(X+Y)$ and $X+Y$ being independent, and $X/(X+Y)$ has a beta distribution, yet X and Y are not independent and neither of the marginal distribution of X and Y is gamma. In particular, when R takes the positive value r_i with probability c_i, $i=1,\ldots,k$, where $k\geqslant 2$, and $\sum_{i=1}^{k}c_i=1$, then (X,Y) and (U,W) have the p.d.f.'s

$$f_{X,Y}(x,y)=\sum_{i=1}^{k}c_i\,\frac{x^{p-1}e^{-x/r_i}}{\Gamma(p)r_i^{p}}\,\frac{y^{q-1}e^{-y/r_i}}{\Gamma(q)r_i^{q}},\quad x,y>0, \tag{5}$$

and

$$f_{U,W}=\left(\sum_{i=1}^{k}c_i\,\frac{u^{p+q-1}e^{-u/r_i}}{\Gamma(p+q)r_i^{p+q}}\right)\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,w^{p-1}(1-w)^{q-1},\quad u>0,\ 0<w<1, \tag{6}$$

respectively. That is the distribution of (X,Y) is the mixture of k distributions $F_1(x,y),\ldots,F_k(x,y)$, where $F_i(x,y)$, $i=1,\ldots,k$, is the joint distribution function of two independent random variables with $\Gamma(p,r_i)$ and $\Gamma(q,r_i)$ distributions, respectively, the distribution of U is the mixture of k distributions $\Gamma(p+q,r_1),\ldots,\Gamma(p+q,r_k)$.

Conversely, let X and U be two random variables. Assume X/U and U are independent, X/U has a $\mathscr{B}e(p,q)$ distribution. Then (X,U) has the p.d.f.

$$f_{X,U}(x,u)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,x^{p-1}u^{1-p-q}(u-x)^{q-1}f_U(u),\quad 0<x<u<T\leqslant\infty, \tag{7}$$

where $f_U(u)$, $0<u<T$, is the p.d.f. of U, $T=\inf\{u: F_U(u)=1\}$, and $F_U(u)$, $u\in\mathscr{R}$, is the distribution function of U. From (7), the marginal p.d.f. of X, and the conditional p.d.f. of U given X can be determined while knowing f_U. That is

$$f_X(x)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,x^{p-1}\int_x^{T}u^{1-p-q}(u-x)^{q-1}f_U(u)\,\mathrm{d}u,\quad 0<x<T \tag{8}$$

and

$$f_{U|X}(u|x) = \frac{u^{1-p-q} f_U(u)}{\int_x^T u^{1-p-q}(u-x)^{q-1} f_U(u)\,\mathrm{d}u}, \quad 0 < x < u < T. \tag{9}$$

When the p.d.f. of (X, Y) is as given in (5), (X, U) has some constant regression properties as the case when X and Y are independent gamma distributed with the same scale parameter. Our first result Theorem 1 is by using

$$E\left(\sum_{i=1}^{n} a_i (U-X)^i | X\right) = b \tag{10}$$

to determine the distribution of U under the assumption X/U and U are independent, and X/U is $\mathscr{B}e(p, 1)$ distributed.

On the other hand, assume X and Y are independent and X has a $\Gamma(p, r)$ distribution, Y has a $\Gamma(q, r)$ distribution. Then it can be shown that the first two conditional moments of U given X have the following forms:

$$E(U|X) = X + qr, \tag{11}$$

$$E(U^2|X) = X^2 + 2qrX + q(1+q)r^2. \tag{12}$$

Note that (11) can be rewritten as $E(U - X|X) = b$. Among others, in this work, we will also use a more general form than (11), that is for some real $n > 0$

$$E((U-X)^n|X) = b, \tag{13}$$

or (12), to determine the distribution of (U, X), under the assumption X/U and U are independent and X/U is $\mathscr{B}e(p, q)$ distributed.

3. Main results

Throughout this section, assume X/U and U are independent and X/U is $\mathscr{B}e(p, q)$ distributed.

Theorem 1. *Assume $q = 1$, (10) holds for some integer $n \geqslant 1$, and constants $b, a_1, \ldots, a_n$, where $a_1^2 + \cdots + a_n^2 \neq 0$, and $b \neq 0$. Assume additionally that $f_U(u)$ is continuous with the support $[0, T]$, where $0 < T \leqslant \infty$. If the equation*

$$bx^n + \sum_{i=1}^{n} a_i (-1)^{i-1} i! x^{n-i} = 0 \tag{14}$$

has negative roots $-1/r_1, \ldots, -1/r_k$, with multiplicities $m_1, \ldots, m_k$, respectively, where $m_1, \ldots, m_k \geqslant 1$, then

$$f_U(u) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \frac{u^{p+j-1} e^{-u/r_i}}{\Gamma(p+j) r_i^{p+j}}, \quad u > 0, \tag{15}$$

and

$$f_X(x) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} \frac{x^{p+j-2} e^{-x/r_i}}{\Gamma(p+j-1) r_i^{p+j-1}}, \quad x > 0, \tag{16}$$

where $\sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} = 1$, such that $f_U(u) > 0, \forall u > 0$.

Proof. Observe that it follows immediately with X/U being $\mathscr{B}e(p, 1)$ distributed, support$(X) = [0, T] \subset [0, \infty)$ (if $T = \infty$ then we write $[0, T)$ instead of $[0, T]$). Since $X \leqslant U$, a.s., if $T < \infty$ then $U = T$ if $X = T$. It follows that $0 = E(\sum_{i=1}^{n} a_i (U - T)^i | X = T) \neq b$ as $b \neq 0$ by assumption. The contradiction implies $T = \infty$.

By letting $g(u)=u^{-p}f_U(u)$, $u>0$, (9) and (10) imply (note that $q=1$ here)

$$b\int_x^\infty g(u)\,\mathrm{d}u-\sum_{i=1}^n a_i\int_x^\infty (u-x)^i g(u)\,\mathrm{d}u=0,\quad x>0. \tag{17}$$

Taking derivatives on both sides of (17) with respect to x $(n+1)$ times, we obtain

$$bg^{(n)}(x)+\sum_{i=1}^n a_i(-1)^{i-1}i!g^{(n-i)}(x)=0,\quad x>0, \tag{18}$$

where $g^{(0)}(x)=g(x)$. Solving the above differential equation, yields

$$g(x)=s_1e^{t_1x}+s_2e^{t_2x}+\cdots+s_ne^{t_nx},\quad x>0, \tag{19}$$

where $s_1,\ldots,s_n$ are constants, and $t_1,\ldots,t_n$ are the roots of Eq. (14).

Now without loss of generality, we assume all the roots of (14) have no multiplicities. The p.d.f. $f_U(u)=u^pg(u)$ is a linear combination of exponential functions, we have $\lim_{u\to\infty}f_U(u)=0$, otherwise f_U cannot be a p.d.f. Hence the coefficient s_i which corresponds to positive t_i must be zero. Consequently

$$f_U(u)=s_1'u^pe^{-u/r_1}+s_2'u^pe^{-u/r_2}+\cdots+s_k'u^pe^{-u/r_k}, \tag{20}$$

where $-1/r_1,\ldots,-1/r_k$ are the distinct negative roots of (14), and s_i' is the constant in (19) which corresponds to the root $-1/r_i$ of (14), $i=1,\ldots,k$. We thus obtain (15) where $c_i=s_i'\Gamma(p+1)(r_i)^{p+1}$, $i=1,\ldots,k$, and $\sum_{i=1}^k c_i=1$.

From (8) with $T=\infty$, we obtain X has the p.d.f. as given in (16). This completes the proof of this theorem. □

Remark 1. It can be checked easily that (10) indeed holds true for f_U as given in (15). On the other hand in order that the function f_U as defined in (15) is a p.d.f., it is allowed that some of $\{c_{ij}\}$ can be negative. We give an example for the case $k=1$ and $m_1=m_2=1$ in the following. This example also demonstrates that solutions (15) are not unique.

Let $p>0$, $0<r_2<r_1$ and $c_1+c_2=1$. Then it can be shown easily that

$$f_U(u)=c_1\frac{u^{p-1}e^{-u/r_1}}{\Gamma(p)r_1^p}+c_2\frac{u^{p-1}e^{-u/r_2}}{\Gamma(p)r_2^p},\quad u>0 \tag{21}$$

is a p.d.f. if and only if $0\leqslant c_1\leqslant r_1^p/(r_1^p-r_2^p)$. This leads to $c_2<0$ when $1<c_1<r_1^p/(r_1^p-r_2^p)$.

In the next three theorems q can be any positive constant.

Theorem 2. *Assume for some real number $n>0$,*

$$E((U-X)^n|X)=b, \tag{22}$$

where $b>0$ is a constant. Then $(X,U)\stackrel{d}{=}(V_1,V_1+V_2)$, where V_1 and V_2 are independent, and V_1 and V_2 are $\Gamma(p,(b\Gamma(q)/\Gamma(q+n))^{1/n})$ and $\Gamma(q,(b\Gamma(q)/\Gamma(q+n))^{1/n})$ distributed, respectively.

Proof. Again $T=\infty$ can be obtained. For $t\geqslant 0$, (22) yields $E(X^t(U-X)^n|X)=bX^t$, i.e.

$$E(U^{t+n}W^t(1-W)^n|X)=bU^tW^t, \tag{23}$$

where $W=X/U$ is independent of U and is $\mathscr{B}e(p,q)$ distributed by assumption. So if $M(t)=E(U^t)$, $t\geqslant 0$, denotes the moment function of U, then taking the expectations of both sides of (23) and using independence of U and W gives

$$E(W^t(1-W)^n)M(t+n)=bE(W^t)M(t). \tag{24}$$

Evaluating the moments of W yields

$$B(p+t,q+n)M(t+n)=bB(p+t,q)M(t). \tag{25}$$

This is valid at least for $0\leqslant t\leqslant n$ and hence, by iteration, for all $t\geqslant 0$.

Let $t=0$ in (25) and divide into the general equation to obtain

$$\frac{B(p+t,q+n)}{B(p,q+n)}\,\frac{B(p,q)}{B(p+t,q)}\,\frac{M(t+n)}{M(n)}=M(t). \tag{26}$$

Expressing the beta functions in terms of gamma functions, canceling and rearranging terms, we find that

$$\frac{B(p+q+t,n)}{B(p+q,n)}\,\frac{M(t+n)}{M(n)}=M(t). \tag{27}$$

This is equivalent to the relation $V\hat{U}_n \stackrel{d}{=} U$, where V is $\mathscr{B}e(p+q,n)$ distributed, and $\hat{U}_n$ denotes the order-n length-biased version U, i.e. its distribution function is $\int_0^x u^n\,\mathrm{d}F_U(u)/M(n)$, $x>0$, and V and $\hat{U}_n$ are independent. But this in-law identity has the known solution $U \stackrel{d}{=} \Gamma(p+q,c)$, for any constant $c>0$, see Pakes (1997, Theorem 4.1). From this and by letting $t=0$ in (25), we obtain

$$M(n)=E(U^n)=\frac{bB(p,q)}{B(p,q+n)}=\frac{c^n\Gamma(p+q+n)}{\Gamma(p+q)}. \tag{28}$$

This in turn implies $c=(b\Gamma(q)/\Gamma(q+n))^{1/n}$. Now that we know U has a gamma law, the joint distribution assertion follows by (7) or by evaluation of

$$E(X^sU^t)=E(W^sU^{s+t})=E\left(\left(\frac{V_1}{V_1+V_2}\right)^s(V_1+V_2)^{s+t}\right)=E(V_1^s(V_1+V_2)^t),\quad s,t\geqslant 0, \tag{29}$$

where V_1 and V_2 are defined in the statement of this theorem. □

Remark 2. Let $q=1$ and n be a positive integer in Theorem 2. Then (22) corresponds to $a_n=1$ and $a_1=\cdots=a_{n-1}=0$ in (10), and (18) becomes $bg^{(n)}(x)=(-1)^n n!g(x)$, $x>0$. Solving the differential equation, we have $g(x)=c_1e^{r_1x}$, $x>0$, where c_1 is a constant, and $r_1=-(n!/b)^{1/n}$ is the only negative root of the equation $bx^n=(-1)^n\,n!$. Therefore U is $\Gamma(p+1,(b/n!)^{1/n})$ distributed and X is $\Gamma(p,(b/n!)^{1/n})$ distributed, which coincides with the result in Theorem 1.

We now present a lemma, which will be useful to prove Theorems 3 and 4.

Lemma 1. *Let F be the distribution function of a positive random variables V, and let $-\infty<\eta<\delta<1$ be constants. Suppose $M(t)=E(V^t)<\infty$ for some $t>0$ and that*

$$L(x)=\int_x^\infty u^\eta(u-x)^{-\delta}\,\mathrm{d}F(u)<\infty,\quad 0\leqslant x\leqslant\beta, \tag{30}$$

where $\beta=\inf\{x:F(x)=1\}$. Then

$$\int_0^\infty x^{t-\eta+\delta-1}L(x)\,\mathrm{d}x=B(t-\eta+\delta,1-\delta)M(t),\quad t\geqslant 0. \tag{31}$$

In addition:

(i) *If $\beta=\infty$ and $L(x)=Ke^{-x/c}$, $0\leqslant x\leqslant\beta$, where $c,K>0$, then*

$$V\stackrel{d}{=}\Gamma(1-\eta,c); \tag{32}$$

and

(ii) *If $0<\beta<\infty$ and $L(x)=K(\beta-x)^c$, $0\leqslant x\leqslant\beta$, where $c,K>0$, then*

$$V\stackrel{d}{=}\beta\mathscr{B}e(1-\eta,\delta+c)\stackrel{d}{=}\mathscr{B}_{\mathrm{I}}(1-\eta,\delta+c,\beta). \tag{33}$$

Proof. Identity (31) follows from a routine reversal of order in the double integral in the left-hand side of (30). Note that the beta function factor $B(t-\eta+\delta, 1-\delta)$ in (31) is finite for every $t \geqslant 0$. For (i), (31) takes the form $B(t-\eta+\delta, 1-\delta)M(t) = Kc^{t-\eta+\delta}\Gamma(t-\eta+\delta)$, that is,

$$M(t) = Kc^{t-\eta+\delta}\,\frac{\Gamma(t+1-\eta)}{\Gamma(1-\delta)}. \tag{34}$$

Since $M(0)=1$, the right-hand side of (34) equals $c^t\Gamma(t+1-\eta)/\Gamma(1-\eta)$, the moment function of $\Gamma(1-\eta, c)$, thus proving (32). The proof of (33) is similar, hence is omitted. □

Note that Lemma 1 can also be applied to prove Theorem 2 for the case when n is a positive integer in (22).

Theorem 3. *Assume*

$$E(U^2|X) = X^2 + 2qrX + q(1+q)r^2 \tag{35}$$

for some positive constant r. Assume additionally that $f_U(u)$ is continuous with the support $[0, T]$, where $0 < T \leqslant \infty$. Then $(X, U) \stackrel{d}{=} (V_1, V_1+V_2)$, where V_1 has a $\Gamma(p, r)$ distribution, V_2 has a $\Gamma(q, r)$ distribution and V_1 and V_2 are independent.

Proof. Again we obtain $T=\infty$. Note that (35) also implies

$$E((U-X)^2|X) + 2XE(U-X|X) = 2qrX + q(1+q)r^2. \tag{36}$$

That X/U is $\mathscr{B}e(p,q)$ distributed and (36) imply

$$\int_x^\infty u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u + 2x\int_x^\infty u^{1-p-q}(u-x)^{q} f_U(u)\,\mathrm{d}u$$
$$= (2qrx + q(q+1)r^2)\int_x^\infty u^{1-p-q}(u-x)^{q-1} f_U(u)\,\mathrm{d}u, \quad x>0. \tag{37}$$

From (37) we have

$$\int_x^\infty u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u - \frac{2x}{q+1}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u$$
$$= \frac{2qrx + q(q+1)r^2}{q(q+1)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_x^\infty u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u, \quad x>0. \tag{38}$$

By letting $h_1(x) = \int_x^\infty u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u$, $x>0$, (38) implies

$$h_1(x) - 2x\frac{1}{q+1}h_1'(x) = (2qrx + q(q+1)r^2)\frac{1}{q(q+1)}h_1''(x), \quad x>0. \tag{39}$$

Furthermore, by letting $h_2(x) = e^{x/r}h_1(x)$, $x>0$, (39) becomes

$$\frac{2x+2r(q+1)}{q+1}h_2'(x) = \frac{2rx + r^2(q+1)}{q+1}h_2''(x), \quad x>0. \tag{40}$$

Solving the above differential equation, we obtain

$$h_2(x) = L_1 + L_2\int_0^x e^{t/r}(2rt + r^2(q+1))^{(q+1)/2}\,\mathrm{d}t, \quad x>0, \tag{41}$$

and it follows that

$$h_1(x) = L_1e^{-x/r} + L_2e^{-x/r}\int_0^x e^{t/r}(2rt + r^2(q+1))^{(q+1)/2}\,\mathrm{d}t, \quad x>0, \tag{42}$$

where L_1, L_2 are constants. Consequently, for every $x > 0$,

$$\int_x^\infty u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = L_1 e^{-x/r} + L_2 e^{-x/r} \int_0^x e^{t/r}(2rt + r^2(q+1))^{(q+1)/2}\,\mathrm{d}t. \tag{43}$$

Let x tend to ∞ in (43), we have $L_2 = 0$. Hence

$$\int_x^\infty u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = L_1 e^{-x/r}, \quad x > 0.$$

The result now follows by (i) of Lemma 1. □

Next we use a form which is slightly different from (35) to characterize the distribution of U.

Theorem 4. *Assume*

$$E((U-X)^2|X) = a(X+b)^2 \tag{44}$$

for some constants a, b, where $a \geqslant 0$. Assume additionally that $f_U(u)$ is continuous with the support $[0, T]$, *where $0 < T \leqslant \infty$. Then only the following cases are possible*:

(i) $b < 0$, *and then $a < 1$ and U is $\mathscr{B}_{\mathrm{I}}(p+q, (\sqrt{4q(q+1)/a+1}-1)/2-q, -b)$ distributed*;
(ii) $b > 0$, *and then U is $\mathscr{B}_{\mathrm{II}}(p+q, (\sqrt{4q(q+1)/a+1}+3)/2-p, b)$ distributed, also $p > 2$, and $a < q(q+1)/((p-1)(p-2))$.*

Proof. First (44) implies $a > 0$. The assumptions, via the Bayes theorem, imply that

$$\int_x^T u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = a(x+b)^2 \int_x^T u^{1-p-q}(u-x)^{q-1} f_U(u)\,\mathrm{d}u, \quad 0 < x < T. \tag{45}$$

The second derivative of the left-hand side of (45) with respect to x is

$$q(q+1)\int_x^T (u-x)^{q-1} u^{1-p-q} f_U(u)\,\mathrm{d}u.$$

Hence (45) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_x^T u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = \frac{q(q+1)}{a}(x+b)^{-2}\int_x^T u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u, \quad 0 < x < T.$$

It follows that

$$\int_x^T u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = D_1|x+b|^{(1+s)/2} + D_2|x+b|^{(1-s)/2}, \quad 0 < x < T, \tag{46}$$

for some constants D_1 and D_2, where $s = \sqrt{4q(q+1)/a+1} > 1$. Before the detailed analysis of the above equation we give some basic properties for the two constants a and b. That $b = 0$ is impossible can be seen easily by letting $x = 0$ in (45), which implies the contradictory result $U \equiv 0$.

Case (i): $b < 0$. In this case we have $T = -b$. To see this we first observe that $T < \infty$. Otherwise, given $X = -b$ in (44), it yields zero for the right-hand side. Yet from (9), the left-hand side of (44) cannot be zero. The contradiction implies $T < \infty$. Now let $X = T$, it follows that $U = T$. Consequently, $0 = E((U-T)^2|X = T) = a(T+b)^2$. This gives $T = -b$. Moreover, let $x = 0$, (45) becomes

$$\int_0^{-b} u^{2-p} f_U(u)\,\mathrm{d}u = ab^2\int_0^{-b} u^{-p} f_U(u)\,\mathrm{d}u.$$

Obviously,

$$\int_0^{-b} u^{2-p} f_U(u)\,\mathrm{d}u < b^2\int_0^{-b} u^{-p} f_U(u)\,\mathrm{d}u.$$

Hence

$$ab^2 \int_0^{-b} u^{-p} f_U(u)\,\mathrm{d}u < b^2 \int_0^{-b} u^{-p} f_U(u)\,\mathrm{d}u,$$

and $a < 1$ follows.

Now (46) becomes

$$\int_x^{-b} u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = D_1(-b-x)^{(1+s)/2} + D_2(-b-x)^{(1-s)/2}, \quad 0 < x < -b. \tag{47}$$

Let $x = -b$, we have $D_2 = 0$ and

$$\int_x^{-b} u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = D_1(-b-x)^{(1+s)/2}, \quad 0 < x < -b. \tag{48}$$

Finally, U is $\mathscr{B}_1(p+q, (s-1)/2-q, -b)$ distributed follows from (ii) of Lemma 1.

Case (ii): $b > 0$. Observe that in this case $T = \infty$. Since if $T < \infty$ it yields the following contradiction: $0 = E((U-T)^2|X=T) = a(T+b)^2 > 0$. Hence (45) is equivalent to

$$\int_x^{\infty} u^{1-p-q}(u-x)^{q+1} f_U(u)\,\mathrm{d}u = a(x+b)^2 \int_x^{\infty} u^{1-p-q}(u-x)^{q-1} f_U(u)\,\mathrm{d}u, \quad x > 0. \tag{49}$$

We now make the following transformation of variables in (49):

$$z = \frac{bu}{b+u}, \quad u > 0 \quad \text{and} \quad t = \frac{bx}{b+x}, \quad x > 0.$$

Then

$$u = u(z) = \frac{bz}{b-z}, \quad 0 < z < b \quad \text{and} \quad x = \frac{bt}{b-t}, \quad 0 < t < b.$$

The left-hand side and the right-hand side of (46) with $T = \infty$ becomes

$$(b-t)^{-(q+1)} b^{q+5-p} \int_t^b (z-t)^{q+1}(b-z)^{p-4} z^{1-p-q} f_U(u(z))\,\mathrm{d}z, \quad 0 < t < b,$$

and

$$D_1\left(\frac{b^2}{b-t}\right)^{(1+s)/2} + D_2\left(\frac{b^2}{b-t}\right)^{(1-s)/2}, \quad 0 < t < b,$$

respectively. Hence

$$\int_t^b (z-t)^{q+1} h_3(z)\,\mathrm{d}z = K_1(b-t)^{q+(1-s)/2} + K_2(b-t)^{q+(1+s)/2}, \quad 0 < t < b, \tag{50}$$

for some constants K_1 and K_2, where

$$h_3(z) = \frac{(b-z)^{p-4} z^{1-p-q} f_U(u(z))}{\int_0^b (b-v)^{p-4} v^{1-p-q} f_U(u(v))\,\mathrm{d}v}, \quad 0 < z < b \tag{51}$$

is a p.d.f.

The left-hand side of (50) is bounded above by $(b-t)^{q+1} \int_t^b h_3(z)\,\mathrm{d}z$. Divide the resulting inequality by $(b-t)^{q+1}$ to get

$$1 \geqslant \int_t^b h_3(z)\,\mathrm{d}z = K_1(b-t)^{-(s+1)/2} + K_2(b-t)^{(s-1)/2}, \quad 0 < t < b.$$

As $t \uparrow b$, $K_2(b-t)^{(s-1)/2} \to 0$, and $K_1(b-t)^{-(s+1)/2} \to \infty$ if $K_1 > 0$. Hence $K_1 = 0$, and (50) becomes

$$\int_t^b (z-t)^{q+1} h_3(z)\,\mathrm{d}z = K_2(b-t)^{q+(s+1)/2}, \quad 0 < t < b. \tag{52}$$

Again by (ii) of Lemma 1 we obtain

$$h_3(z) = \frac{(s-1)(b-z)^{(s-3)/2}}{2b^{(s-1)/2}}, \quad 0 < z < b.$$

It follows from (51) that U is $\mathscr{B}_{\mathrm{II}}(p+q, (s+3)/2 - p, b)$ distributed as required. Being a parameter of the second kind beta distribution, $(s+3)/2 - p$ must be positive. To this end obviously, $p > 2$, and $a < q(q+1)/((p-1)(p-2))$ follows immediately.

This completes the proof. □

Remark 3. The result of the case $b > 0$ in Theorem 4 can be stated in the following equivalent way: The random variables $S_1 = X/(b+X)$ and $S_2 = U/(b+U)$ are $Be(p, \sqrt{(4q(q+1)/a)+1} - 1)/2 - q)$ and $Be(p+q, (\sqrt{(4q(q+1)/a)+1} - 1)/2 - q)$ distributed, respectively. Then by applying the above transformation of variables to the case X/U and U are independent and X/U has a $\mathscr{B}e(p,q)$ distribution, and using (44), it follows that

$$\left.\frac{S_1(1-S_1)}{S_2(1-S_2)}\right| S_2 \sim Be(p,q), \tag{53}$$

and

$$E\left(\left.\left(\frac{S_2 - S_1}{1 - S_2}\right)^2 \right| S_2\right) = a. \tag{54}$$

Hence suppose that (S_1, S_2) is a random vector satisfying (53) and (54) for some constant a, and assume additionally that the p.d.f. of S_2 is continuous. Then S_2 has a $\mathscr{B}e(p+q, (\sqrt{(4q(q+1)/a)+1} - 1)/2 - q)$ distribution.

The final result which first appeared in Huang and Wong (1998) is about the homoscedasticity of the conditional distribution of U given X, i.e. $\mathrm{Var}(U|X) = b$. We give a more complete proof here.

Theorem 5. *Assume* $q = 1$, *and*

$$\mathrm{Var}(U|X) = b \tag{55}$$

for some positive constant b. *Assume additionally that* $f_U(u)$ *is continuous with the support* $[0, T]$, *where* $0 < T \leqslant \infty$. *Then* $(X, U) \overset{d}{=} (V_1, V_1 + V_2)$, *where* V_1 *has a* $\Gamma(p, \sqrt{b})$ *distribution*, V_2 *has a* $\Gamma(1, \sqrt{b})$ *distribution and* V_1 *and* V_2 *are independent*.

Proof. As before first we can obtain $T = \infty$. Now (55) yields

$$E(U^2|X) + (E(U|X))^2 = b. \tag{56}$$

This together with (9) imply

$$\int_x^\infty u^2 \frac{u^{-p} f_U(u)}{\int_x^\infty u^{-p} f_U(u)\,\mathrm{d}u}\,\mathrm{d}u + \left(\int_x^\infty u \frac{u^{-p} f_U(u)}{\int_x^T u^{-p} f_U(u)\,\mathrm{d}u}\,\mathrm{d}u\right)^2 = b, \quad x > 0. \tag{57}$$

Again let $g(u) = u^{-p} f_U(u)$, $u > 0$, in (57) to obtain

$$\int_x^\infty u^2 g(u)\,\mathrm{d}u \int_x^\infty g(u)\,\mathrm{d}u - \left(\int_x^\infty u g(u)\,\mathrm{d}u\right)^2 = b\left(\int_x^\infty g(u)\,\mathrm{d}u\right)^2, \quad x > 0. \tag{58}$$

Taking the derivatives of both sides of (58) with respect to x four times, we obtain $bg''(x) - g(x) = 0$. The solution of the above differential equation is

$$g(x) = c_1 e^{-x/\sqrt{b}} + c_2 e^{x/\sqrt{b}}, \quad x > 0, \tag{59}$$

where c_1 and c_2 are constants. Hence $f_U(u) = c_1 u^p e^{-u/\sqrt{b}}$, $u > 0$, and the assertions follow immediately. □

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