Characterizations of the order statistics point process by the relations between its conditional moments

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Abstract

Let $A \equiv \{A(t), t \geq 0\}$ be an order statistics point process, with $E(A(t)) = m(t)$ being the mean value function of $A(t), t \geq 0$. It is known that $m(t)$ determines the distribution of the process A. In this work, we give some characterizations of $m(t)$, by using certain relations between the conditional moments of the last jump time or current life of A at time t. It is interesting that some results are parallel to those characterizations of Poisson process as a renewal process. Finally, we present some extensions of the results about record values given in Abu-Youssef (2003).

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1 Introduction

Let $\{A(t), t \geq 0\}$ with $A(0) = 0$, $A(t) < \infty$, $\forall t \geq 0$, be a point process with right continuous sample paths having successive unit steps at times $S_1, S_2, \ldots$. The process $\{A(t), t \geq 0\}$ is said to have the order statistics property or called an order statistics point process if for every $t > 0$ and integer $n \geq 1$, whenever $P(A(t) = n) > 0$, given $A(t) = n$, the successive jump times $(S_1, \ldots, S_n)$ are distributed as the order statistics of n independent and identically distributed (i.i.d.) random variables with distribution function $F_t(\cdot)$ supported on $[0, t]$. It is well known that nonhomogeneous Poisson processes have the order statistics property. Nonhomogeneous Poisson processes are widely used models for the occurrence of events in time, for example, the failure times in software reliability (see Joe (1989), Kuo and Yang (1996) and Huang *et al.* (2003)). Order statistics property provides a nice way to simulate nonhomogeneous Poisson processes. This and some other useful applications motivate many authors to investigate the intrinsic properties within the class of order statistics point processes.

Properties and characterizations of order statistics point processes have been studied by Nawrotzki (1962), Westcott (1973), Crump (1975), Kallenberg (1976), Feigin (1979), Puri (1982), Huang and Su (1999) and Shaked *et al.* (2004) and many others. In particular, Crump (1975) proves that the order statistics point processes are Markovian and for every $t > 0$, the associated distribution function $F_t(\cdot)$ is continuous, $F_t(x) = E(A(x))/E(A(t))$, $\forall 0 \leq x \leq t$, if $E(A(t)) < \infty$. On the other hand, Puri (1982) shows that the class of order statistics point processes with $E(A(t)) < \infty$, $\forall t > 0$, is characterized by mixed Poisson processes (up to a time-scale transformation) if $\lim_{t\to\infty} E(A(t)) = \infty$, or mixed sample processes if $\lim_{t\to\infty} E(A(t)) < \infty$.

As mentioned in Nagaraja (1988), Deheuvels (1984), Gupta (1984) and

Huang and Su (1999), it is known that record values and order statistics of a sequence of i.i.d. random variables are closely related. Let $\{W_i, i \geq 1\}$ be a sequence of i.i.d. random variables having continuous distribution function H with $H(0) = 0$. Define the sequence of record times $\{L(n), n \geq 1\}$ by $L(1) = 1$ and $L(n) = \min\{j | W_j > W_{L(n-1)}\}, n \geq 2$. Let $Y_n = W_{L(n)}, n \geq 1$, then $\{Y_n, n \geq 1\}$ is called the sequence of record values corresponding to $\{W_i, i \geq 1\}$. Denote $N(t)$ as the number of $Y_n \leq t, t \geq 0$. Shorrock (1972a,b) prove that the point process $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with mean value function $E(N(t)) = -\ln(1 - H(t))$. On the other hand, for every $m \geq 1$, let $X_{1,m} \leq X_{2,m} \leq ... \leq X_{m,m}$ be the order statistics from a random sample $X_1, X_2, ..., X_m$ having distribution function F with $F(0) = 0$. Then $\{M_m(t), t \geq 0\}$, where $M_m(t)$ is the number of $X_{k,m} \leq t, 1 \leq k \leq m$, is called the sample process generated by m and F. Note that the mean value function $E(M_m(t)) = mF(t)$. It is worth to mention that when record values and order statistics are viewed as point processes, the two processes both share the order statistics property. Huang and Su (1999) give properties for an order statistics point process $\{A(t), t \geq 0\}$ and establish some characterizations of $E(A(t))$, by using certain conditional moments of the spacings of the jump times. They also show that for every $t > 0$ and integer $n \geq 1$, whenever $P(t-\delta < S_{n+1} \leq t+\delta) > 0, \forall \delta > 0$, the conditional distribution of $(S_1, ..., S_n)$ given $A(t) = n$ is the same as that given $S_{n+1} = t$. This explains why record values and order statistics have many similar characterization results by using conditional moments. Also the characterizations of $E(A(t))$ in Huang and Su (1999) can deduce the corresponding characterization results of record values and order statistics as special cases.

In this work, for an order statistics point process $\{A(t), t \geq 0\}$, we give some characterizations of the mean value function $E(A(t))$, by using certain relations between the conditional moments of the jump times or current lives of $\{A(t), t \geq 0\}$. In Sections 2 and 3, some theorems are motivated by characterizations of the homogeneous Poisson process as a renewal process, and some characterization results are extensions of results in Huang and Su (1999). Also these characterization results can reduce to the corresponding characterizations of distributions via record values and order statistics, respectively. Recently, Abu-Youssef (2003) establishes the characterizations through conditional moments of record values. The corresponding characterization results for order statistics point processes are given in Section 4.

2 Characterizations by using conditional moments of jump times of the process

First, we give the following lemma, which can be found in Boyce and DiPrima (1997), and will be used to prove the main results of this section.

Lemma 1. *Consider the Euler equation*:

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0, \tag{1}$$

in any interval not containing the origin, where α and β are some fixed real numbers. Then

$$y(t) = \begin{cases} c_1 |t|^{\frac{(1-\alpha)+\sqrt{(1-\alpha)^2-4\beta}}{2}} + c_2 |t|^{\frac{(1-\alpha)-\sqrt{(1-\alpha)^2-4\beta}}{2}} & , \textit{if } \ (1-\alpha)^2 > 4\beta, \\ (c_3 + c_4 \log |t|)|t|^{\frac{(1-\alpha)}{2}} & , \textit{if } \ (1-\alpha)^2 = 4\beta, \end{cases}$$

is the general solution of (1), *where c_1, c_2, c_3, c_4 are arbitrary constants.*

Throughout this section, let $\{B(t), t \geq 0\}$ be a renewal process and $\{T_i, i \geq 1\}$ be the sequence of arrival times of $\{B(t), t \geq 0\}$. Li *et al.* (1994)

and Huang and Su (1997) characterize $\{B(t), t \geq 0\}$ to be a Poisson process by assuming that, for certain pairs of (r, s),

$$E(T_i^r | B(t) = n) = at^r \text{ and } E(T_i^s | B(t) = n) = bt^s, t \geq 0, \tag{2}$$

where i, n are some fixed integers, $1 \leq i \leq n$, a and b are independent of t. Huang and Su (1997) and Chou and Huang (2003) obtain similar characterizations under the conditions that

$$E(T_i^{r+1} | B(t) = n) = atE(T_i^r | B(t) = n), t > 0, \tag{3}$$

and

$$E(T_i^{r+s+1} | B(t) = n) = btE(T_i^{r+s} | B(t) = n), t > 0, \tag{4}$$

where $s \in \{1, 2\}, r \in (-\infty, \infty)$, i and n are some fixed integers, $1 \leq i \leq n$, and a, b are independent of t.

It turns out that there are similar characterization results within the class of order statistics point processes. In this section, let $\{A(t), t \geq 0\}$ be an order statistics point process with $E(A(t)) = m(t) < \infty, t \geq 0$, and $\{S_i, i \geq 1\}$ be the successive jump times of $\{A(t), t \geq 0\}$. As mentioned in Section 1, it is known that the order statistics property implies that for every $t > 0$, and integer $n \geq 1$, whenever $P(A(t) = n) > 0$, given $A(t) = n$, the conditional distribution function of the last jump time S_n is

$$P(S_n \leq s | A(t) = n) = (m(s)/m(t))^n, \ 0 \leq s \leq t.$$

Huang and Su (1999) prove that, given a nondecreasing function G, if

$$E(G(S_n) | A(t) = n) = cG(t), 0 < t < \eta, \tag{5}$$

holds for a positive integer n, $c > 0$, and $0 < \eta \leq \infty$, then $m(\cdot)$ can be determined. Actually from the proof it can be found that the assumption G

being nondecreasing is not needed. Next by using a condition similar to (3) with $i = n$, $m(\cdot)$ can be characterized within the class of order statistics point processes. We state and prove the result in the following theorem.

Theorem 1. *Let* $m(\cdot)$ *be positive, twice differentiable in* $(0, \eta)$ *and* $m'(\cdot)$ *be positive on* $(0, \eta)$*, where* $0 < \eta \leq \infty$*. Assume for some fixed integer* $n \geq 1$*,* $k \in (-\infty, \infty), r > 0$*, whenever* $P(A(t) = n) > 0$*,*

$$E(S_n^{k+r}|A(t) = n) = bt^r E(S_n^k|A(t) = n), 0 < t < \eta, \tag{6}$$

where b *is independent of* t*. Then*

(i) $k/(k+r) < b < 1$ *if* $k \geq 0$*, and* $0 < b < 1$ *if* $k < 0$;

(ii) $m(t) = \lambda t^{\frac{1}{n}(\frac{br}{1-b} - k)}, 0 < t < \eta$*, where* $\lambda > 0$ *is a constant.*

In particular, if $b = (n+k)/(n+k+r)$*, then* $m(t) = \lambda t, 0 < t < \eta$*.*

Proof. First from (6), we obtain

$$\int_0^t s^{k+r} (m(s))^{n-1} m'(s) ds = bt^r \int_0^t s^k (m(s))^{n-1} m'(s) ds, 0 < t < \eta.$$

Taking the derivatives of both sides with respect to t yields

$$\int_0^t s^k (m(s))^{n-1} m'(s) ds = \frac{(1-b)}{br} t^{k+1} (m(t))^{n-1} m'(t), 0 < t < \eta. \tag{7}$$

As the left-hand side of (7) is positive for $0 < t < \eta$, $(1-b)/(br)$ must be positive, and $0 < b < 1$ follows, since $r > 0$. Now taking the derivatives of both sides of (7) with respect to t, we obtain

$$\begin{aligned} &nt^2 m(t) m''(t) + n(n-1)t^2 (m'(t))^2 \\ &+(1 + k - \frac{br}{1-b}) ntm(t)m'(t) = 0, 0 < t < \eta. \end{aligned} \tag{8}$$

Let $u(t) = (m(t))^n$, then (8) becomes

$$t^2 u''(t) + \left(1 + k - \frac{br}{1-b}\right) tu'(t) = 0, 0 < t < \eta. \tag{9}$$

We now show that $br/(1-b) \neq k$. Assume $br/(1-b) = k$. By Lemma 1, the general solution of (9) is

$$u(t) = c_1 + c_2 \log t, 0 < t < \eta, \tag{10}$$

where c_1 and c_2 are constants. Consequently,

$$m(t) = (u(t))^{1/n} = (c_1 + c_2 \log t)^{1/n}, 0 < t < \eta,$$

which contradicts to the fact that $m(t)$ is a monotone positive function. Hence $br/(1-b) \neq k$, and Lemma 1 implies the general solution of (9) is

$$u(t) = c_3 t^{\frac{br}{1-b}-k} + c_4, 0 < t < \eta, \tag{11}$$

where c_3 and c_4 are constants. As $m(t)$ is positive and $m(0^+) = 0$, it turns out $c_3 > 0$, $c_4 = 0$ and $br/(1-b) > k$. From this and recalling that $0 < b < 1$, we obtain that $k/(k+r) < b < 1$ if $k \geq 0$, and $0 < b < 1$ if $k < 0$. Finally the assertion (ii) follows by letting $\lambda = c_3^{1/n}$. □

It is known that when $\{A(t), t \geq 0\}$ is a Poisson process, one can deduce that

$$E(S_n^2|A(t) = n) = ((n+1)^2 / (n(n+2)))E^2(S_n|A(t) = n), t > 0. \tag{12}$$

Inspired by this, in the next theorem, we characterize $m(\cdot)$ within the class of order statistics point processes by a condition similar to (12).

Theorem 2. *Let $m(\cdot)$ be positive and twice differentiable in $(0, \eta)$, where $0 < \eta \leq \infty$. Assume that for some fixed integer $n \geq 1$, whenever $P(A(t) = n) > 0$,*

$$E(S_n^2|A(t) = n) = bE^2(S_n|A(t) = n), 0 < t < \eta, \tag{13}$$

where b is independent of t. Then

(i) $b > 1$;

(ii) $m(t) = \lambda t^{\frac{1}{n}\left(\sqrt{\frac{b}{b-1}}-1\right)}, 0 < t < \eta$, *where $\lambda > 0$ is a constant.*

In particular, if $b = (n+1)^2/(n(n+2))$, then $m(t) = \lambda t, 0 < t < \eta$.

Proof. Again the assertion (i) can be obtained immediately from (13). Next (13) can be rewritten as

$$m^n(t)\int_0^t s^2\,(m(s))^{n-1}\,m'(s)ds = bn\left(\int_0^t s\,(m(s))^{n-1}\,m'(s)ds\right)^2, 0 < t < \eta.$$

Taking the third derivatives of both sides with respect to t yields

$$\begin{aligned} &nt^2 m(t)m''(t) + n(n-1)t^2(m'(t))^2 \\ &+3ntm(t)m'(t) - \frac{1}{b-1}m^2(t) = 0, 0 < t < \eta. \end{aligned} \tag{14}$$

Let $u(t) = (m(t))^n$, (14) leads to

$$t^2u''(t) + 3tu'(t) - \frac{1}{b-1}u(t) = 0, 0 < t < \eta. \tag{15}$$

By Lemma 1, the general solution of (15) is

$$u(t) = c_1 t^{-1+\sqrt{b/(b-1)}} + c_2 t^{-1-\sqrt{b/(b-1)}}, 0 < t < \eta, \tag{16}$$

where c_1 and c_2 are constants. Using the assumption $m(t)$ is positive, $m(0^+) = 0$, and the fact $\sqrt{b/(b-1)} > 1$, we obtain $c_1 > 0, c_2 = 0$. The rest of the proof follows immediately. □

Huang *et al.* (1993) and Li *et al.* (1994) characterize a renewal process $\{B(t), t \geq 0\}$ to be a Poisson process through some conditional expectations about current life $\delta_t = t - S_{B(t)}$. More precisely, Huang *et al.* (1993) prove that, given a monotone function G, under some mild conditions, as long as

$$E(G(\delta_t)|B(t) = n) = E(G(X_1)|B(t) = n), t \geq 0, \tag{17}$$

holds for a single positive integer n, then $\{B(t), t \geq 0\}$ is a Poisson process. Within the class of order statistic point processes, Huang and Su (1999) use the assumption similar to (17) to characterize $m(\cdot)$. In the following, we give two characterization results based on the conditional moments of current life δ_t.

Theorem 3. *Let $m(\cdot)$ be positive and differentiable in $(0, \eta)$, where $0 < \eta \leq \infty$. Assume that for some fixed integers $n \geq 1, k \geq 1$, whenever $P(A(t) = n) > 0$,*

$$E(\delta_t^k | A(t) = n) = btE(\delta_t^{k-1} | A(t) = n), 0 < t < \eta, \tag{18}$$

where b is independent of t. Then
(i) $0 < b < 1$;
(ii) $m(t) = \lambda t^{(1-b)k/(nb)}, 0 < t < \eta$, *where $\lambda > 0$ is a constant.*
In particular, if $b = k/(n+k)$, then $m(t) = \lambda t, 0 < t < \eta$.

Proof. First the assertion (i) can be obtained from (18). Next the following is a consequence of (18),

$$\int_0^t ((1-b)t - s)(t-s)^{k-1} (m(s))^{n-1} m'(s) ds = 0, 0 < t < \eta. \tag{19}$$

Taking the kth derivatives of both sides of (19) with respect to t yields

$$\frac{m'(t)}{m(t)} = \frac{(1-b)\,k}{nb}\frac{1}{t}, 0 < t < \eta.$$

Hence

$$m(t) = \lambda t^{(1-b)k/nb}, 0 < t < \eta,$$

where $\lambda > 0$ is a constant. This completes the proof of this theorem. □

Huang and Su (1999) give the following result. If

$$E(\delta_t^2 | A(t) = n) = at^2, 0 < t < \eta,$$

where $n \geq 1$ is an integer and $0 < a < 1$ is independent of t, then $m(\cdot)$ can be determined. The following is an extension.

Theorem 4. *Let $m(\cdot)$ be positive, twice differentiable in $(0, \eta)$ where $0 < \eta \leq \infty$. Assume for some fixed integers $n \geq 1$, $k \geq 2$, whenever $P(A(t) = n) > 0$,*

$$E(\delta_t^k | A(t) = n) = bt^2 E(\delta_t^{k-2} | A(t) = n), 0 < t < \eta, \tag{20}$$

where b is independent of t. Then

(i) $0 < b < 1$;

(ii) $m(t) = \lambda t^{\left(-k+1/2+\sqrt{1+4k(k-1)/b}/2\right)/n}, 0 < t < \eta$, *where $\lambda > 0$ is a constant. In particular, if $b = k(k-1)/((k+n)(k+n-1))$, then $m(t) = \lambda t, 0 < t < \eta$.*

Proof. Again we only prove the assertion (ii). First (20) can be rewritten as

$$\int_0^t \left[(1-b)\, t^2 - 2st + s^2\right] (t-s)^{k-2} \left(m(s)\right)^{n-1} m'(s) ds = 0, 0 < t < \eta. \tag{21}$$

After taking the kth derivatives of both sides of (21), we obtain

$$\begin{aligned} &nt^2 m(t) m''(t) + n\,(n-1)\, t^2 \left(m'(t)\right)^2 + 2kntm(t)m'(t) \\ &- ((1-b)/b)\, k\,(k-1)\, m^2(t) = 0, 0 < t < \eta. \end{aligned} \tag{22}$$

By letting $u(t) = m^n(t)$, (22) leads to

$$t^2 u''(t) + 2ktu'(t) - ((1-b)/b)\, k\,(k-1)\, u(t) = 0, 0 < t < \eta. \tag{23}$$

By Lemma 1, the general solution of (23) is

$$u(t) = c_1 t^{-k+1/2+\sqrt{1+4k(k-1)/b}/2} + c_2 t^{-k+1/2-\sqrt{1+4k(k-1)/b}/2}, 0 < t < \eta, \tag{24}$$

where c_1 and c_2 are constants. From the assumption $m(t)$ is positive and $m(0^+) = 0$, and the fact $\sqrt{1+4k\,(k-1)\,/b}/2 > k - 1/2$, we obtain $c_1 > 0$, $c_2 = 0$. Let $\lambda = c_1^{1/n} > 0$, then

$$m(t) = \lambda t^{\left(-k+1/2+\sqrt{1+4k(k-1)/b}/2\right)/n}, 0 < t < \eta.$$

as required. □

For the general case, assume for some fixed integers $n \geq 1$, $k \geq r \geq 3$, whenever $P(A(t) = n) > 0$,

$$E(\delta_t^k | A(t) = n) = bt^r E(\delta_t^{k-r} | A(t) = n), 0 < t < \eta,$$

where b is independent of t. Then along the lines of the proofs of Theorems 3 and 4, an rth differential equation of $m(t)$ can be obtained. Yet the differential equation is unable to be solved.

When $\{A(t), t \geq 0\}$ is a Poisson process, it is known that

$$E(\delta_t^2 | A(t) = n) = (2(n+1)/(n+2)) E^2(\delta_t | A(t) = n), t > 0.$$

We have the following characterization result by using δ_t within the class of order statistics point processes, the proof is omitted.

Theorem 5. *Let $m(\cdot)$ be positive, twice differentiable in $(0, \eta)$ where $0 < \eta \leq \infty$. Assume for some fixed integer $n \geq 1$, whenever $P(A(t) = n) > 0$,*

$$E(\delta_t^2 | A(t) = n) = bE^2(\delta_t | A(t) = n), 0 < t < \eta, \tag{25}$$

where b is independent of t. Then

(i) $1 < b < 2$;

(ii) $m(t) = \lambda t^{2(b-1)/(n(2-b))}, 0 < t < \eta$, *where $\lambda > 0$ is a constant.*

In particular, if $b = 2(n+1)/(n+2)$, then $m(t) = \lambda t, 0 < t < \eta$.

3 Characterizations based on relationship of conditional moments of jump time and current life

Given $A(t) = n$, by using some simple forms of conditional expectations of S_n and δ_t can also characterize $m(\cdot)$. We present three theorems in this section.

Theorem 6. *Let $m(\cdot)$ be positive, twice differentiable in $(0,\eta)$, where $0 < \eta \leq \infty$. Assume for some fixed integer $n \geq 1$, whenever $P(A(t)=n) > 0$,*

$$E(S_n^2|A(t)=n) = bE(\delta_t^2|A(t)=n), 0 < t < \eta, \tag{26}$$

where b is independent of t. Then

$$m(t) = \lambda t^{(-1+\sqrt{1+8b})/(2n)}, 0 < t < \eta,$$

where $\lambda > 0$ is a constant. In particular, if $b = n(n+1)/2$, then $m(t) = \lambda t, 0 < t < \eta$.

Proof. From (26), we obtain

$$\int_0^t s^2 (m(s))^{n-1} m'(s)ds = b\int_0^t (t-s)^2 (m(s))^{n-1} m'(s)ds, 0 < t < \eta. \tag{27}$$

Then, taking the derivatives twice of both sides of (27) yields

$$\begin{aligned} &nt^2 m(t)m''(t) + n(n-1)t^2 (m'(t))^2 + 2ntm(t)m'(t) \\ &-2b(m(t))^2 = 0, 0 < t < \eta. \end{aligned} \tag{28}$$

Again let $u(t) = (m(t))^n$, (28) leads to

$$t^2 u''(t) + 2tu'(t) - 2bu(t) = 0, 0 < t < \eta. \tag{29}$$

By Lemma 1, the general solution of (29) is

$$u(t) = c_1 t^{(-1+\sqrt{1+8b})/2} + c_2 t^{(-1-\sqrt{1+8b})/2}, 0 < t < \eta,$$

where c_1 and c_2 are constants. Using the assumption that $m(t)$ is nonnegative, $m(0^+) = 0$ and the fact $b > 0$, we obtain $c_1 > 0, c_2 = 0$. Let $\lambda = c_1^{1/n} > 0$, and the assertion (ii) follows. □

As both proofs of the following Theorems 7 and 8 are similar to that of Theorem 6, they are omitted.

Theorem 7. *Let $m(\cdot)$ be positive, twice differentiable in $(0,\eta)$, where $0<\eta\leq\infty$. Assume for some fixed integer $n\geq 1, k\geq 1$, whenever $P(A(t)=n)>0$,*

$$E(S_n^k|A(t)=n)=bt^{k-1}E(\delta_t|A(t)=n), 0<t<\eta, \tag{30}$$

where b is independent of t. Then

$$m(t)=\lambda t^{\left(b-1+\sqrt{(b-1)^2+4bk}\right)/(2n)}, 0<t<\eta,$$

where $\lambda>0$ is a constant. In particular, if $b=n\,(n+1)\,/\,(n+k)$, then $m(t)=\lambda t, 0<t<\eta$.

Theorem 8. *Let $m(\cdot)$ be positive, twice differentiable in $(0,\eta)$, where $0<\eta\leq\infty$. Assume for some fixed integer $n\geq 1$, whenever $P(A(t)=n)>0$,*

$$E(\delta_t^2|A(t)=n)=btE(S_n|A(t)=n), 0<t<\eta, \tag{31}$$

where b is independent of t. Then

$$m(t)=\lambda t^{\left(\sqrt{1+2/b}-1\right)/n}, 0<t<\eta,$$

where $\lambda>0$ is a constant. In particular, if $b=2/(n\,(n+2))$, then $m(t)=\lambda t, 0<t<\eta$.

4 Some characterizations related to Abu-Youssef (2003)

Recently, a general class of distributions has been characterized through conditional expectation of record values in Abu-Youssef (2003). As expected there is a version for the class of order statistics point processes.

Theorem 9. *Let $m(\cdot)$ be positive and differentiable in $(0,\eta)$ and $\phi(\cdot)$ be differentiable in $(0,\eta)$* and $\phi(0^+)$ exists, *where $0<\eta\leq\infty$. Thus for some fixed integer $n\geq 1$, whenever $P(A(t)=n)>0$ and $P(A(t)=n+1)>0$,*

$$\begin{aligned}
& E(\phi(S_{n+1})|A(t)=n+1) \\
= & -\frac{c(n+1)}{m(t)}E(\phi(S_n)|A(t)=n)+\frac{c(n+1)}{m(t)}\phi(t)-b, 0<t<\eta,
\end{aligned}\tag{32}$$

where $b,c\neq 0$ are independent of t, if and only if

$$m(t)=c\log\frac{\phi(t)+b}{\phi(0^+)+b}, 0<t<\eta. \tag{33}$$

Proof. First, we have for every $k\geq 1$, whenever $P(A(t)=k)>0$,

$$\begin{aligned}
E(\phi(S_k)|A(t)=k) &= \int_0^t \phi(s)n\left(\frac{m(s)}{m(t)}\right)^{k-1}\frac{m'(s)}{m(t)}ds \\
&= \phi(t)-\int_0^t\left(\frac{m(s)}{m(t)}\right)^{k}\phi'(s)ds, 0<t<\eta,
\end{aligned}\tag{34}$$

by using the order statistics property.

(Necessary part.) By (34), (32) is equivalent to

$$\begin{aligned}
& \phi(t)-\int_0^t\left(\frac{m(s)}{m(t)}\right)^{n+1}\phi'(s)ds \\
= & -\frac{c(n+1)}{m(t)}\int_0^t\phi(s)n\left(\frac{m(s)}{m(t)}\right)^{n-1}\frac{m'(s)}{m(t)}ds+\frac{c(n+1)}{m(t)}\phi(t)-b, 0<t<\eta.
\end{aligned}$$

Multiplying by $m^{n+1}(t)$ in both sides of the above equation, it becomes

$$\begin{aligned}
& \phi(t)m^{n+1}(t)-\int_0^t m^{n+1}(s)\phi'(s)ds \\
= & -c(n+1)\int_0^t\phi(s)nm^{n-1}(s)m'(s)ds+c(n+1)\phi(t)m^n(t) \\
& -bm^{n+1}(t), 0<t<\eta.
\end{aligned}\tag{35}$$

After differentiating with respect to t in both sides of (35), it follows

$$m'(t)=\frac{c\phi'(t)}{\phi(t)+b}, 0<t<\eta.$$

This in turn implies that $m(t) = \log(k\,|\phi(t) + b|^c), 0 < t < \eta$, where $k > 0$ is a constant.

Now because of $m(0^+) = 0$ we have $k = |\phi(0^+) + b|^{-c}$. Since $m(t)$ is increasing in $(0, \eta)$, so does $|\phi(t) + b|^c$. Thus $\phi(t) + b$ is either positive for every $t \in (0, \eta)$ or negative for every $t \in (0, \eta)$. Therefore,

$$m(t) = c \log((\phi(t) + b)/(\phi(0^+) + b)), 0 < t < \eta,$$

is obtained. The proof of the necessary part is completed.

(Sufficient part.) For some fixed integer $n \geq 1$, whenever $P(A(t) = n) > 0$, by (34), we have

$$E(\phi(S_n)|A(t) = n) = \phi(t) - \int_0^t \left(\frac{m(s)}{m(t)}\right)^n \phi'(s)ds. \tag{36}$$

Taking the derivatives of both sides of (33) with respect to t yields

$$\phi'(t) = (\phi(t) + b)\, m'(t)/c, 0 < t < \eta. \tag{37}$$

Substituting (37) into (36), (32) is obtained. □

Next we give a result analogous to Theorem 2.4 of Abu-Youssef (2003).

Theorem 10. *Let $m(\cdot)$ be positive and differentiable in $(0, \eta)$, and $\phi(\cdot)$ be differentiable in $(0, \eta)$ and $\phi(0^+)$ exists, where $0 < \eta \leq \infty$. Thus for some fixed integer $n \geq 1$, whenever $P(A(t) = n) > 0$ and $P(A(t) = n + 1) > 0$,*

$$\begin{aligned} &E(\phi(S_n)|A(t) = n) \\ = \;& \phi(t) + \frac{bm(t)}{c(n+1)} E(e^{-c\phi(S_{n+1})}|A(t) = n + 1) + \frac{m(t)}{c(n+1)}, 0 < t < \eta, \end{aligned} \tag{38}$$

where $b, c \neq 0$ are independent of t, if and only if

$$m(t) = \log \frac{e^{c\phi(0^+)} + b}{e^{c\phi(t)} + b}, 0 < t < \eta. \tag{39}$$

Proof. (Necessary part.) Again (38) can be rewritten as

$$\begin{aligned} & \int_0^t \phi(s) n \left(\frac{m(s)}{m(t)}\right)^{n-1} \frac{m'(s)}{m(t)} ds \\ = \; & \phi(t) + \frac{bm(t)}{c\,(n+1)} \int_0^t e^{-c\phi(s)} (n+1) \left(\frac{m(s)}{m(t)}\right)^{n} \frac{m'(s)}{m(t)} ds \\ & + \frac{m(t)}{c(n+1)}, 0 < t < \eta. \end{aligned}$$

Taking the derivatives of both sides with respect to t yields

$$m'(t) = -\frac{c\phi'(t) e^{c\phi(t)}}{e^{c\phi(t)} + b}, 0 < t < \eta.$$

This implies that $m(t) = \log\left(k/|e^{c\phi(t)} + b|\right), 0 < t < \eta$, where $k > 0$ is a constant.

Because of $m(0^+) = 0$, $k = |e^{c\phi(0^+)} + b|$ is obtained. Since $m(t)$ is increasing in $(0, \eta)$, so does $|e^{c\phi(t)} + b|$. Thus $e^{c\phi(t)} + b$ is either positive for every $t \in (0, \eta)$ or negative for every $t \in (0, \eta)$. Therefore $m(t) = \log((e^{c\phi(0^+)} + b)/(e^{c\phi(t)} + b)), 0 < t < \eta$ as required.

(Sufficient part.) Taking the derivatives of both sides of (39) with respect to t yields

$$\phi'(t) = -e^{-c\phi(t)} \left(e^{c\phi(t)} + b\right) m'(t)/c, 0 < t < \eta. \tag{40}$$

By (36) and (40), (38) is obtained. The proof is finished. □

Obviously, (32) can be referred to as a recurrence relation. According to (32), for every $n \geq 1$, for a pair of integers $(n, l), n > l$, whenever $P(A(t) = i) > 0$, $i = l, ..., n$,

$$\begin{aligned} E(\phi(S_n)|A(t) = n) \; = \; & (-1)^{n-l} \frac{\Gamma(n+1)}{\Gamma(l+1)} \left(\frac{c}{m(t)}\right)^{n-l} E(\phi(S_l)|A(t) = l) \\ + \; & \phi(t) \sum_{i=1}^{n-l} (-1)^{i-1} \frac{\Gamma(n+1)}{\Gamma(n-i+1)} \left(\frac{c}{m(t)}\right)^{i} \\ + \; & b \sum_{i=0}^{n-l-1} (-1)^{i-1} \frac{\Gamma(n+1)}{\Gamma(n-i+1)} \left(\frac{c}{m(t)}\right)^{i}, \; 0 < t < \eta. \end{aligned}$$

The following theorem gives a characterization of the mean value function $E(A(t)), t \geq 0$, by using the conditional expectations of S_n and S_l, giving $A(t) = n$ and $A(t) = l$ respectively. The proof is standard hence is omitted.

Theorem 11. *Let $G(\cdot)$ and $m(\cdot)$ be positive and differentiable in $(0, \eta)$, where $0 < \eta \leq \infty$. Also let $G(\cdot)$ be non-constant and $m'(\cdot)$ be positive in $(0, \eta)$. Assume for some fixed integers $1 \leq l < n$, whenever $P(A(t) = n) > 0$ and $P(A(t) = l) > 0$,*

$$E(G(S_n)|A(t) = n) = bE(G(S_l)|A(t) = l), \forall 0 < t < \eta, \qquad (41)$$

where b is independent of t. Then

$$m(t) = \lambda \left(G(t)\right)^{(n-lb)/(nl(b-1))}, 0 < t < \eta,$$

where $\lambda > 0$ is a constant.

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