A Study of Inverses of Thinned Renewal Processes

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Abstract

We study properties of thinning and Markow chain thinning of renewal processes. Among others, for some special renewal processes we investigate whether these processes can be obtained through Markov chain thinning.

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1. Introduction

In this work we will investigate the problem of inverses of thinned renewal processes. Let $\mathcal{N} \equiv \{N(t),\ t \geq 0\}$ be a point process, and denote by $\mathcal{N}_p \equiv \{N_p(t),\ t \geq 0\}$ the point process obtained by retaining (in the same location) every point of $\mathcal{N}$ with a constant probability p and deleting it with probability $1-p$, independent of all other points and independent of the point process $\mathcal{N}$. $\mathcal{N}_p$ is called the p-thinning of $\mathcal{N}$, and $\mathcal{N}$ is called the p-inverse of $\mathcal{N}_p$. As mentioned in Yannaros (1988a), the p-inverse of any thinned point process is unique in distributional meaning, and it is also called the original process.

Now let $\mathcal{N}$ be a renewal process, and $\{\chi_i,\ i \geq 1\}$, independent of $\mathcal{N}$, be a sequence of binary variables which form a stationary Markov chain with marginal distribution

$$P(\chi_i = 1) = p = 1 - P(\chi_i = 0), \quad 0 < p \leq 1,$$

and transition probabilities

$$P(\chi_{i+1} = 1 | \chi_i = 1) = \alpha_1 = 1 - P(\chi_{i+1} = 0 | \chi_i = 1),$$

$$P(\chi_{i+1} = 1 | \chi_i = 0) = \alpha_0 = 1 - P(\chi_{i+1} = 0 | \chi_i = 0),$$

where $0 \leq \alpha_0,\ \alpha_1 \leq 1,\ i \geq 1$. The stationarity of the chain imposes that $\alpha_0,\ \alpha_1$ and p satisfy the following constraint

$$p = \alpha_1 p + \alpha_0 (1-p)\ . \tag{1}$$

Then a thinned point process $\mathcal{A} \equiv \{A(t),\ t \geq 0\}$ can be obtained by retaining the i-th point of $\mathcal{N}$ if $\chi_i = 1$ and deleting it if $\chi_i = 0$. $\mathcal{A}$ is called the Markov chain thinning of $\mathcal{N}$. Generating this way, it can be proved easily that $\mathcal{A}$ is a delayed renewal process. Conversely, we are interested in knowing that given a stationary Markov chain $\{\chi_i,\ i \geq 1\}$ and a delayed renewal process $\mathcal{A}$, under what conditions there exists an ordinary renewal process $\mathcal{N}$, say the M-inverse of $\mathcal{A}$, such that $\mathcal{A}$ can be obtained through the Markov chain thinning of $\mathcal{N}$.

When the sequence $\{\chi_i,\ i \geq 1\}$ are independent and identically distributed (i.i.d.), namely $\alpha_0 = \alpha_1 = p$, the Markov chain thinning becomes p-thinning, and the above inverse problem has been studied by many authors. Yannaros (1988a) proved that the p-inverse of a thinned renewal process is unique and is also a renewal process. Next in (1988b), Yannaros characterized when an ordinary gamma renewal process has a p-inverse. He also gave necessary and sufficient conditions for a delayed gamma renewal process can be obtained through p-thinning. Later, in (1991), Yannaros extended the above model to the thinned random walks, and gave the limit behaviour of p-thinned random walks, as $p \to 0$. Yannaros (1994) investigated the class of renewal processes with Weibull lifetime distribution from the point of view of the general theory of point processes. On the other hand, Isham (1980), Chandramohan et al. (1985) discussed Markov chain thinning in various problems. That is the motivation in this note we will study properties of thinning and Markov chain thinning of renewal processes. Also we will investigate whether some special renewal processes can be obtained through Markov chain thinning.

In Section 2, we present some properties of completely monotone functions and Laplace transforms. In Section 3, we give some simple properties related to the Markov chain thinning. In Sections 4 and 5, when $\mathcal{A}$ is a delayed renewal process, we give conditions such that the M-inverse exists, with interarrival times being gamma-c or negative binomial distributed, respectively. Here a random variable X is said to be $\Gamma_c(a, \lambda)$ distributed, if it has the Laplace transform $(1 + \lambda s^c)^{-a}$, for some $0 < c \le 1,\ \lambda > 0,\ a > 0$, for every $s \ge 0$, and the delayed gamma-c renewal process can be defined similarly. Gamma-c distribution was studied by Huang and Chen (1989) and (1991). Let Z be a nonnegative random variable having the distribution function H so that H has support in $[0, \infty)$, namely $H(0-) = 0$. The Laplace transform of Z or H is the function $\hat{h}$ on $[0, \infty)$ given by

$$\hat{h}(s) = E(e^{-sZ}) = \int_0^\infty e^{-sx} dH(x) \ .$$

In Section 6, we give an example of delayed renewal process which does not belong to any of the two classes discussed in Sections 4 and 5, and a stationary Markov chain such that the M-inverse exists. Finally, in Section 7, we discuss some unsolved problems of inverses of thinned renewal processes.

When $\mathcal{N}$ is a delayed renewal process, let $\{X_i,\ i \ge 1\}$ be the sequence of interarrival times with G being the distribution function of X_1 and F being the distribution function of $\{X_i,\ i \ge 2\}$, where $F(0) = G(0) = 0$. Also let $\hat{g}(s)$ and $\hat{f}(s)$ denote the Laplace transforms of G and F, respectively. Given the stationary Markov chain $\{\chi_i,\ i \ge 1\}$, it can be derived easily (see, e.g., Isham (1980)) that

$$\hat{\xi}(s) = \frac{p\hat{g}(s) + (\alpha_0 - p)\hat{f}(s)\hat{g}(s)}{1-(1-\alpha_0)\hat{f}(s)}, \tag{2}$$

$$\hat{\eta}(s) = \frac{\alpha_1\hat{f}(s) + (\alpha_0 - \alpha_1)\hat{f}^2(s)}{1-(1-\alpha_0)\hat{f}(s)}, \tag{3}$$

where $\hat{\xi}(s) = E(e^{-sY_1})$, $\hat{\eta}(s) = E(e^{-sY_2})$ and $\{Y_i,\ i \geq 1\}$ is the sequence of interarrival times of the thinned point process $\mathcal{A}$ which is a delayed renewal process also. Note that the delayed renewal process $\mathcal{N}$ is stationary if and only if

$$G(x) = \frac{\int_0^x (1-F(y))dy}{E(X_2)}, \quad x \geq 0, \tag{4}$$

or equivalently

$$\hat{g}(s) = \frac{1}{E(X_2)s}(1-\hat{f}(s)), \quad s > 0\ . \tag{5}$$

2. Preliminaries

First a function ψ on $(0,\infty)$ is called completely monotone if it possesses derivatives $\psi^{(n)}$ of all orders and

$$(-1)^n\psi^{(n)}(s) \geq 0, \tag{6}$$

for each $n \geq 0$ and each s in $(0,\infty)$ .The following is a useful characterization of Laplace transforms of measures on $(0,\infty)$ due to Chung (1974).

Theorem 1. A function ψ on $(0,\infty)$ is the Laplace transform of a distribution function B, namely

$$\psi(s) = \int_0^\infty e^{-sx}dB(x),$$

if and only if it is completely monotone in $(0,\infty)$ with $\psi(0+) = 1$.

In the following we give two criteria of completely monotone functions which can be found in books such as Feller (1971).

Criterion 1. If ψ and ϕ are completely monotone so is their product $\psi\phi$.

Criterion 2. If ψ is completely monotone and ϕ a positive function with a completely monotone derivative then $\psi(\phi)$ is completely monotone.

The next result was proved by Kolsrud (1986), which gives a simple consequence of Bernstein functions. Here a function ϕ on $(0, \infty)$ is said to be a Bernstein function if it has a completely monotone derivative, i.e. if $(-1)^n \phi^{(n)}(s) \leq 0$, $\forall s > 0$, for $n = 1, 2, \cdots$.

Lemma 1. If ϕ is a Bernstein function with $\phi(0+) = 1$, then for any α in $(0, 1]$, $0 < p < 1$, the function $(p + (1-p)\phi^\alpha)^{-1}$ is the Laplace transform of a probability measure.

As mentioned in Section 1, let $\mathcal{N}$ be a renewal process with interarrival distribution function F, then $\mathcal{N}_p$ is also a renewal process with interarrival distribution function G. Let $\hat{f}$ and $\hat{g}$ be the Laplace transforms of F and G, respectively. Yannaros (1988c) gave the relation of $\hat{f}$ and $\hat{g}$, and proved the following lemma.

Lemma 2. The function $\hat{f} = \hat{g}/(p + (1-p)\hat{g})$ is completely monotone for every $p \in (0, 1]$, if and only if $\hat{g} = 1/(1+\phi)$, where ϕ is a Bernstein function with $\phi(0+) = 0$.

In Lemma 2, if $\hat{g} = 1/(1 + \phi)$, then $\hat{f} = 1/(1 + p\phi)$, which gives a description of the class of completely monotone functions.

3. Some basic properties for thinning via a Markov chain

In this section we give some elementary theorems. First we characterize the class of ordinary renewal processes.

Theorem 2. Assume $\mathcal{N}$ is an ordinary renewal process which is thinned by a stationary Markov chain $\{\chi_i,\ i \geq 1\}$ as defined in Section 1. Then the thinned point process $\mathcal{A}$ is an ordinary renewal process if and only if $\{\chi_i,\ i \geq 1\}$ is an independent sequence.

Proof. From (2) and (3) we find that $\mathcal{A}$ is an ordinary renewal process if and only if

$$\alpha_1 \hat{f}(s) + (\alpha_0 - \alpha_1)\hat{f}^2(s) = p\hat{f}(s) + (\alpha_0 - p)\hat{f}^2(s), \quad s > 0, \tag{7}$$

or

$$(p - \alpha_1)\hat{f}(s)(\hat{f}(s) - 1) = 0, \quad s > 0 \ . \tag{8}$$

As $0 < \hat{f}(s) < 1, \ s > 0$, (8) is equivalent to $\alpha_1 = p$.

On the other hand, the assumption that $\{\chi_i, \ i \geq 1\}$ is a stationary Markov chain gives $p = \alpha_1 p + \alpha_0(1-p)$. This together with $\alpha_1 = p$ implies $p = 1$ or $\alpha_0 = p$. Obviously, either $\alpha_0 = \alpha_1 = p$ or $p = 1$, implies $\{\chi_i, \ i \geq 1\}$ is an independent sequence.

Conversely, it is easy to see that if $\{\chi_i, \ i \geq 1\}$ forms an independent sequence then $\mathcal{A}$ is an ordinary renewal process. This completes the proof.

The "if" part of the following corollary is well known, the "only if" part can be proved by using Theorem 2 and the fact that Poisson process is also an ordinary renewal process.

Corollary 1. Let $\mathcal{N}$ be a Poisson process and $\{\chi_i, \ i \geq 1\}$ be a stationary Markov chain. Then $\mathcal{A}$ is Poisson if and only if $\{\chi_i, \ i \geq 1\}$ is an independent sequence.

The next theorem is about stationary renewal process.

Theorem 3. Let $\mathcal{N}$ be a delayed renewal process, $\{\chi_i, \ i \geq 1\}$ be a stationary Markov chain. Then $\mathcal{A}$ is a stationary renewal process if and only if $\mathcal{N}$ is a stationary renewal process.

Proof. If $\mathcal{N}$ is stationary, then $\hat{g}(s) = (E(X_2)s)^{-1}(1 - \hat{f}(s))$. Substituting this into (2), yields

$$\hat{\xi}(s) = \frac{(p + (\alpha_0 - p)\hat{f}(s))(1 - \hat{f}(s))}{E(X_2)s(1 - (1 - \alpha_0)\hat{f}(s))} \ . \tag{9}$$

The stationarity of $\{\chi_i, \ i \geq 1\}$ in turn implies $p = \alpha_1 p + \alpha_0(1-p)$, or $\alpha_0 - p = (\alpha_0 - \alpha_1)p$. Hence (9) can be rewritten as, by replacing $\alpha_0 - p$ by $(\alpha_0 - \alpha_1)p$,

$$\hat{\xi}(s) \quad = \quad \frac{p}{E(X_2)s}\left(1 - \frac{\alpha_1 \hat{f}(s) + (\alpha_0 - \alpha_1)\hat{f}^2(s)}{1 - (1 - \alpha_0)\hat{f}(s)}\right) \tag{10}$$

$$= \quad \frac{p}{E(X_2)s}(1 - \hat{\eta}(s)) \ .$$

This proves $\mathcal{A}$ is a stationary renewal process and the "if " part is obtained.

Conversely, assume $\mathcal{A}$ is a stationary renewal process, then

$$\hat{\xi}(s) = \frac{1}{E(Y_2)s}(1 - \hat{\eta}(s)) \ . \tag{11}$$

In view of (2) and (3), (11) implies

$$\frac{p\hat{g}(s) + (\alpha_0 - p)\hat{f}(s)\hat{g}(s)}{1 - (1 - \alpha_0)\hat{f}(s)} = \frac{p}{E(X_2)s}(1 - \frac{\alpha_1 \hat{f}(s) + (\alpha_0 - \alpha_1)\hat{f}^2(s)}{1 - (1 - \alpha_0)\hat{f}(s)}) \ . \tag{12}$$

Note that since $\{\chi_i, \ i \geq 1\}$ is stationary,

$$E(Y_2) = \frac{1 - \alpha_1 + \alpha_0}{\alpha_0} E(X_2) = \frac{1}{p} E(X_2) \ . \tag{13}$$

Again substituting $\alpha_0 - p$ by $(\alpha_0 - \alpha_1)p$ in the left side of (12) and after some simplifications, gives

$$\hat{g}(s) = \frac{1}{E(X_2)s}(1 - \hat{f}(s)) \ . \tag{14}$$

Therefore $\mathcal{N}$ is a stationary renewal process as required. This completes the proof of this theorem.

We also have the following immediate consequence which can be compared with Corollary 1.

Corollary 2. Let $\mathcal{N}$ be an ordinary renewal process, $\{\chi_i, \ i \geq 1\}$ be a stationary Markov chain. Then $\mathcal{A}$ is a stationary renewal process if and only if $\mathcal{N}$ is Poisson process.

Proof. From Theorem 3 we obtain that $\mathcal{A}$ is a stationary renewal process if and only if $\mathcal{N}$ is a stationary renewal process. Yet the only ordinary stationary renewal process is Poisson. This completes the proof.

4. Delayed gamma-c renewal process

It is desirable to know that given an arbitrarily delayed renewal process $\mathcal{A}$ and a sequence of Markov chain $\{\chi_i, \ i \geq 1\}$, whether there exists an original process $\mathcal{N}$, i.e., the M-inverse of $\mathcal{A}$, such that $\mathcal{A}$ can be obtained through the Markov chain thinning. When $\{\chi_i,\ i \geq 1\}$ is an i.i.d. sequence with $P(\chi_i = 1) = p$, Yannaros (1988a) has shown that a renewal process cannot be obtained through the thinning of a non-renewal process for any $p \leq 1$. That is the class of renewal processes is closed under inverse thinning. Yannaros (1988b) also gave necessary and sufficient condition for a delayed gamma renewal process to be a Cox process. Note that a Cox process can be viewed as a Poisson process with a random intensity.

In the following let $\mathcal{A}$ be a delayed gamma-c renewal process as defined in Section 1 with interarrival distribution functions H and K, which are both gamma-c with shape parameters β and α, respectively, and for simplicity we assume both scale parameters equal to 1 (hence $\hat{\xi}(s) = \int_0^\infty e^{-sx} dH(x) = (1+s^c)^{-\beta}$ and $\hat{\eta}(s) = \int_0^\infty e^{-sx} dK(x) = (1+s^c)^{-\alpha}$); and let $\{\chi_i,\ i \geq 1\}$ be a stationary Markov chain as defined in Section 1. We find conditions for the existence of an ordinary renewal process to be the M-inverse of $\mathcal{A}$. In the special case $c = 1$, $\mathcal{A}$ becomes a delayed gamma renewal process.

Case 1. $\alpha = \beta$.

In this case $\mathcal{A}$ becomes an ordinary renewal process. Hence by Theorem 2, we obtain $\{\chi_i,\ i \geq 1\}$ must be an independent sequence. So this reduces to the problem of determining the p-inverse. The case $p = 1$ is trivial, $\mathcal{A}$ is the inverse of itself for every $\alpha > 0$. For every $0 < p < 1$, from (2) and (3) with $\hat{g}(s) = \hat{f}(s)$, we obtain

$$\hat{f}(s) = \frac{\hat{\xi}(s)}{p + (1-p)\hat{\xi}(s)} \ .$$

Since $\hat{\xi}(s) = \hat{\eta}(s) = (1+s^c)^{-\alpha}$, it yields

$$\hat{f}(s) = \frac{1}{p(1+s^c)^\alpha + (1-p)} \ . \qquad (15)$$

As $\hat{f}(0+) = 1$, being a Laplace transform, $\hat{f}(s)$ must be completely monotone. In order to determine the conditions such that $\hat{f}(s)$ is completely monotone, we consider the following three situations: $0 < \alpha \leq 1$, $1 < \alpha \leq 1/c$, and $\alpha > 1/c$, where $0 < c \leq 1$.

Firstly, we study the case $0 < \alpha \leq 1$. Let $\phi(s) = 1 + s^c$, then $\phi(0+) = 1$. Since for $0 < c \leq 1$,

$$(-1)^n \phi^{(n)}(s) \leq 0, \quad s > 0,$$

$\phi(s)$ is a Bernstein function. Consequently, $\hat{f}(s)$ as defined in (15), is a Laplace transform by Lemma 1.

Although for the case $1 < \alpha \leq 1/c$, we are unable to determine whether the function $\hat{f}(s)$ is a Laplace transform, we have some partial result. Let α be an integer, and $\phi(s) = (1+s^c)^\alpha - 1$, then $\phi(0+) = 0$. Applying the Binomial theorem, we have

$$\phi(s) = (1+s^c)^\alpha - 1 = \sum_{j=0}^{\alpha} \binom{\alpha}{j} s^{cj} - 1,$$

hence

$$\phi'(s) = \sum_{j=0}^{\alpha} \binom{\alpha}{j} cjs^{cj-1} .$$

It is easy to see that $\phi(s)$ has a completely monotone derivative if $cj \leq 1$ for every $j = 0, 1, \cdots, \alpha$. Thus $\phi(s)$ is a Berstein function if $c\alpha \leq 1$. From Lemma 2, $\hat{f}(s)$ as defined in (15), is a Laplace transform.

Finally, consider the case $\alpha > 1/c$. Again let $\phi(s) = (1+s^c)^\alpha - 1$, then $\phi(0+) = 0$. It is easy to get

$$\begin{aligned}
\phi'(s) &= \alpha(1+s^c)^{\alpha-1} cs^{c-1}, \\
\phi''(s) &= \alpha(\alpha-1)(1+s^c)^{\alpha-2} c^2 s^{2c-2} + \alpha(1+s^c)^{\alpha-1} c(c-1)s^{c-2} \\
&= c\alpha(1+s^c)^{\alpha-2}\{(c\alpha-1)s^c + (c-1)\} .
\end{aligned}$$

Thus, $\phi''(s) \geq 0$ when s is large enough. Hence $\phi(s)$ is not a Bernstein function. From Lemma 2, $\hat{f}(s)$ as defined in (15), is not a Laplace transform.

Case 2. $\alpha > \beta$.

First note that as $\mathcal{A}$ is not an ordinary renewal process, by Theorem 2, $\{\chi_i,\ i \geq 1\}$ cannot be an independent sequence. That is $\alpha_0 \neq \alpha_1$. From (2) and (3) with $\hat{g}(s) = \hat{f}(s)$, we obtain

$$\frac{\hat{\xi}(s)}{\hat{\eta}(s)} = \frac{p + (\alpha_0 - p)\hat{f}(s)}{\alpha_1 + (\alpha_0 - \alpha_1)\hat{f}(s)} . \tag{16}$$

Substituting $\hat{\xi}(s) = (1+s^c)^{-\beta}$, $\hat{\eta}(s) = (1+s^c)^{-\alpha}$ and $\alpha_0 - p = p(\alpha_0 - \alpha_1)$ into (16), then solving for $\hat{f}(s)$, yields

$$\hat{f}(s) = \frac{p(1+s^c)^{-\alpha} - \alpha_1(1+s^c)^{-\beta}}{(\alpha_0 - \alpha_1)((1+s^c)^{-\beta} - p(1+s^c)^{-\alpha})} . \tag{17}$$

Being a Laplace transform,

$$\hat{f}'(s) = \frac{p(1-\alpha_1)(\beta-\alpha)}{\alpha_0-\alpha_1} \cdot \frac{(1+s^c)^{\beta-\alpha-1}}{(1-p(1+s^c)^{\beta-\alpha})^2} \leq 0 \ . \tag{18}$$

Thus $\alpha_0 > \alpha_1$. Again (17) can be rewritten as

$$\hat{f}(s) = \frac{1}{\alpha_0-\alpha_1} \cdot \frac{p-\alpha_1(1+s^c)^{\alpha-\beta}}{(1+s^c)^{\alpha-\beta}-p} \ . \tag{19}$$

For $\alpha_1 \neq 0$, since $\alpha > \beta$, the function $\{(1+s^c)^{\alpha-\beta}-p\}$ is positive for every $s > 0$, and the function $\{p-\alpha_1(1+s^c)^{\alpha-\beta}\}$ is negative for s large enough. Consequently, $\hat{f}(s) < 0$, $\hat{f}(s)$ as defined in (19), is not a Laplace transform for $\alpha_1 \neq 0$.

For the special case $\alpha_1 = 0$, we investigate when the function

$$\hat{f}(s) = \frac{1}{\alpha_0} \cdot \frac{p}{(1+s^c)^{\alpha-\beta}-p} \tag{20}$$

is a Laplace transform. By (1), (20) becomes

$$\hat{f}(s) = \frac{1-p}{(1+s^c)^{\alpha-\beta}-p} \ . \tag{21}$$

Again, let

$$\phi(s) = \frac{1}{p(1-p)}(1+s^c)^{\alpha-\beta} - \frac{1}{p(1-p)} \ . \tag{22}$$

Since $\phi(0+) = 0$, the problem becomes to determine when the function $\phi(s)$ is a Bernstein function. It can be seen that as in Case 1 by considering the three situations $0 < \alpha-\beta \leq 1$, $1 < \alpha-\beta \leq 1/c$, and $\alpha-\beta > 1/c$, parallel results can be obtained.

Case 3. $\alpha < \beta$.

Again (17) can be rewritten as

$$\hat{f}(s) = \frac{1}{\alpha_0-\alpha_1} \cdot \frac{p(1+s^c)^{\beta-\alpha}-\alpha_1}{1-p(1+s^c)^{\beta-\alpha}} \ . \tag{23}$$

Since $\alpha_0 < \alpha_1$, if both $\{p(1+s^c)^{\beta-\alpha}-\alpha_1\}$ and $\{1-p(1+s^c)^{\beta-\alpha}\}$ are positive, then $\hat{f}(s) < 0$. It is easy to obtain the inequality

$$\{(\frac{\alpha_1}{p})^{\frac{1}{\beta-\alpha}}-1\}^{\frac{1}{c}} < \ s \ < \{(\frac{1}{p})^{\frac{1}{\beta-\alpha}}-1\}^{\frac{1}{c}}, \ \ \alpha_1 \neq 1 \ .$$

Therefore, $\hat{f}(s) < 0$, $\hat{f}(s)$ as defined in (23), is not a Laplace transform for $\alpha_1 \neq 1$.

We now consider the special case $\alpha_1 = 1$. From (1), we have

$$p = \frac{\alpha_0}{1 + \alpha_0 - \alpha_1} = \frac{\alpha_0}{1 + \alpha_0 - 1} = 1 \; .$$

This shows that $\mathcal{A}$ is the inverse of itself. In other words, given a stationary Markov chain, the M-inverse exists if $\alpha_1 = 1$ when $\alpha < \beta$.

5. Delayed negative binomial renewal process

In the above section, we consider renewal process with continuous interarrival distribution function. In this section we consider the discrete situation. More precisely, we consider a delayed renewal process with interarrival times being $NB(k,\theta)$ and $NB(r,\theta)$ distributed (hence $\hat{\xi}(s) = (\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^k$ and $\hat{\eta}(s) = (\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^r$); and let $\{\chi_i, \; i \geq 1\}$ be a stationary Markov chain as defined in Section 1. We find conditions for the existence of an ordinary renewal process to be the M-inverse of $\mathcal{A}$.

Case 1. $k = r$.

In this case $\mathcal{A}$ becomes an ordinary renewal process. Hence by Theorem 2, we obtain $\{\chi_i, \; i \geq 1\}$ is an independent sequence. So this reduces to the problem of determining the p-thinning. The case $p = 1$ is trivial, $\mathcal{A}$ is the inverse of itself for every $\alpha > 0$. For every $0 < p < 1$, from (2) and (3) with $\hat{g}(s) = \hat{f}(s)$, we obtain

$$\hat{f}(s) = \frac{\hat{\xi}(s)}{p + (1-p)\hat{\xi}(s)} \; .$$

Since $\hat{\xi}(s) = \hat{\eta}(s) = (\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^r$, it yields

$$\hat{f}(s) = \frac{\theta^r}{p(e^s - (1-\theta))^r + (1-p)\theta^r} \; . \tag{24}$$

Let

$$\phi(s) = (\frac{e^s - (1-\theta)}{\theta})^r - 1 \; . \tag{25}$$

It is easy to obtain

$$\phi'(s) \;\; = \;\; (\frac{1}{\theta})^r r(e^s - (1-\theta))^{r-1} e^s,$$

$$\begin{aligned}\phi''(s) &= (\frac{1}{\theta})^r r\{(r-1)(e^s-(1-\theta))^{r-2}e^{2s} + (e^s-(1-\theta))^{r-1}e^s\} \\ &= (\frac{1}{\theta})^r r(e^s-(1-\theta))^{r-2}e^s\{re^s-(1-\theta)\}\ .\end{aligned}$$

Since r is an integer, $\phi''(s)$ is positive for every $s > 0$. Hence $\phi(s)$ is not a Bernstein function. Consequently, by Lemma 2, $\hat{f}(s)$ as defined in (24) is not a Laplace transform.

Case 2. $k < r$.

First note that as $\mathcal{A}$ is not an ordinary renewal process, by Theorem 2 $\{\chi_i,\ i \geq 1\}$ cannot be an independent sequence. Hence $\alpha_0 \neq \alpha_1$. From (2) and (3) with $\hat{g}(s) = \hat{f}(s)$, we obtain

$$\frac{\hat{\xi}(s)}{\hat{\eta}(s)} = \frac{p + (\alpha_0 - p)\hat{f}(s)}{\alpha_1 + (\alpha_0 - \alpha_1)\hat{f}(s)}\ . \tag{26}$$

Substituting $\hat{\xi}(s) = (\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^k$, $\hat{\eta}(s) = (\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^r$ and $\alpha_0 - p = p(\alpha_0 - \alpha_1)$ into (26), then solving for $\hat{f}(s)$, yields

$$\begin{aligned}\hat{f}(s) &= \frac{1}{\alpha_0-\alpha_1}\cdot\frac{\alpha_1(\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^k - p(\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^r}{p(\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^r - (\frac{\theta e^{-s}}{1-(1-\theta)e^{-s}})^k} \\ &= \frac{1}{\alpha_0-\alpha_1}\cdot\frac{\alpha_1\theta^k(e^s-(1-\theta))^r - p\,\theta^r(e^s-(1-\theta))^k}{p\,\theta^r(e^s-(1-\theta))^k - \theta^k(e^s-(1-\theta))^r}\ . \end{aligned}\tag{27}$$

Being a Laplace transform,

$$\hat{f}'(s) = \frac{p(1-p)(k-r)}{\alpha_0-\alpha_1}\cdot\frac{\theta^{k+r}(e^s-(1-\theta))^{k+r-1}}{(p\,\theta^r(e^s-(1-\theta))^k - \theta^k(e^s-(1-\theta))^r)^2} \leq 0 \tag{28}$$

Thus $\alpha_0 > \alpha_1$, and from (1) we have $\alpha_0 > p > \alpha_1$.

Again (27) can be rewritten as

$$\hat{f}(s) = \frac{1}{\alpha_0-\alpha_1}\cdot\frac{\alpha_1(e^s-(1-\theta))^{r-k} - p\,\theta^{r-k}}{p\,\theta^{r-k} - (e^s-(1-\theta))^{r-k}}\ . \tag{29}$$

If $\alpha_1 \neq 0$, since $\theta \leq e^s - (1-\theta)$, $\{\alpha_1(e^s-(1-\theta))^{r-k} - p\,\theta^{r-k}\}$ is positive for s large enough and $\{p\,\theta^{r-k} - (e^s-(1-\theta))^{r-k}\}$ is negative for every $s > 0$, we obtain $\hat{f}(s) < 0$ for s large enough. Therefore, $\hat{f}(s)$ as defined in (29), is not a Laplace transform when $\alpha_1 \neq 0$.

Now consider the case $\alpha_1 = 0$. In this case

$$\hat{f}(s) = \frac{1}{\alpha_0} \cdot \frac{p\,\theta^{r-k}}{(e^s - (1-\theta))^{r-k} - p\,\theta^{r-k}} \ .$$

Furthermore, by using (1), we obtain

$$\hat{f}(s) = \frac{1-p}{(\frac{e^s-(1-\theta)}{\theta})^{r-k} - p} \ . \tag{30}$$

Again, let

$$\phi(s) = \frac{1}{p(1-p)}(\frac{e^s - (1-\theta)}{\theta})^{r-k} - \frac{1}{p(1-p)} \ .$$

Then

$$\begin{aligned}
\phi'(s) &= \frac{1}{p(1-p)\theta^{r-k}}(r-k)(e^s - (1-\theta))^{r-k-1}e^s, \\
\phi''(s) &= \frac{r-k}{p(1-p)\theta^{r-k}}\{(r-k-1)(e^s-(1-\theta))^{r-k-2}e^{2s} + (e^s-(1-\theta))^{r-k-1}e^s\} \\
&= \frac{(r-k)(e^s-(1-\theta))^{r-k-2}e^s}{p(1-p)\theta^{r-k}}\{(r-k)e^s - (1-\theta)\} \ .
\end{aligned}$$

Since $r-k$ is an integer, $\phi''(s)$ is positive for every $s > 0$. Hence $\phi(s)$ is not a Bernstein function. Consequently, $\hat{f}(s)$ as defined in (30), is not a Laplace transform by Lemma 2. This shows that when $r > k$, for any Markov chain, there does not exist an ordinary renewal process $\mathcal{N}$ such that $\mathcal{A}$ can be obtained through this Markov chain thinning.

Case 3. $k > r$.

Again from (27), by a similar argument as in Case 2, we obtain

$$\hat{f}(s) = \frac{1}{\alpha_0 - \alpha_1} \cdot \frac{\alpha_1\,\theta^{k-r} - p(e^s-(1-\theta))^{k-r}}{p(e^s-(1-\theta))^{k-r} - \theta^{k-r}}, \tag{31}$$

here $\alpha_0 < \alpha_1$, and from (1), we have $\alpha_0 < p < \alpha_1$.

Similarly, if both $\{\alpha_1\,\theta^{k-r} - p(e^s-(1-\theta))^{k-r}\}$ and $\{p(e^s-(1-\theta))^{k-r} - \theta^{k-r}\}$ are negative, then $\hat{f}(s) < 0$. It is easy to obtain the inequality

$$((\frac{\alpha_1}{p})^{\frac{1}{k-r}} - 1)\theta + 1 < e^s < ((\frac{1}{p})^{\frac{1}{k-r}} - 1)\theta + 1, \ \ \alpha_1 \neq 1 \ .$$

Therefore, $\hat{f}(s) < 0$, and $\hat{f}(s)$ is not a Laplace transform when $\alpha_1 \neq 1$.

Now consider the special case $\alpha_1 = 1$. From (1), we have

$$p = \frac{\alpha_0}{1+\alpha_0-\alpha_1} = \frac{\alpha_0}{1+\alpha_0-1} = 1 \ .$$

This shows that $\mathcal{A}$ is the inverse of itself. In other words when $k > r$, given a Markov chain, the M-inverse exists if and only if $\alpha_1 = 1$.

6. An example

In Sections 4 and 5, where $\mathcal{A}$ is a delayed renewal process, we give conditions such that the M-inverse exists, with interarrival times being gamma-c or negative binomial distributed, respectively. For some special delayed renewal process $\mathcal{A}$, which does not belong to any of the above two classes, the M-inverse may also exist. We give an example in this section.

Let $\mathcal{A}$ be a delayed renewal process with distribution functions of the interarrival times $\{Y_i,\ i \geq 1\}$ being :

$$\begin{aligned} \mathrm{P}(Y_1 \leq x) &= 1 - \frac{1}{2}(2+x)e^{-2x}, \quad x > 0, \\ \mathrm{P}(Y_k \leq x) &= 1 - (1+x)e^{-2x}, \quad x > 0,\ k \geq 2, \end{aligned}$$

and $P(Y_k \leq x) = 0$, if $x < 0$. Then the Laplace transforms of Y_1 and Y_2 are

$$\hat{\xi}(s) = \mathrm{E}(e^{-sY_1}) = \frac{4+1.5s}{(2+s)^2}, \quad s > 0, \tag{32}$$

and

$$\hat{\eta}(s) = \mathrm{E}(e^{-sY_2}) = \frac{4+s}{(2+s)^2}, \quad s > 0, \tag{33}$$

respectively. In the following we find the conditions that given $\mathcal{A}$, and a stationary Markov chain $\{\chi_i,\ i \geq 1\}$, as defined in Section 1, when the M-inverse of $\mathcal{A}$ exists.

From (2) and (3) with $\hat{g}(s) = \hat{f}(s)$, we obtain

$$\frac{\hat{\xi}(s)}{\hat{\eta}(s)} = \frac{p + (\alpha_0 - p)\hat{f}(s)}{\alpha_1 + (\alpha_0 - \alpha_1)\hat{f}(s)} \ . \tag{34}$$

Substituting (32), (33) and $\alpha_0 - p = p(\alpha_0 - \alpha_1)$ into (34), then solving for $\hat{f}(s)$, yields

$$\hat{f}(s) = \frac{1}{\alpha_0 - \alpha_1} \cdot \frac{(p - 1.5\alpha_1)s + 4(p - \alpha_1)}{(1.5 - p)s + 4(1 - p)} \ . \tag{35}$$

Being a Laplace transform,

$$\hat{f}'(s) = \frac{1}{\alpha_0 - \alpha_1} \cdot \frac{2p(\alpha_1 - 1)}{((1.5-p)s + 4(1-p))^2} \leq 0 \ . \tag{36}$$

Thus $\alpha_0 > \alpha_1$. Note that from (1) we have $\alpha_0 > p > \alpha_1$.

Since $(-1)^n \hat{f}^{(n)}(s) \geq 0, \forall\ n \geq 1, s > 0$, with $\alpha_0 > \alpha_1$, we only need to find the conditions such that whether $\hat{f}(s) \geq 0, \forall s > 0$. The problem is equivalent to determining when $p - 1.5\alpha_1 \geq 0$. Solving the above inequality with $p = \dfrac{\alpha_0}{1 + \alpha_0 - \alpha_1}$ and noting that $0 \leq \alpha_0, \alpha_1 \leq 1$, yields

$$1 \geq \alpha_0 \geq \frac{3\alpha_1(1-\alpha_1)}{2 - 3\alpha_1} \qquad \text{and} \qquad \alpha_1 \leq \frac{3 - \sqrt{3}}{3} \ . \tag{37}$$

(37) is then a necessary and sufficient condition for $\hat{f}(s)$ being a Laplace transform. This is also the necessary and sufficient condition for $\mathcal{A}$ having an M-inverse.

7. Discussion

As mentioned in Theorem 1, a function ψ on $(0, \infty)$ is a Laplace transform if and only if it is completely monotone with $\psi(0+) = 0$. Usually, it is difficult to determine whether a function is completely monotone. It is also difficult to determine whether the function $\phi(s) = (1 + s^c)^\alpha$, $0 < c \leq 1,\ 1 < \alpha \leq 1/c$ and $s > 0$, is a Bernstein function. In this work we have solved the problem for the case that α is an integer. The case that α is not an integer will be investigated in the future work.

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