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On characterizations of the gamma and generalized inverse Gaussian distributions☆

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Abstract

Given two independent non-degenerate positive random variables X and Y, Letac and Wesolowski (Ann. Probab. 28 (2000) 1371) proved that $U=(X+Y)^{-1}$ and $V=X^{-1}-(X+Y)^{-1}$ are independent if and only if X and Y are generalized inverse Gaussian (GIG) and gamma distributed, respectively. Note that $X=(U+V)^{-1}$ and $Y=U^{-1}-(U+V)^{-1}$. This interesting transformation between (X,Y) and (U,V) preserves a bivariate probability measure which is a product of GIG and gamma distributions.

In this work, characterizations of the GIG and gamma distributions through the constancy of regressions of V^r on U are considered.

MSC: Primary 60E05; 62E10

Keywords: Characterization; Constancy of regression; Gamma distribution; Generalized inverse Gaussian distribution; Laplace transform

1. Introduction

Lukacs (1955) characterized two independent non-degenerate positive random variables to be gamma distributed by the independence of their quotient and sum. Since then, there are many further investigations. Among others the following are some basic directions: (i) Weakening the independence condition to constancy of regressions—see Bolger and Harkness (1965), Hall and

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Simons (1969), Wesolowski (1990), Li et al. (1994) Huang and Su (1997), Bobecka and Wesolowski (2002a), Chou and Huang (2003). (ii) considering the renewal process—see Wesolowski (1989), Li et al. (1994), Huang and Su (1997), Chou and Huang (2003). (iii) Considering the bivariate cases —see Wang (1981), Bobecka (2002), Pusz (2002), Chou and Huang (2004). (iv) Considering the matrix variates—see Olkin and Rubin (1962), Casalis and Letac (1996), Letac and Massam (1998), Bobecka and Wesolowski (2002b).

Letac and Wesolowski (2000) (LW in the sequel) gave a similar characterization of generalized inverse Gaussian (GIG) and gamma distributions via independence condition: given two independent non-degenerate positive random variables X and Y, if $U=(X+Y)^{-1}$ and $V=X^{-1}-(X+Y)^{-1}$ are independent, then X is GIG distributed and Y is gamma distributed. The readers may refer to Pusz (1997) and Matsumoto and Yor (2003) for some related works of GIG distribution. Again, there are at least two directions developed after LW: (i) Weakening the independence condition to constancy of regressions (see Seshadri and Wesolowski (2001) (SW in the sequel), and Wesolowski (2002)). (ii) Considering the matrix variates (see Wesolowski, 2002).

The GIG distribution $\mu_{p,a,b}$ is defined

$$\mu_{p,a,b}(\mathrm{d}x) = Cx^{p-1}\exp(-ax - b/x)I_{(0,\infty)}(x)\,\mathrm{d}x,$$

where C is the norming constant. The family of GIG distribution can be partitioned into the following three classes according to the parameter (p,a,b):

(i) Class I: $a>0, b>0, p\in R$.
(ii) Class II: $a>0, b=0, p>0$.
(iii) Class III: $a=0, b>0, p<0$.

Note that if X is $\mu_{p,a,b}$ distributed, then X^{-1} is $\mu_{-p,b,a}$ distributed. Class I contains the inverse Gaussian (IG) (with $p=-\frac{1}{2}$), reciprocal inverse Gaussian (RIG) ($p=\frac{1}{2}$), hyperbolic ($p=1$) and hyperbola ($p=0$) distributions. Class II is the class of gamma distributions. Class III is the class of reciprocal gamma distributions. For the details of the IG distribution characteristics and various statistical methods, see Chhikara and Folks (1989), Seshadri (1999) and the references therein. A random variable X is RIG (or reciprocal gamma) distributed, if and only if X^{-1} is IG (or gamma) distributed.

In this work we consider only Class II and a subclass of Class I. More precisely, we consider the gamma distribution $\Gamma(q,c)$ (*i.e.* $\mu_{q,c,0}$), where $q,c>0$,

$$\gamma_{q,c}(\mathrm{d}y) = \frac{c^q}{\Gamma(q)}\, y^{q-1}\mathrm{e}^{-cy}I_{(0,\infty)}(y)\,\mathrm{d}y, \tag{1}$$

and the GIG distribution

$$\mu_{-p,a,b}(\mathrm{d}x) = \frac{(a/b)^{-p/2}}{2K_{-p}(2\sqrt{ab})}\, x^{-p-1}\exp(-ax-b/x)I_{(0,\infty)}(x)\,\mathrm{d}x, \tag{2}$$

where $p,a,b>0$ and K_{-p} is a modified Bessel function with

$$K_{-p}(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^{-p}\int_0^\infty u^{p-1}\exp\left(-u-\frac{z^2}{4u}\right)\,\mathrm{d}u.$$

Note that the definition in (2) is the same as that in Wesolowski (2002) while somewhat different from that in SW in parameters a and b. In fact, our $\mu_{-p,a,b}$ is the same as $\mu_{-p,2a,2b}$ in SW.

Given $Y \sim \Gamma(p,a)$, SW characterized that $X \sim \mu_{-p,a,b}$ under the assumptions that for $r=0$ or -1, if $r > -p$,

$$E(V^{r+1}|U) = c_r E(V^r|U), \tag{3}$$

holds for some constant c_r. In this work we will prove that the above result holds if (3) is true for some fixed real $r > -p$. On the other hand, SW also characterized that $Y \sim \Gamma(p,a)$ if $X \sim \mu_{-p,a,b}$, and (3) holds for $r=0$ or -1, if $r > -p$, with $c_0 = E(V),\ c_{-1} = 1/E(V^{-1})$. Again, we prove that the result still holds if (3) is true for some $r > -p$.

Simultaneous characterizations of the distributions of X and Y are considered in Wesolowski (2002), he characterized X to be GIG distributed and Y gamma distributed under the assumption that for $r=-1$, (3) and

$$E(V^{r+2}|U) = c_{r+1} E(V^{r+1}|U), \tag{4}$$

hold for some constants c_r and c_{r+1}. Some further extension will be given in this work.

Before going into the details, we define the following transforms, if they exist,

$$f_X(s) = E(X^{-r-1}\mathrm{e}^{sX}), \tag{5}$$

$$g_X(s) = E(X^{-r-2}\mathrm{e}^{sX}), \tag{6}$$

and

$$h_Y(s) = E(Y^r\mathrm{e}^{sY}), \tag{7}$$

where $s \leqslant 0$. Note that if $Y \sim \Gamma(p,a)$, then for every $r > -p$,

$$h_Y(s) = E(Y^r)(1-s/a)^{-(p+r)}, \quad s \leqslant 0. \tag{8}$$

2. Characterization of GIG distribution given that Y is gamma distributed

In this section, we characterize X to be GIG distributed given the distribution of Y is gamma and Eq. (3) holds for some fixed $r > -p$.

The following theorem generalizes Theorems 1 and 3 of SW.

Theorem 1. *Let $Y \sim \Gamma(p,a)$ and for some fixed $r > -p$, $E(X^{-r-1}), E(X^{-r+1}) < \infty$. Assume that (3) holds for some constant c_r. Then $c_r > 0$ and $X \sim \mu_{-p,a,b}$, where $b = (p+r)/c_r > 0$.*

Proof. First (3) implies $c_r > 0$ immediately. Next from the definitions of U and V, (3) is equivalent to

$$E\left(\left(\frac{Y}{X(X+Y)}\right)^{r+1} |(X+Y)^{-1}\right) = c_r E\left(\left(\frac{Y}{X(X+Y)}\right)^{r} |(X+Y)^{-1}\right),$$

which in turns implies

$$E(X^{-r-1}Y^{r+1}\mathrm{e}^{s(X+Y)}) = c_r E(X^{-r+1}Y^r\mathrm{e}^{s(X+Y)} + X^{-r}Y^{r+1}\mathrm{e}^{s(X+Y)}), \quad s \leqslant 0. \tag{9}$$

In view of (5), (7) and the independence of X and Y, (9) can be rewritten as

$$f_X(s)h'_Y(s) = c_r f''_X(s)h_Y(s) + c_r f'_X(s)h'_Y(s), \quad s \leqslant 0. \tag{10}$$

Since $Y \sim \Gamma(p, a)$, substituting (8) into (10) we obtain

$$(a - s)f''_X(s) + (p + r)f'_X(s) - bf_X(s) = 0, \quad s \leqslant 0, \tag{11}$$

where $b = (p + r)/c_r > 0$.

Now let F_X denote the distribution function of X, W be a random variable having a distribution function G with

$$G(x) = \int_0^x \eta u^{-r}\, \mathrm{d}F_X(u),\ x \geqslant 0, \quad G(x) = 0,\ x < 0, \tag{12}$$

where $\eta^{-1} = E(X^{-r}) < \infty$, and define the following transform:

$$k_W(s) = E(W^{-1}\mathrm{e}^{sW}). \tag{13}$$

Then (5), (12) and (13) together yield

$$k_W(s) = \eta f_X(s). \tag{14}$$

Substituting this into (11), yields

$$(a - s)k''_W(s) + (p + r)k'_W(s) - bk_W(s) = 0. \tag{15}$$

Since (15) is exactly the same as (4) of SW, where the complete solution is given, we obtain immediately that $W \sim \mu_{-(p+r),a,b}$. This together with (2) and (12) yield that $X \sim \mu_{-p,a,b}$ and the theorem follows. □

3. Characterization of gamma distribution given that X is GIG distributed

In Theorem 1, knowing the distribution of Y can characterize the distribution of X. Alternatively, in this section we characterize the distribution of Y when the distribution of X is known.

The following theorem generalizes Theorems 2 and 4 of SW.

Theorem 2. *Let $X \sim \mu_{-p,a,b}$, with $p, a, b > 0$. Assume that for some fixed $r > -p, E(Y^r), E(Y^{r+1}) < \infty$, and (3) holds with $c_r = (p + r)/b > 0$. Then $Y \sim \Gamma(p, a)$.*

Proof. First define W as in Theorem 1. From (2) and (12) it can be seen that $W \sim \mu_{-(p+r),a,b}$. Next let F_Y denote the distribution function of Y, Z be a random variable having a distribution function H with

$$H(y) = \int_0^y \zeta u^r\, \mathrm{d}F_Y(u),\ y \geqslant 0, \quad H(y) = 0, y < 0, \tag{16}$$

where $\zeta^{-1} = E(Y^r) < \infty$, and define the Laplace transform

$$l_Z(s) = E(\mathrm{e}^{sZ}). \tag{17}$$

Then (7), (16) and (17) together yield

$$l_Z(s) = \zeta h_Y(s). \tag{18}$$

Since (3) is equivalent to (10), substituting (14) and (18) into (10) we obtain

$$k_W(s) l_Z'(s) = c_r k_W''(s) l_Z(s) + c_r k_W'(s) l_Z'(s), \quad s \leqslant 0, \tag{19}$$

where $c_r = (p+r)/b$. Now (19) is exactly the same as Eq. (3) of SW. Note that the role of W and Z in (19) correspond to that of X and Y, respectively, in (3) of SW. From Theorem 2 of SW we obtain immediately that $Z \sim \Gamma(p+r, a)$. This together with (16) yield that $Y \sim \Gamma(p, a)$ and the theorem follows. □

4. Simultaneous characterization of GIG and gamma distributions given two conditional expectations

In this section, we give a simultaneous characterization of the distributions of X and Y. The following theorem extends Theorem 1 of Wesolowski (2002).

Theorem 3. *Assume that $E(X^{-r-2}), E(X^{-r}), E(Y^r)$ and $E(Y^{r+2}) < \infty$ for some fixed r. If (3) and (4) hold for some constants c_r and c_{r+1}, then*

(i) $c_{r+1} > c_r > 0$;
(ii) *there exists $a > 0$ such that $X \sim \mu_{-p,a,b}$ and $Y \sim \Gamma(p, a)$, where $p = c_r/(c_{r+1} - c_r) - r > 0$ and $b = 1/(c_{r+1} - c_r) > 0$.*

Proof. First (3) and (4) yield that $c_{r+1}, c_r > 0$ immediately. Next from (4) we have

$$E\left(\left(\frac{1}{X} - \frac{1}{X+Y}\right)\left(\frac{Y}{X}\right)^{r+1} |(X+Y)^{-1}\right) = c_{r+1} E\left(\left(\frac{Y}{X}\right)^{r+1} |(X+Y)^{-1}\right), \tag{20}$$

which in turn implies

$$E\left(\left(\frac{1}{X} - c_{r+1}\right)\left(\frac{Y}{X}\right)^{r+1} |X+Y\right) = E\left(\frac{1}{X+Y}\left(\frac{Y}{X}\right)^{r+1} |X+Y\right). \tag{21}$$

On the other hand, from (3) we have

$$c_r E\left(\frac{X(X+Y)}{Y}\left(\frac{Y}{X}\right)^{r+1} |X+Y\right) = E\left(\left(\frac{Y}{X}\right)^{r+1} |X+Y\right),$$

hence

$$c_r E\left(\frac{X}{Y}\left(\frac{Y}{X}\right)^{r+1} |X+Y\right) = E\left(\frac{1}{X+Y}\left(\frac{Y}{X}\right)^{r+1} |X+Y\right). \tag{22}$$

Eqs. (21) and (22) yield

$$E\left(\left(\frac{1}{X}-c_{r+1}\right)\left(\frac{Y}{X}\right)^{r+1}|X+Y\right)=c_r E\left(\frac{X}{Y}\left(\frac{Y}{X}\right)^{r+1}|X+Y\right). \tag{23}$$

Furthermore (20) implies

$$E\left(\frac{Y}{X}\left(\frac{Y}{X}\right)^{r+1}|X+Y\right)=c_{r+1}E\left((X+Y)\left(\frac{Y}{X}\right)^{r+1}|X+Y\right),$$

hence

$$E\left(Y\left(\frac{1}{X}-c_{r+1}\right)\left(\frac{Y}{X}\right)^{r+1}|X+Y\right)=c_{r+1}E\left(X\left(\frac{Y}{X}\right)^{r+1}|X+Y\right). \tag{24}$$

From (23) and (24) we have

$$E\left(\left(\frac{1}{X}-c_{r+1}\right)\left(\frac{Y}{X}\right)^{r+1}\mathrm{e}^{s(X+Y)}\right)=c_r E\left(\frac{X}{Y}\left(\frac{Y}{X}\right)^{r+1}\mathrm{e}^{s(X+Y)}\right) \tag{25}$$

and

$$E\left(Y\left(\frac{1}{X}-c_{r+1}\right)=\left(\frac{Y}{X}\right)^{r+1}\mathrm{e}^{s(X+Y)}\right)=c_{r+1}E\left(X\left(\frac{Y}{X}\right)^{r+1}\mathrm{e}^{s(X+Y)}\right),\quad s\leqslant 0. \tag{26}$$

In view of (6) and (7), and independence of X and Y, (25) and (26) can be rewritten as

$$h'_Y(s)(g_X(s)-c_{r+1}g'_X(s))=c_r g''_X(s)h_Y(s), \tag{27}$$

and

$$h''_Y(s)(g_X(s)-c_{r+1}g'_X(s))=c_{r+1}g''_X(s)h'_Y(s). \tag{28}$$

Now (27) and (28) together imply

$$\frac{h''_Y(s)}{h'_Y(s)}=\frac{c_{r+1}}{c_r}\frac{h'_Y(s)}{h_Y(s)}. \tag{29}$$

Define Z as in Theorem 2. Substituting (18) into (29) yields

$$\frac{l''_Z(s)}{l'_Z(s)}=\frac{c_{r+1}}{c_r}\frac{l'_Z(s)}{l_Z(s)}. \tag{30}$$

Solving (30) we obtain $c_{r+1}>c_r$ and

$$l_Z(s)=(1-s/a)^{(1-c_{r+1}/c_r)^{-1}}, \tag{31}$$

for some positive constant a. Upon substituting this into (18) yields

$$h_Y(s)=\zeta^{-1}(1-s/a)^{(1-c_{r+1}/c_r)^{-1}}. \tag{32}$$

From (31) we have $Z\sim\Gamma(c_r/(c_{r+1}-c_r),a)$. Once again this together with (16) yield that $Y\sim\Gamma(p,a)$, where $p=c_r/(c_{r+1}-c_r)-r>0$. Substituting (32) into (27), we arrive at

$$(a-s)g''_X(s)+(p+r)g'_X(s)-bg_X(s)=0, \tag{33}$$

where $b = 1/(c_{r+1} - c_r)$. Let F_X denote the distribution function of X, I be a random variable having a distribution function J with

$$J(x) = \int_0^x \tau u^{-r-1}\, \mathrm{d}F_X(u),\ x \geqslant 0, \quad J(x) = 0,\ x < 0, \tag{34}$$

where $\tau^{-1} = E(X^{-r-1}) < \infty$, and define the transform

$$k_I(s) = E(I^{-1} \mathrm{e}^{sI}). \tag{35}$$

Then (6), (34) and (35) together imply $k_I(s) = \tau g_X(s)$. Substituting this into (33) yields

$$(a - s)k_I''(s) + (p + r)k_I'(s) - bk_I(s) = 0. \tag{36}$$

Again, (36) is exactly the same as Eq. (4) of SW. We obtain immediately that $I \sim \mu_{-(p+r),a,b}$. This together with (2) and (34) lead to $X \sim \mu_{-p,a,b}$ and the theorem follows. □

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