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Journal of Statistical Planning and Inference ■■■ (■■■■) ■■■–■■■

journal of statistical planning and inference

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A note on characterizations of the bivariate gamma distribution☆

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Received 28 August 2002; accepted 6 September 2003

Abstract

Given two independent non-degenerate positive random variables X and Y, Lukacs (1955) proved that $X/(X+Y)$ and $X+Y$ are independent if and only if X and Y are gammally distributed with the same scale parameter.

In this work, properties of bivariate gamma distribution are studied. Certain regression version of Lukacs's theorem are given for the bivariate case. Furthermore, characterization of bivariate gamma distribution by the conditions of constancy regression of quadratic statistics is also given.

MSC: primary 62E10

Keywords: Bivariate gamma distribution; Characterization; Constancy of regression; Laplace transform

1. Introduction

Let X and Y be two independent non-degenerate positive random variables. Lukacs (1955) proved that $X/(X+Y)$ and $X+Y$ are independent if and only if X and Y are gammally distributed with the same scale parameter. However, in the bivariate case such a property does not hold in general. Note that a positive random vector $\bar{X}=(X_1,X_2)$ has a bivariate gamma distribution $\mathrm{BG}(p,\bar{\lambda})$ (denote it by $\bar{X}\sim\mathrm{BG}(p,\bar{\lambda})$), with shape parameter p and scale parameter $\bar{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$, if it has the Laplace transform of

☆ Support for this research was provided in part by the National Science Council of the Republic of China, Grant No. NSC 90-2118-M390-001.

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0378-3758/$ - see front matter
doi:10.1016/j.jspi.2003.09.034

the form,

$$E(\exp(s_1X_1 + s_2X_2)) = (1 - \lambda_1 s_1 - \lambda_2 s_2 + \lambda_3 s_1 s_2)^{-p}, \quad s_1, s_2 \leqslant 0,$$

$$\lambda_1 s_1 + \lambda_2 s_2 - \lambda_3 s_1 s_2 < 1,$$

where $p, \lambda_1, \lambda_2 > 0$ and $\lambda_1\lambda_2 \geqslant \lambda_3 \geqslant 0$. The case $\lambda_1\lambda_2 = \lambda_3$ and $\lambda_3 = 0$ corresponds to the condition that X_1, X_2 are independent and $P(X_2 = (\lambda_2/\lambda_1)X_1) = 1$, respectively.

Let $\bar{X} = (X_1, X_2)$ and $\bar{Y} = (Y_1, Y_2)$ be independent non-degenerate positive random vectors.

Bobecka (2002) gave a bivariate version of Lukacs theorem by showing that $(X_1/(X_1 + Y_1), X_2/(X_2 + Y_2))$ and $(X_1 + Y_1, X_2 + Y_2)$ are independent, if and only if $\bar{X} \sim \mathrm{BG}(p, \bar{\lambda})$, $\bar{Y} \sim \mathrm{BG}(q, \bar{\lambda})$ with $P(X_2 = (\lambda_2/\lambda_1)X_1) = P(Y_2 = (\lambda_2/\lambda_1)Y_1) = 1$, or $\bar{X}, \bar{Y}$ have independent gamma components. On the other hand, when $\bar{X} \sim \mathrm{BG}(p, \bar{\lambda})$ and $\bar{Y} \sim \mathrm{BG}(q, \bar{\lambda})$, for $r = 1, 2, -1$, or -2, if $r > -p$, Bobecka (2002) proved that for some constants c_r:

$$E\left(\left(\frac{X_j}{X_j + Y_j}\right)^r \middle| \bar{X} + \bar{Y}\right) = c_r, \quad j = 1, 2. \tag{1}$$

Conversely, for $(u, v) = (1, 2)$, $(1, -1)$ or $(-1, -2)$, under the assumptions,

$$E\left(\left(\frac{X_j}{X_j + Y_j}\right)^u \middle| \bar{X} + \bar{Y}\right) = a_j, \tag{2}$$

$$E\left(\left(\frac{X_j}{X_j + Y_j}\right)^v \middle| \bar{X} + \bar{Y}\right) = b_j, \tag{3}$$

hold for some constants a_j and b_j, $j = 1, 2$. Bobecka (2002) characterized $\bar{X}$ and $\bar{Y}$ to be bivariate gammally distributed with the same scale parameter. This generalized the results of Bolger and Harkness (1965), Wesolowski (1990) and Li et al. (1994), where univariate cases were considered. Instead of (2) and (3), under the following weaker assumptions, a characterization of bivariate gamma distribution in a much more general form is given: For some fixed integer r,

$$E\left(\left(\frac{X_j}{X_j + Y_j}\right)^{r+1} \middle| \bar{X} + \bar{Y}\right) = \alpha_j E\left(\left(\frac{X_j}{X_j + Y_j}\right)^{r} \middle| \bar{X} + \bar{Y}\right), \tag{4}$$

$$E\left(\left(\frac{X_j}{X_j + Y_j}\right)^{r+2} \middle| \bar{X} + \bar{Y}\right) = \beta_j E\left(\left(\frac{X_j}{X_j + Y_j}\right)^{r+1} \middle| \bar{X} + \bar{Y}\right), \tag{5}$$

hold for some constants α_j, β_j, $j = 1, 2$. Note that (1) holds for every integer $r > -p$, and $(u, v) = (1, 2)$, $(1, -1)$ and $(-1, -2)$ in (2) and (3) corresponds to $r = 0, -1$ and -2, respectively. This also extends the result of Huang and Su (1997) in the univariate case.

In Bobecka (2002), the following two equations has also been used to characterize bivariate gamma distribution:

$$E\left(\left(\frac{X_j}{X_j + Y_j}\right)^2 \middle| \bar{X} + \bar{Y}\right) = d_j, \tag{6}$$

and

$$E\left(\left(\frac{Y_j}{X_j+Y_j}\right)^2\middle|\bar{X}+\bar{Y}\right)=e_j, \tag{7}$$

hold for some constants d_j and e_j, $j=1,2$. Note that under the next two equations with conditions of constancy of quadratic regression

$$E(a_jX_j^2+b_jX_jY_j+c_jY_j^2|\bar{X}+\bar{Y})=0, \tag{8}$$

and

$$E(d_jX_j^2+e_jX_jY_j+f_jY_j^2|\bar{X}+\bar{Y})=0, \tag{9}$$

hold for some constants a_j, b_j, c_j, d_j, e_j and f_j, with vectors (a_j,b_j,c_j) linearly independent of (d_j,e_j,f_j), $j=1,2$, it can also characterize bivariate gamma distribution. It is interesting to find that although the forms in (8) and (9) seem to be more general, they are actually equivalent to (6) and (7).

The case that $\bar{X}_i=(X_{i1},X_{i2})$, $i=1,2,\ldots,n$, are independent and identically distributed (i.i.d.) has also been considered by some authors. For example, Huang and Hu (1999, 2000) and Theorems 6.2.8 and 6.2.9 of Kagan et al. (1973) characterized univariate gamma distribution, while Theorem 3 of Pusz (2002) characterized bivariate gamma distribution using the following assumptions:

$$E\left(\sum_{i=1}^n\sum_{k=1}^n a_{ik}X_{i1}X_{k1}+\sum_{i=1}^n b_iX_{i1}\middle|\sum_{i=1}^n\bar{X}_i\right)=0, \tag{10}$$

and

$$E\left(\sum_{i=1}^n\sum_{k=1}^n c_{ik}X_{i2}X_{k2}+\sum_{i=1}^n d_iX_{i2}\middle|\sum_{i=1}^n\bar{X}_i\right)=0, \tag{11}$$

with $a_{ik}=c_{ik}$, $b_i=d_i$ for all $i,k=1,\ldots,n$. In Section 5, without assuming that $a_{ik}=c_{ik}$, $b_i=d_i$, $i,k=1,\ldots,n$, we give a similar characterization.

2. Preliminaries

In this section, we list three lemmas which will be used in proving our theorems, where Lemma 1 is due to Kagan et al. (1973) and Lemma 3 is due to Bobecka (2002).

Lemma 1. *Let U be a positive random variable and $\bar{V}=(V_1,\ldots,V_n)$ a random vector with positive components. Suppose that $E(U)$ and $E(\bar{V})$exist. Then U has a linear regression on $\bar{V}$,*

$$E(U|\bar{V})=\alpha+\sum_{i=1}^n\beta_iV_i,$$

if and only if the relation

$$E\left(U\exp\left(\sum_{i=1}^{n}s_iV_i\right)\right)=\alpha E\left(\exp\left(\sum_{i=1}^{n}s_iV_i\right)\right)+\sum_{i=1}^{n}\beta_iE\left(V_i\exp\left(\sum_{i=1}^{n}s_iV_i\right)\right),$$

holds for all vectors $\bar{s}=(s_1,\dots,s_n)$, $s_i\leqslant 0$, $i=1,\dots,n$, *where* α,β_i, $i=1,\dots,n$, *are constants.*

Lemma 2. *Let* $\bar{X}=(X_1,X_2)$ *be a non-degenerate positive random vector, with* $E(X_1^r)<\infty$ *for some integer* $r\neq 0$. *Assume that for some* $l<0$,

$$E(X_1^r\exp(s_1X_1+s_2X_2))=B_r(s_2)(1+A(s_2)s_1)^l,$$

for every $s_1,s_2\leqslant 0$, *where* $B_r(s_2)>0$ *and* $A(s_2)<0$ *for every* $s_2\leqslant 0$. *Then*

(i) $l<-r$,
(ii) $E(X_1^i\exp(s_1X_1+s_2X_2))=B_i(s_2)(1+A(s_2)s_1)^{l+r-i}$, *where*

$$B_i(s_2)=\begin{cases}[(l+1)(l+2)\cdots(l+r-i)]^{-1} & \\ \quad\times(A(s_2))^{i-r}B_r(s_2), & i=0,1,\dots,r-1, \\ & r>0, \\ l(l-1)\cdots(l+r-i+1) & \\ \quad\times(A(s_2))^{i-r}B_r(s_2), & i=r+1,r+2,\dots,0, \\ & r<0.\end{cases}$$

Proof. First as $E(X_1^r)<\infty$, for every $s_1,s_2\leqslant 0$, $E(X_1^i\exp(s_1X_1+s_2X_2))$ exists, for $0\leqslant|i|\leqslant|r|$, $ir\geqslant 0$. For convenience, denote $E(X_1^i\exp(s_1X_1+s_2X_2))$ by $H_i(s_1,s_2)$ if it exists. Hence $0<H_i(s_1,s_2)<\infty$, $\forall 0\leqslant|i|\leqslant|r|$, $ir\geqslant 0$, since $X_1>0$. Throughout this proof, we only consider $s_1,s_2\leqslant 0$.

The proof will be divided into the following two cases.

Case 1: $r>0$.

From the assumption, we have

$$H_r(s_1,s_2)=B_r(s_2)(1+A(s_2)s_1)^l,$$

and in view of the definition of $H_i(s_1,s_2)$, it yields

$$\begin{aligned}H_{r-1}(s_1,s_2)&=\int_{-\infty}^{s_1}H_r(t,s_2)\,\mathrm{d}t\\&=\begin{cases}(A(s_2))^{-1}B_r(s_2)\log(1+A(s_2)t)|_{-\infty}^{s_1}, & l=-1,\\(l+1)^{-1}(A(s_2))^{-1}B_r(s_2)(1+A(s_2)t)^{l+1}|_{-\infty}^{s_1}, & l\neq-1.\end{cases}\end{aligned}$$

We conclude immediately (recall that $B_r(s_2)>0$, $A(s_2)<0$ and $l<0$) that $l<-1$, otherwise $H_{r-1}(s_1,s_2)$ cannot be finite. Consequently

$$H_{r-1}(s_1,s_2)=(l+1)^{-1}(A(s_2))^{-1}B_r(s_2)(1+A(s_2)s_1)^{l+1}.$$

It follows

$$H_{r-1}(s_1,s_2)=B_{r-1}(s_2)(1+A(s_2)s_1)^{l+1},$$

where

$$B_{r-1}(s_2)=(l+1)^{-1}(A(s_2))^{-1}B_r(s_2)>0.$$

Along the lines of the above discussion, we obtain $l<-r$, and

$$H_i(s_1,s_2)=B_i(s_2)(1+A(s_2)s_1)^{l+r-i}, \quad i=0,1,\ldots,r-1,$$

where

$$B_i(s_2)=[(l+1)(l+2)\cdots(l+r-i)]^{-1}(A(s_2))^{i-r}B_r(s_2)>0.$$

This proves Case 1.

Case 2: $r<0$.

Obviously, assertion (i) holds in this case. Next, we have

$$H_{r+1}(s_1,s_2)=\frac{\partial H_r(s_1,s_2)}{\partial s_1}=lA(s_2)B_r(s_2)(1+A(s_2)s_1)^{l-1}.$$

It follows

$$H_{r+1}(s_1,s_2)=B_{r+1}(s_2)(1+A(s_2)s_1)^{l-1},$$

where

$$B_{r+1}(s_2)=lA(s_2)B_r(s_2)>0.$$

As in Case 1, it yields

$$H_i(s_1,s_2)=B_i(s_2)(1+A_2(s_2)s_1)^{l+r-i}, \quad i=r+1,r+2,\ldots,-1,0,$$

where

$$B_i(s_2)=l(l-1)\cdots(l+r-i+1)(A(s_2))^{i-r}B_r(s_2)>0.$$

The proof is completed. □

Lemma 3. *Let M_1, N_1, M_2 and N_2 be real functions defined on $(-\infty,0]$, where N_1, N_2 are non-positive and $M_1(0)\neq 0$. Suppose that for some $t_1,t_2>0$,*

$$M_1(s_1)(1+N_1(s_1)s_2)^{t_1}=M_2(s_2)(1+N_2(s_2)s_1)^{t_2}, \tag{12}$$

holds for every $s_1,s_2\leqslant 0$. Denote $M_1(0)$, $N_1(0)$, $M_2(0)$ and $N_2(0)$ by m_1,n_1,m_2 and n_2, respectively. Then $m_1=m_2=m$, $M_1(s_1)=m(1+n_2s_1)^{t_2}$, $M_2(s_2)=m(1+n_1s_2)^{t_1}$, $\forall s_1,s_2\leqslant 0$, and there are two possible cases: either

(i) *$t_1=t_2$, then*

$$N_1(s_1)=\frac{fs_1}{1+n_2s_1}+n_1, \quad N_2(s_2)=\frac{fs_2}{1+n_1s_2}+n_2, \ \forall s_1,s_2\leqslant 0,$$

where f is a constant; or

(ii) *$t_1\neq t_2$, then $N_1(s_1)=n_1$, $N_2(s_2)=n_2$, $\forall s_1,s_2\leqslant 0$.*

3. Constant regression property of bivariate gamma distribution

In this section, we prove a constant regression property of bivariate gamma distribution.

Proposition 1. *Let $\bar{X} = (X_1, X_2) \sim \mathrm{BG}(p, \bar{\lambda})$ and $\bar{Y} = (Y_1, Y_2) \sim \mathrm{BG}(q, \bar{\lambda})$ be independent. Then (1) holds for all integers $r > -p$, where $c_0 = 1$, and*

$$c_r = \begin{cases} \dfrac{p(p+1)\cdots(p+r-1)}{(p+q)(p+q+1)\cdots(p+q+r-1)}, & r > 0, \\ \dfrac{(p+q-1)(p+q-2)\cdots(p+q+r)}{(p-1)(p-2)\cdots(p+r)}, & -p < r < 0. \end{cases}$$

Proof. The case for $r = 0$ is obvious. We now prove the case for $r \neq 0$. Also by symmetry, we only need to prove for every non-zero integer $r > -p$,

$$E\left(\left(\frac{X_1}{X_1 + Y_1} \right)^r \middle| \bar{X} + \bar{Y} \right) = c_r. \tag{13}$$

First note that both $E(\bar{X} + \bar{Y})$ and $E(X_1/(X_1 + Y_1))^r$ exist for every integer $r > 0$. By using the Binomial Theorem and the independence of X_1 and Y_1, we have

$$E\left(\frac{X_1}{X_1 + Y_1} \right)^r = E\left(1 + \frac{Y_1}{X_1} \right)^{-r} = \sum_{i=0}^{-r} C_i^{-r} (EY_1^i)(EX_1^{-i}), \quad -p < r < 0.$$

Here every term in the right side is finite as for a random variable W having a $\Gamma(\alpha, \beta)$ distribution, $\alpha, \beta > 0$, with density function

$$\gamma_{\alpha,\beta}(\mathrm{d}x) = \frac{x^{\alpha-1} \mathrm{e}^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} I_{(0,\infty)}(x)\, \mathrm{d}x,$$

$E(W^k)$ exists, for every $k > -\alpha$. Thus $E(X_1/(X_1 + Y_1))^r$ exists for every integer $r > -p$ and Lemma 1 can be applied (with $U = (X_1/(X_1 + Y_1))^r$ and $\bar{V} = \bar{X} + \bar{Y}$).

By Lemma 1 and the independence of $\bar{X}$ and $\bar{Y}$, Eq. (13) is equivalent to

$$\begin{aligned} &E\left(\left(\frac{X_1}{X_1 + Y_1} \right)^r \exp(s_1 X_1 + s_2 X_2 + s_1 Y_1 + s_2 Y_2) \right) \\ &\quad = c_r E(\exp(s_1 X_1 + s_2 X_2)) E(\exp(s_1 Y_1 + s_2 Y_2)), \quad \forall r > -p, s_1, s_2 \leqslant 0. \end{aligned} \tag{14}$$

Since $\bar{X} \sim \mathrm{BG}(p, \bar{\lambda})$ and $\bar{Y} \sim \mathrm{BG}(q, \bar{\lambda})$, we have

$$E(\exp(s_1 X_1 + s_2 X_2)) = (1 - \lambda_1 s_1 - \lambda_2 s_2 + \lambda_3 s_1 s_2)^{-p} \tag{15}$$

and

$$E(\exp(s_1 Y_1 + s_2 Y_2)) = (1 - \lambda_1 s_1 - \lambda_2 s_2 + \lambda_3 s_1 s_2)^{-q}, \quad \forall s_1, s_2 \leqslant 0. \tag{16}$$

Substituting (15) and (16) in (14) it follows that (13) is equivalent to

$$E\left(\left(\frac{X_1}{X_1+Y_1}\right)^r \exp(s_1X_1+s_2X_2+s_1Y_1+s_2Y_2)\right)$$
$$=c_r(1-\lambda_1 s_1-\lambda_2 s_2+\lambda_3 s_1 s_2)^{-(p+q)}, \quad \forall r>-p, s_1, s_2 \leqslant 0. \tag{17}$$

Note that $E(X_1^i \exp(s_1X_1+s_2X_2)) \leqslant E(X_1^i) < \infty$, $\forall i > -p, s_1, s_2 \leqslant 0$, we have

$$E(X_1^r \exp(s_1X_1+s_2X_2))$$
$$=\begin{cases} \dfrac{\partial^r}{\partial s_1^r} E(\exp(s_1X_1+s_2X_2)), & r>0, \\ \displaystyle\int_{-\infty}^{s_1}\int_{-\infty}^{t_{-\tau}}\cdots\int_{-\infty}^{t_2} E(\exp(t_1X_1+s_2X_2))\,\mathrm{d}t_1\cdots\mathrm{d}t_{-r-1}\,\mathrm{d}t_{-r}, & -p<r<0, \end{cases}$$

$\forall s_1, s_2 \leqslant 0$. This together with (15) and (16) imply

$$E(X_1^r \exp(s_1X_1+s_2X_2+s_1Y_1+s_2Y_2))$$
$$=\begin{cases} p(p+1)\cdots(p+r-1)(\lambda_1-\lambda_3 s_2)^r \\ \quad \times(1-\lambda_1 s_1-\lambda_2 s_2+\lambda_3 s_1 s_2)^{-(p+q+r)}, & r>0, \\ ((p-1)(p-2)\cdots(p+r))^{-1}(\lambda_1-\lambda_3 s_2)^r \\ \quad \times(1-\lambda_1 s_1-\lambda_2 s_2+\lambda_3 s_1 s_2)^{-(p+q+r)}, & -p<r<0, \end{cases}$$

$\forall s_1, s_2 \leqslant 0$. Again by using the Binomial Theorem and the independence of X_1 and Y_1, we have for $s_1, s_2 \leqslant 0$, $0 \leqslant |i| \leqslant |r|$ and $ir \geqslant 0$,

$$E\left(\frac{X_1^r}{(X_1+Y_1)^i}\exp(s_1X_1+s_2X_2+s_1Y_1+s_2Y_2)\right) \leqslant E\left(\frac{X_1^r}{(X_1+Y_1)^i}\right) < \infty.$$

Using a similar argument, we obtain for every $s_1, s_2 \leqslant 0$,

$$E\left(\left(\frac{X_1}{X_1+Y_1}\right)^r \exp(s_1X_1+s_2X_2+s_1Y_1+s_2Y_2)\right)$$
$$=E(X_1^r(X_1+Y_1)^{-r}\exp(s_1X_1+s_2X_2+s_1Y_1+s_2Y_2))$$
$$=\begin{cases} \displaystyle\int_{-\infty}^{s_1}\int_{-\infty}^{t_\tau}\cdots\int_{-\infty}^{t_2} E(X_1^r\exp(t_1(X_1+Y_1) \\ \quad +s_2(X_2+Y_2))\,\mathrm{d}t_1\cdots\mathrm{d}t_{r-1}\,\mathrm{d}t_r, & r>0, \\ \dfrac{\partial^{(-r)}}{\partial s_1^{(-r)}} E(X_1^r\exp(s_1X_1+s_2X_2+s_1Y_1+s_2Y_2)), & -p<r<0. \end{cases}$$

This in turn implies for every $s_1, s_2 \leqslant 0$,

$$E\left(\left(\frac{X_1}{X_1+Y_1}\right)^r \exp(s_1X_1+s_2X_2+s_1Y_1+s_2Y_2)\right)$$

$$=\begin{cases} \dfrac{p(p+1)\cdots(p+r-1)}{(p+q)(p+q+1)\cdots(p+q+r-1)} \\ \quad\times(1-\lambda_1s_1-\lambda_2s_2+\lambda_3s_1s_2)^{-(p+q)}, & r>0, \\ \dfrac{(p+q-1)(p+q-2)\cdots(p+q+r)}{(p-1)(p-2)\cdots(p+r)} \\ \quad\times(1-\lambda_1s_1-\lambda_2s_2+\lambda_3s_1s_2)^{-(p+q)}, & -p<r<0. \end{cases}$$

This proves (17) and the proposition follows.

4. Characterizations of bivariate gamma distribution by conditional moments

In Section 3, we have proved a constant regression property of bivariate gamma random vectors. Next we will investigate whether this constant regression property can characterize bivariate gamma distribution. Theorem 1 extends Theorems 7, 8, and 9 of Bobecka (2002) simultaneously.

Theorem 1. *Let* $\bar{X}=(X_1,X_2)$ *and* $\bar{Y}=(Y_1,Y_2)$ *be independent non-degenerate bivariate positive random vectors such that for some fixed integer* r, $E(Y_j)<\infty$, $E(X_j^r)<\infty$, $E(X_j^{r+2})<\infty$, $j=1,2$, *and the conditions* (4), (5) *hold for some real* α_j, β_j, $j=1,2$. *Let*

$$p_j=\frac{\alpha_j-\beta_j\alpha_j}{\beta_j-\alpha_j}-r, \quad q_j=\frac{\alpha_j-\beta_j\alpha_j}{\beta_j-\alpha_j}\,\frac{1-\alpha_j}{\alpha_j}, \quad j=1,2.$$

Then p_j, q_j *are well defined,* $p_j>0$, $q_j>0$, *and there are two possible cases: either*

(i) $\alpha_1=\alpha_2$ *and* $\beta_1=\beta_2$ *and then* $\bar{X},\bar{Y}$ *have bivariate gamma distributions* $\mathrm{BG}(p,\bar{\lambda})$, $\mathrm{BG}(q,\bar{\lambda})$, *where* $p=p_j$, $q=q_j$, $j=1,2$, *or*
(ii) $\alpha_1\neq\alpha_2$ *or* $\beta_1\neq\beta_2$ *and then* $\bar{X},\bar{Y}$ *have independent gamma components* $X_j\sim\gamma_{\lambda_j,p_j}$, $Y_j\sim\gamma_{\lambda_j,q_j}$, $j=1,2$.

Proof. Assumptions (4) and (5) imply that for $s_1,s_2\leqslant 0$,

$$\begin{aligned} &E(X_j^{r+1}\exp(s_1X_1+s_2X_2)\exp(s_1Y_1+s_2Y_2)) \\ &\quad=\alpha_jE(X_j^r(X_j+Y_j)\exp(s_1X_1+s_2X_2)\exp(s_1Y_1+s_2Y_2)), \end{aligned} \tag{18}$$

and

$$\begin{aligned} &E(X_j^{r+2}\exp(s_1X_1+s_2X_2)\exp(s_1Y_1+s_2Y_2)) \\ &\quad=\beta_jE(X_j^{r+1}(X_j+Y_j)\exp(s_1X_1+s_2X_2)\exp(s_1Y_1+s_2Y_2)), \quad j=1,2. \end{aligned} \tag{19}$$

Let

$$F_j(s_1,s_2) = E(X_j^r \exp(s_1X_1 + s_2X_2)), \quad j = 1,2,$$

$$F_Y(s_1,s_2) = E(\exp(s_1Y_1 + s_2Y_2)),$$

and

$$F_X(s_1,s_2) = E(\exp(s_1X_1 + s_2X_2)), \quad s_1,s_2 \leqslant 0.$$

Using the independence of $\bar{X}$ and $\bar{Y}$ and in view of the definition of $F_j(s_1,s_2)$ and $F_Y(s_1,s_2)$, after some routine computations, (18) and (19) imply

$$\frac{1-\alpha_j}{\alpha_j}\frac{\partial F_j(s_1,s_2)}{\partial s_j}F_Y(s_1,s_2) = F_j(s_1,s_2)\frac{\partial F_Y(s_1,s_2)}{\partial s_j}, \tag{20}$$

and

$$\frac{1-\beta_j}{\beta_j}\frac{\partial^2 F_j(s_1,s_2)}{\partial s_j^2}F_Y(s_1,s_2) = \frac{\partial F_j(s_1,s_2)}{\partial s_j}\frac{\partial F_Y(s_1,s_2)}{\partial s_j}, \quad j = 1,2. \tag{21}$$

For $j = 1$, solving (20) and (21) yield

$$F_1(s_1,s_2) = C_1(s_2)(1 + C(s_2)s_1)^{(\alpha_1-\beta_1\alpha_1)/(\alpha_1-\beta_1)}, \tag{22}$$

and

$$F_Y(s_1,s_2) = C_2(s_2)(1 + C(s_2)s_1)^{(\alpha_1-\beta_1\alpha_1)/(\alpha_1-\beta_1)(1-\alpha_1)/\alpha_1}, \tag{23}$$

for some functions $C_1(s_2), C_2(s_2) > 0$ and $C(s_2) < 0$. In view of Lemma 2 and (22), it turns out that

$$F_X(s_1,s_2) = C_3(s_2)(1 + C(s_2)s_1)^{(\alpha_1-\beta_1\alpha_1)/(\alpha_1-\beta_1)+r}, \tag{24}$$

for some functions $C_3(s_2) > 0$, where $(\alpha_1 - \beta_1\alpha_1)/(\alpha_1 - \beta_1) + r < 0$. Denote $(\alpha_1 - \beta_1\alpha_1)/(\beta_1 - \alpha_1)(1 - \alpha_1)/\alpha_1$ by $q_1 > 0$, and $(\alpha_1 - \beta_1\alpha_1)/(\beta_1 - \alpha_1) - r$ by $p_1 > 0$, respectively. We rewrite (23) and (24) as

$$F_Y(s_1,s_2) = C_2(s_2)(1 + C(s_2)s_1)^{-q_1}, \tag{25}$$

and

$$F_X(s_1,s_2) = C_3(s_2)(1 + C(s_2)s_1)^{-p_1}. \tag{26}$$

On the other hand, for $j=2$ in (20) and (21), along the lines of the above arguments it yields for some functions $D_2(s_1), D_3(s_1) > 0$, $D(s_1) < 0$,

$$F_Y(s_1,s_2) = D_2(s_1)(1 + D(s_1)s_2)^{-q_2}, \tag{27}$$

and

$$F_X(s_1,s_2) = D_3(s_1)(1 + D(s_1)s_2)^{-p_2}, \tag{28}$$

where $q_2 = (\alpha_2 - \beta_2\alpha_2)/(\beta_2 - \alpha_2)(1 - \alpha_2)/(\alpha_2) > 0$ and $p_2 = (\alpha_2 - \beta_2\alpha_2)/(\beta_2 - \alpha_2) - r > 0$. Combining (25)–(28) and Lemma 3, the theorem follows immediately. □

5. Characterizations by the conditions of constancy regression of sampling quadratic statistics on sample mean

For a sequence of i.i.d. random vectors, we can characterize bivariate gamma distribution by the conditions of constancy of quadratic regression with a similar argument as in Theorem 1. We state the theorem without proof.

Theorem 2. *Let $\bar{X}_i = (X_{i1}, X_{i2})$, $i = 1, 2, \ldots, n, n \geqslant 2$, be i.i.d. non-degenerate bivariate positive random vectors. Assume that $E(X_{11}^2) < \infty$, $E(X_{12}^2) < \infty$, and (10) and (11) hold for some constants a_{ik}, b_i, c_{ik} and $d_i, i, k = 1, 2, \ldots, n$, with*

$$\begin{cases} a^2 + b^2 \neq 0, \\ c = 0, \\ d^2 + e^2 \neq 0, \\ f = 0, \end{cases} \quad \text{where} \begin{cases} a = \sum_{i=1}^{n} a_{ii}, & d = \sum_{i=1}^{n} c_{ii}, \\ b = \sum_{i \neq k} a_{ik}, & e = \sum_{i \neq k} c_{ik}, \\ c = \sum_{i=1}^{n} b_i, & f = \sum_{i=1}^{n} d_i. \end{cases}$$

Then $ab \neq 0$ and $cd \neq 0$. Furthermore, let

$$l_1 = \frac{1}{-b/a - 1}, \quad l_2 = \frac{1}{-e/d - 1}.$$

Then $l_j > 0$, $j = 1, 2$, and there are two possible cases: either

(i) *$l_1 = l_2 = l$, that is $b/a = e/d$, then $\bar{X}_1 \sim \mathrm{BG}(l, \bar{\lambda})$; or*
(ii) *$l_1 \neq l_2$, that is $b/a \neq e/d$, then $\bar{X}_1$ has independent gamma components $X_{1j} \sim \gamma_{l_j, \lambda_j}$, $j = 1, 2$.*

Remark. Case (i) of Theorem 2 is essentially given in Theorem 4.2 of Bar-Lev et al. (1994).

Next, as a special case of Theorem 2, we characterize bivariate gamma distribution by the conditions of constancy regression of sample coefficient of variation on sample mean. First let $Z_j = \sum_{i=1}^{n} X_{ij}/n$ and $S_j = \sqrt{\sum_{i=1}^{n} (X_{ij} - Z_j)^2/(n-1)}$, $j = 1, 2$.

Corollary. *Let $\bar{X}_i = (X_{i1}, X_{i2})$, $i = 1, 2, \ldots, n$, $n \geqslant 2$, be i.i.d. non-degenerate bivariate positive random vectors. Assume that $E(X_{11}^2) < \infty$, $E(X_{12}^2) < \infty$, and the conditions*

$$E\left(\left(\frac{S_j}{Z_j} \right)^2 \middle| (Z_1, Z_2) \right) = e_j,$$

hold for some constants e_j, $j = 1, 2$. Let

$$l_j = \frac{1}{e_j} - \frac{1}{n}, \quad j = 1, 2.$$

Then $l_j > 0,\ j = 1, 2$, *and there are two possible cases*: *either*

(i) $l_1 = l_2 = l$, *that is* $e_1 = e_2$, *then* $\bar{X}_1 \sim \mathrm{BG}(l, \bar{\lambda})$; *or*
(ii) $l_1 \neq l_2$, *that is* $e_1 \neq e_2$, *then* $\bar{X}_1$ *has independent gamma components* $X_{1j} \sim \gamma_{l_j,\lambda_j},\ j = 1, 2$.

Acknowledgements

We are grateful to the referee for the valuable comments and suggestions, which have greatly improved the presentation of the results. We also thank Professors Wesolowski and Pusz for sending us the preprints of Bobecka (2002) and Pusz (2002), respectively.

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