CHARACTERIZATIONS OF THE GAMMA DISTRIBUTION VIA
CONDITIONAL MOMENTS

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SUMMARY. In this work we characterize two independent non-degenerate positive
random variables $X$ and $Y$ to be gamma distributed with the same scale parameter by
the assumptions $E(X^{r+s}|X + Y) = \alpha(X + Y)E(X^r|X + Y)$ and $E(X^{r+s}|X + Y) = \beta(X + Y)E(X^{r+s}|X + Y)$ for some fixed integer $r$ and $s = 2$. Furthermore, let $A \equiv \{A(t), t \geq 0\}$ be a renewal process with $\{S_k, k \geq 1\}$ being the sequence of arrival times,
under the assumptions $E(S_k^{r+s}|A(t) = n) = atE(S_k^r|A(t) = n)$ and $E(S_k^{r+s}|A(t) = n) = btE(S_k^{r+s}|A(t) = n)$ for fixed integers $r, k, n$, where $1 \leq k \leq n$, and $s = 2$, we prove that $A$
has to be a Poisson process. In the case that $s = 1$ the above two results were proved by
Huang and Su (1997).

On the other hand, recently characterizations of gamma distribution by the so-called
dual regression schemes were investigated by Bobecka and Wesolowski (2001). More
precisely, they considered the constancy of regressions of $X$ and $Y$, while independence of
$X/(X + Y)$ and $X + Y$ is assumed instead of independence of $X$ and $Y$. They characterized
$X$ and $Y$ to be gamma distributed by the assumptions $E(Y^r|X) = c$ and $E(Y^r|X) = d$,
for $(u, v) = (1, 2), (1, -1)$ or $(1, -2)$, where $c$ and $d$ are constants. As a generalization,
we prove that $X$ and $Y$ are gamma distributed with the same scale parameter under the
assumptions $E(Y^{r+s}|X) = cE(Y^r|X)$ and $E(Y^{r+s}|X) = dE(Y^{r+s}|X)$, for some fixed
integer $r$, where $c$ and $d$ are constants. Note that $(u, v) = (1, 2), (1, -1)$ and $(1, -2)$
corresponds to $r = 0, -1$ and $-2$, respectively.

1. Introduction

Given two independent non-degenerate positive random variables $X$ and $Y$, Lukacs (1955) proved that $X/(X + Y)$ and $X + Y$ are independent

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if and only if $X$ and $Y$ are gamma distributed with the same scale parameter. Since then many papers considered different extensions. Among others, Belger and Harkness (1965), Wesolowski (1990) and Li et al. (1994) replaced the independence condition of $X/(X + Y)$ and $X + Y$ by the following regression assumptions:

$$E(X^u|X + Y) = a(X + Y)^u,$$

and

$$E(X^v|X + Y) = b(X + Y)^v,$$

where $(u, v) = (1, 2), (1, -1)$ or $(-1, -2)$. Huang and Su (1997) generalized the above results, and obtained similar characterization, under the weaker conditions:

$$E(X^{r+1}|X + Y) = a(X + Y)E(X^r|X + Y),$$

and

$$E(X^{r+s+1}|X + Y) = b(X + Y)E(X^{r+s}|X + Y),$$

with $s = 1$, $r$ being some fixed integer, and $a$ and $b$ being some constants. Note that $(u, v) = (1, 2), (1, -1)$ and $(-1, -2)$ corresponds to $r = 0, -1$ and $-2$, respectively. In this work we prove the case of $s = 2$.

There are also parallel characterizations for Poisson process. Let $A \equiv \{A(t), t \geq 0\}$ be a renewal process, with $\{S_k, k \geq 1\}$ being the sequence of arrival times, and $F$ being the common distribution function of the inter-arrival times. Li et al. (1994) characterized $A$ to be a Poisson process by the assumptions:

$$E(S^u_k|A(t) = n) = at^u,$$

and

$$E(S^v_k|A(t) = n) = bt^v,$$

for some fixed integers $k$ and $n, 1 \leq k \leq n$, and constants $a$ and $b$, where $(u, v) = (1, 2), (1, -1)$ or $(-1, -2)$. Huang and Su (1997) characterized $A$ to be Poisson under the weaker conditions:

$$E(S^{r+1}_k|A(t) = n) = atE(S^r_k|A(t) = n),$$
and

\[ E(S_k^{r+1}|A(t) = n) = bt E(S_k^r|A(t) = n), \]

with \( s = 1, r, k, n \) being some integers, where \( 1 \leq k \leq n \), and \( a, b \) being some constants. Again, we prove the case of \( s = 2 \).

On the other hand, Hall and Simons (1969) and Huang and Su (1997), characterized gamma distributions by using

\[ E(X^n|X + Y) = a(X + Y)^n, \]

and

\[ E(Y^n|X + Y) = b(X + Y)^n, \]

for \( u = 2 \) or \(-1\).

Recently, Bobecka and Wesolowski (2001) generalized the Lukacs theorem in another direction, namely, under the so-called dual regression schemes. That is, they assumed the constancy of regressions for \( X \) and \( Y \), while independence of \( U = X/(X + Y) \) and \( V = X + Y \) is assumed instead of independence of \( X \) and \( Y \). More precisely, they characterized \( X \) and \( Y \) to be gamma distributed by the assumptions:

\[ E(Y^n|X) = c, \quad \text{and} \quad E(Y^r|X) = d, \]

or

\[ E(X^n|Y) = c, \quad \text{and} \quad E(X^r|Y) = d, \]

where \((u, v) = (1, 2), (1, -1) \) or \((-1, -2)\). In Section 3, we extend the above results by using

\[ E(Y^{r+1}|X) = cE(Y^r|X), \quad \text{and} \quad E(Y^{r+2}|X) = dE(Y^{r+1}|X), \]

or

\[ E(X^{r+1}|Y) = cE(X^r|Y), \quad \text{and} \quad E(X^{r+2}|Y) = dE(X^{r+1}|Y), \]

for some fixed integer \( r \) and constants \( c \) and \( d \), to characterize gamma distributions of \( X \) and \( Y \). Again \((u, v) = (1, 2), (1, -1) \) and \((-1, -2)\) corresponds to \( r = 0, -1 \) and \(-2\), respectively.
2. Characterizations of Gamma Distribution and Poisson Process by Conditional Moments

Introduce first the notation for two distributions which play the important role in this paper. Denote by $\Gamma(a, b)$ the gamma distribution defined by the density

$$f(x) = \frac{\theta^a e^{-\theta b}}{\Gamma(a)} I_{(0, \infty)}(x),$$

where $a, b$ are positive numbers with $a$ being the shape parameter and $b$ being the scale parameter,

$\text{Be}(p, q)$ the beta distribution defined by the density

$$f(x) = \frac{\Gamma(p + q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} I_{(0,1)}(x),$$

where $p, q$ are positive numbers.

We now state and prove a lemma, which will be used to prove the main results of this section.

**Lemma 1.** Suppose that $Q$ is a twice differentiable positive function of $\theta$ on $[0, \infty)$, with $Q(\theta) < 0, \forall \theta \geq 0$. Furthermore, assume that $\lim_{\theta \to \infty} Q(\theta) = \lim_{\theta \to \infty} \frac{d}{d\theta} Q(\theta) = 0$ and $Q''(\theta) = cQ'(\theta)$, where $c$ and $l$ are two constants with $l \neq 1$. Then

(i) $c > 0$ and $l > 1$;

(ii) $Q(\theta) = (c_1 \theta + c_2)^{-\frac{2}{l-1}}$, where $c_1 = \frac{l-1}{2} \sqrt{\frac{2\pi}{l+1}}$ and $c_2$ is some positive constant.

**Proof.** From the assumption $Q''(\theta) = cQ'(\theta)$, it yields

$$\frac{d}{d\theta} [Q'(\theta)]^2 = 2Q'(\theta)Q''(\theta) = 2cQ'(\theta)Q'(\theta).$$

This in turn implies

$$[Q'(\theta)]^2 = \begin{cases} \frac{2\pi}{l+1} Q'^1(\theta) + C_1, & l \neq -1, \\ 2c \log Q(\theta) + C_2, & l = -1, \end{cases}$$

(1)
where \( C_1, C_2 \) are constants. Letting \( \theta \to \infty \) in (1) and using the assumption \( \lim_{\theta \to \infty} Q(\theta) = \lim_{\theta \to \infty} Q'(\theta) = 0 \), we obtain \( l \neq -1 \) and \( C_1 = 0 \). Consequently, we have \( \frac{2c}{l+1} > 0 \), and

\[
Q'(\theta) = -\left(\frac{2c}{l+1}\right)^{1/2} Q^{1/2}(\theta). \tag{2}
\]

Solving (2), yields

\[
[Q(\theta)]^{-\frac{l-1}{2}} = \frac{l-1}{2} \sqrt{\frac{2c}{l+1}} \theta + c_2, \tag{3}
\]

where \( c_2 \) is a constant. As \( Q(\theta) > 0, \forall \theta > 0 \), we have \( l > 1, c > 0 \) and \( c_2 > 0 \). By letting \( c_1 = \frac{l-1}{2} \sqrt{\frac{2c}{l+1}} \), it follows

\[
Q(\theta) = (c_1 \theta + c_2)^{-\frac{2}{l-1}}. \tag{4}
\]

This completes the proof.

The following lemma is due to Huang and Su(1997).

**Lemma 2.** Let the common distribution function of the inter-arrival times of the renewal process \( A \) be \( \Gamma(\alpha, \beta) \) distributed. Given integers \( s, r, k, n \), where \( s > 0, r > -k \alpha \) and \( 1 \leq k \leq n \), if for some constant \( a > 0 \),

\[
E(S_k^r|A(t) = n) = at^r E(S_k^r|A(t) = n), \forall t > 0, \tag{5}
\]

then \( \alpha = 1 \), namely \( A \) becomes a Poisson process, and \( a = \frac{e^{r+s}k}{e^{r+s}k^n} \Pi_{j=1}^s (n + r + j)^{-1} \), where for \( u = r \) or \( r + s \),

\[
c_{u,k} = \begin{cases} 
\prod_{j=0}^{u-1}(k+j), & u \geq 1, \\
1, & u = 0, \\
\prod_{j=1}^{-u}(k-j)^{-1}, & -1 \leq u \leq -k.
\end{cases}
\]

**Theorem 1.** Let \( X \) and \( Y \) be two independent non-degenerate positive random variables with \( E(X^{r+3}) < \infty \) and \( E(X^r) < \infty \) for some integer \( r \). If the conditions

\[
E(X^{r+1}|X+Y) = a(X+Y)E(X^r|X+Y), \tag{6}
\]

and

\[
E(X^{r+3}|X+Y) = b(X+Y)E(X^{r+2}|X+Y) \tag{7}
\]

hold for some constants \( a \neq b \), then
(i) $0 < a, b < 1$;

(ii) $X$ and $Y$ have gamma distributions with the same scale parameter.

Proof. From (6) and (7), the assertion (i) is obtained immediately. We now prove the assertion (ii). First (6) and (7) imply

$$E(X^{r+1}e^{-\theta(X+Y)}) = aE((X^{r+1} + X^rY)e^{-\theta(X+Y)}), \tag{8}$$

and

$$E(X^{r+3}e^{-\theta(X+Y)}) = bE((X^{r+3} + X^{r+2}Y)e^{-\theta(X+Y)}). \tag{9}$$

Let

$$H(\theta) = E(e^{-\theta X}), Q(\theta) = E(X^r e^{-\theta X}) \text{ and } I(\theta) = E(e^{-\theta Y}), \theta > 0.$$

Note that

$$Q(\theta) = (-1)^r(H(\theta))^{(r)}, r > 0, \tag{10}$$

and

$$(Q(\theta))^{(-r)} = (-1)^rH(\theta), r \leq 0. \tag{11}$$

After some simple computations, (8) and (9) imply

$$(a^{-1} - 1)\frac{Q'(\theta)}{Q(\theta)} = \frac{I'(\theta)}{I(\theta)}, \tag{12}$$

and

$$(b^{-1} - 1)\frac{Q''(\theta)}{Q'(\theta)} = \frac{I'(\theta)}{I(\theta)}. \tag{13}$$

As $a$ and $b$ both are less than $1$, (12) and (13) in turn imply

$$\frac{Q''(\theta)}{Q'(\theta)} = \frac{b - ab Q'(\theta)}{a - ab Q(\theta)}. \tag{14}$$

From this we obtain

$$Q'(\theta) = cQ^{\frac{b - ab}{a - ab}}(\theta),$$

which, by Lemma 1, has the solution $Q(\theta) = (m_1\theta + m_2)^e$, where $m_1, m_2$ are constants and $e = 2(a - ab)/(a - b)$. This together with (10), if $r > 0$, or (11), if $r \leq 0$, imply $H(\theta) = (1 + \theta/\beta)^{-\alpha}$, where $\alpha$ and $\beta$ are some positive constants, and the assertion (ii) follows.
REM 

1. In Theorem 1, if \( a = b \), then \( 0 < a < 1 \) and \( P(X = c) = P(Y = c(a^{-1} - 1)) = 1 \), where \( c \) is a positive constant.

The following is the process version of Theorem 1.

**Theorem 2.** Assume for some fixed integers \( r, k, n \), where \( 1 \leq k \leq n \),

\[
E(S_k^{r+1} | A(t) = n) = atE(S_k^r | A(t) = n), \tag{15}
\]

and

\[
E(S_k^{r+2} | A(t) = n) = btE(S_k^r | A(t) = n) \tag{16}
\]

hold for some constants \( a \neq b \), for every \( t > 0 \) whenever \( P(A(t) = n) > 0 \).

Also assume \( E(X_1^{r+3}) < \infty \) if \( r > 0 \), or \( E(S_k^r) < \infty \) and \( E(S_k^{r+3}) < \infty \) if \( r \leq 0 \). Then we have

(i) \( r > -k, a = (k + r)/(n + r + 1), b = (k + r + 2)/(n + r + 3) \);

(ii) \( A \) is a Poisson process.

**Proof.** From (15) and (16), it follows

\[
\int_0^t x^{r+1} (F_{n-k}(t-x) - F_{n+1-k}(t-x))dF_k(x)
= at \int_0^t x^r (F_{n-k}(t-x) - F_{n+1-k}(t-x))dF_k(x), \tag{17}
\]

and

\[
\int_0^t x^{r+3} (F_{n-k}(t-x) - F_{n+1-k}(t-x))dF_k(x)
= bt \int_0^t x^{r+2} (F_{n-k}(t-x) - F_{n+1-k}(t-x))dF_k(x), \tag{18}
\]

where for \( j \geq 0 \), \( F_j \) is the \( j \)-fold convolution of \( F \) with itself. Taking the Laplace transformations of both sides of (17) and (18) with respect to \( \theta \), respectively, we obtain, after some simple computations, for every \( \theta > 0 \),

\[
\frac{(H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta}{(H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta} = (a^{-1} - 1) \frac{F'(\theta)}{F(\theta)}, \tag{19}
\]

and

\[
\frac{(H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta}{(H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta} = (b^{-1} - 1) \frac{F''(\theta)}{F'(\theta)}, \tag{20}
\]
where

\[ P(\theta) = \int_0^\infty x^r e^{-\theta x} dF_k(x), \]

and

\[ H(\theta) = \int_0^\infty e^{-\theta x} dF(x). \]

Again

\[ P(\theta) = (-1)^r (H^k(\theta))^{(r)}, \quad r > 0, \]

and

\[ (P(\theta))^{(-r)} = (-1)^r H^k(\theta), \quad r \leq 0. \]

Also, from (15) and (16) it can be seen immediately that both \(a\) and \(b\) are less than 1. Hence

\[ \frac{P''(\theta)}{P'(\theta)} = \frac{b - ab P'(\theta)}{a - ab P(\theta)}. \tag{21} \]

a differential equation which exactly has the same form as (14). Along the lines of the proof of Theorem 1, we obtain the solution \(H(\theta) = (1 + \theta/\beta)^{-\alpha}\), where \(\alpha\) and \(\beta\) are some positive constants. The assertions (i) and (ii) now follow from Lemma 2.

3. Characterization of the Gamma Distribution by Dual Regression Schemes

Let \(U = X/(X + Y)\) and \(V = X + Y\), where \(X\) and \(Y\) are two non-degenerate positive random variables. In this section we consider the dual regression schemes, while independence of \(U\) and \(V\) is assumed instead of independence of \(X\) and \(Y\). We now give an extension of Bobecka and Wesolowski (2001):

**Theorem 3.** Let \(U\) and \(V\) be independent. Assume for some fixed integer \(r, E(Y^r) < \infty,\) and \(E(U^{-r}) < \infty\). Also assume

\[ E(Y^{r+1}|X) = cE(Y^r|X), \tag{22} \]

and

\[ E(Y^{r+2}|X) = dE(Y^{r+1}|X) \tag{23} \]

hold for some constants \(c\) and \(d\). Then
(i) \( d > c; \)

(ii) \( V \) is \( \Gamma(a,(d-c)^{-1}) \) distributed, where \( a = (d-c)^{-1}EV; \)

(iii) \( U \) is \( Be(p,q) \) distributed, and \( X \) and \( Y \) are independent and have \( \Gamma(p,(d-c)^{-1}) \) and \( \Gamma(q,(d-c)^{-1}) \) distribution, respectively, where \( p = a - c(d-c)^{-1} + r > 0, \) and \( q = c(d-c)^{-1} - r > 0. \)

PROOF. First (22) and (23) imply
\[
E(V^{r+1}(1 - U)^{r+1}|VU) = cE(V^r(1 - U)^r|VU),
\]  
(24)

and
\[
E(V^{r+2}(1 - U)^{r+2}|VU) = dE(V^{r+1}(1 - U)^{r+1}|VU).
\]  
(25)

From (24) and (25) we have for every integer \( k \geq 0, \)
\[
E(V^{k+1}(1 - U)^{r+1}U^{k-r}) = cE(V^k(1 - U)^r U^{k-r}),
\]  
(26)

and
\[
E(V^{k+2}(1 - U)^{r+2}U^{k-r}) = dE(V^{k+1}(1 - U)^{r+1}U^{k-r}).
\]  
(27)

We prove that both \( E(V^k) \) and \( E[(\frac{v}{u})^r U^k] \) exist, \( \forall k \geq 0, \) in the following.

First the assumptions that \( Y = V(1 - U), U \) and \( V \) are independent, and \( E(Y^r) < \infty, \) imply \( E(V^r) < \infty \) and \( E(1 - U)^r < \infty, \) which together with the assumption that \( E(U^{-r}) < \infty, \) yield for every \( k \geq 0, \)
\[
E \left[ \left( \frac{1 - U}{U} \right)^r U^k \right] < E \left[ \left( \frac{1 - U}{U} \right)^r \right] < \begin{cases} E(U^{-r}) < \infty, & r > 0, \\ E(1 - U)^r < \infty, & r \leq 0. \end{cases}
\]  
(28)

Next, by using the fact that \( E(V^r) < \infty \) and \( E(1 - U)^r < \infty \) we will prove that \( E(V^k) < \infty, \forall k \geq 0, \) by induction. Assume \( E(V^m) < \infty \) for some integer \( m \geq r, \) then
\[
E(V^m(1 - U)^r U^{m-r}) < E(V^m(1 - U)^r) = E(V^m)E(1 - U)^r < \infty.
\]

This together with (22) imply
\[
E(V^{m+1}(1 - U)^r U^{m-r}) = E(Y^{r+1}(UV)^{m-r})
\]
\[
= cE(Y^r(UV)^{m-r}) = cE(V^m(1 - U)^r U^{m-r}) < \infty.
\]  
(29)

Consequently \( E(V^{m+1}) < \infty. \) This proves that \( E(V^k) < \infty, \forall k \geq r. \) Now, if \( r \leq 0 \) then \( E(V^k) < \infty, \forall k \geq 0, \) and if \( r > 0 \) then \( E(V^i) < \infty, i = 0,1, \cdots, r-1, \) which in turn yields that \( E(V^k) < \infty, \forall k \geq 0, \) in either case.
Now (26) and (27) can be rewritten as
\[
E(V^{k+1})E \left[ \left( \frac{1 - U}{U} \right)^r U^k (1 - U) \right] = c E(V^k)E \left[ \left( \frac{1 - U}{U} \right)^r U^k \right],
\]
(30)
and
\[
E(V^{k+2})E \left[ \left( \frac{1 - U}{U} \right)^r U^k (1 - 2U + U^2) \right] = d E(V^k)E \left[ \left( \frac{1 - U}{U} \right)^r U^k (1 - U) \right],
\]
(31)
respectively. For every \( k \geq 0 \), let
\[
h(k) = \frac{E(V^{k+1})}{E(V^k)},
\]
(32)
and
\[
g(k) = \frac{E[(\frac{1 - U}{U})^r U^{k+1}]}{E[(\frac{1 - U}{U})^r U^k]}.
\]
Then (30) and (31) lead to
\[
c = h(k)[1 - g(k)],
\]
(33)
and
\[
dh(k)[1 - g(k)] = h(k)h(k+1)[1 - 2g(k) + g(k)g(k+1)], \forall k \geq 0.
\]
(34)
Comparing (33) and (34) we have
\[
bd = h(k)h(k+1)[1 - g(k) - g(k)(1 - g(k+1))] = ch(k+1) - ch(k)g(k),
\]
\( \forall k \geq 0. \) Hence \( h(k+1) - h(k) = d - c, \forall k \geq 0. \) Consequently, \( h(k) = h(0) + k(d - c), \) where \( h(0) = E(V). \)

As \( h(k) > 0, \forall k \geq 0, \) we have \( d \geq c. \) If \( d = c, \) then \( h(k) = h(0), \forall k \geq 0. \) By (32), the constancy of \( h(k) \) implies \( Var(V) = E(V^2) - E^2(V) = 0. \) Hence \( V \) is degenerate, namely \( Y = e - X, \) a.s., for some constant \( e. \) Substitute \( Y = e - X \) into (22) yields \( X = e, \) a.s., or \( X = e - c, \) a.s., which contradicts to the assumption that \( X \) is non-degenerate. Therefore \( d \neq c \) and the assertion (i) follows.

Let \( b = (d - c)^{-1} \) and \( a = b E(V) > 0. \) Then by using (32) recursively we obtain
\[
E(V^k) = \frac{\Gamma(a + k)}{b^k \Gamma(a)}, \forall k \geq 0.
\]
(35)
Hence, by the uniqueness of the moments sequence for the gamma distribution, we obtain that $V$ is $\Gamma(a, b)$ distributed. This proves the assertion (ii).

Next, we prove the assertion (iii). From (24) we have

$$E(V^{r+k+1}U^k(1-U)^{r+1}|VU) = cE(V^{r+k+1}U^k(1-U)^r|VU), \forall k \geq 0.$$ 

Thus

$$E(V^{r+k+1})E[U^k(1-U)^{r+1}] = cE(V^{r+k})E[U^k(1-U)^r], \forall k \geq 0.$$ 

This in turn implies

$$\frac{E((1-U)^{r+1}U^k)}{E((1-U)^rU^k)} \frac{cE(V^{r+k})}{E(V^{r+k+1})} = \frac{bc}{a+r+k}, \forall k \geq 0. \quad (36)$$

Therefore

$$\frac{E((1-U)^rU^{k+1})}{E((1-U)^rU^k)} = 1 - \frac{bc}{a+r+k} = \frac{a+r+k - bc}{a+r+k}, \forall k \geq 0. \quad (37)$$

Since $U$ is a random variable between 0 and 1, we have

$$0 < \frac{bc}{a+r+k} < 1, \forall k \geq 0.$$ 

This leads to $a+r > bc > 0$. Let $F_U$ denote the distribution function of $U$. Define a new probability measure on (0,1) by

$$\eta(1-u)^rF_U(du) = G(du), \quad (38)$$

where $\eta^{-1} = E(1-U)^r < \infty$. It yields that $G$ is a distribution function. Let $Z$ be a random variable having a distribution function $G$. Then (37) and (38) yield

$$\frac{E(Z^{k+1})}{E(Z^k)} = \frac{E(\eta(1-U)^rU^{k+1})}{E(\eta(1-U)^rU^k)} = \frac{a+r+k - bc}{a+r+k} \equiv m(k), \forall k \geq 0.$$ 

Consequently, all the moments of $Z$ are uniquely determined, hence its distribution is uniquely determined by the function $m$ since $Z$ is in (0,1), which together with the fact that $a - bc + r > 0$ and $bc > 0$ yield that $Z$ is $Be(a - bc + r, bc)$ distributed. In view of (38), we obtain $bc > r$. This in turn implies $U$ is $Be(a - bc + r, bc - r) = Be(p, q)$ distributed, where $p \equiv a - bc + r = a - c(d - c)^{-1} + r > 0$ and $q \equiv bc - r = c(d - c)^{-1} - r > 0$. Finally,
by computing the joint density of \( (X, Y) = (UV, (1-U)V) \), we conclude that
\( X \) and \( Y \) are independent and have \( \Gamma(p, (d - c)^{-1}) \) and \( \Gamma(q, (d - c)^{-1}) \) distribution, respectively. This completes the proof.

We also have the following generalization of Theorems 2, 4 and 6 of Bobecka and Wesolowski (2001).

**Theorem 4.** As in Theorem 3, assume for some integer \( r \), \( E(X^r) < \infty \), and \( E(1 - U)^{-r} < \infty \). Also assume

\[
E(X^{r+1}|Y) = cE(X^r|Y),
\]

and

\[
E(X^{r+2}|Y) = dE(X^{r+1}|Y)
\]

hold for some constants \( c \) and \( d \). Then

(i) \( d > c \);
(ii) \( V \) is \( \Gamma(a, (d - c)^{-1}) \) distributed, where \( a = (d - c)^{-1}EV \);
(iii) \( U \) is \( Be(q, p) \) distributed, and \( X \) and \( Y \) are independent and have \( \Gamma(p, (d - c)^{-1}) \) and \( \Gamma(q, (d - c)^{-1}) \) distribution, respectively, where
\[
p = a - c(d - c)^{-1} + r > 0 \quad \text{and} \quad q = c(d - c)^{-1} - r > 0.
\]

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**References**


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