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CHARACTERIZATIONS OF THE GAMMA DISTRIBUTION VIA CONDITIONAL MOMENTS

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SUMMARY. In this work we characterize two independent non-degenerate positive random variables X and Y to be gamma distributed with the same scale parameter by the assumptions $E(X^{r+1}|X+Y) = a(X+Y)E(X^r|X+Y)$ and $E(X^{r+s+1}|X+Y) = b(X+Y)E(X^{r+s}|X+Y)$ for some fixed integer r and $s = 2$. Furthermore, let $A \equiv \{A(t), t \geq 0\}$ be a renewal process with $\{S_k, k \geq 1\}$ being the sequence of arrival times, under the assumptions $E(S_k^{r+1}|A(t) = n) = atE(S_k^r|A(t) = n)$ and $E(S_k^{r+s+1}|A(t) = n) = btE(S_k^{r+s}|A(t) = n)$ for fixed integers r, k, n, where $1 \leq k \leq n$, and $s = 2$, we prove that A has to be a Poisson process. In the case that $s = 1$ the above two results were proved by Huang and Su (1997).

On the other hand, recently characterizations of gamma distribution by the so-called dual regression schemes were investigated by Bobecka and Wesolowski (2001). More precisely, they considered the constancy of regressions of X and Y, while independence of $X/(X+Y)$ and $X+Y$ is assumed instead of independence of X and Y. They characterized X and Y to be gamma distributed by the assumptions $E(Y^u|X) = c$ and $E(Y^v|X) = d$, for $(u, v) = (1, 2), (1, -1)$ or $(-1, -2)$, where c and d are constants. As a generalization, we prove that X and Y are gamma distributed with the same scale parameter under the assumptions $E(Y^{r+1}|X) = cE(Y^r|X)$ and $E(Y^{r+2}|X) = dE(Y^{r+1}|X)$, for some fixed integer r, where c and d are constants. Note that $(u, v) = (1, 2), (1, -1)$ and $(-1, -2)$ corresponds to $r = 0, -1$ and -2, respectively.

1. Introduction

Given two independent non-degenerate positive random variables X and Y, Lukacs (1955) proved that $X/(X+Y)$ and $X+Y$ are independent

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if and only if X and Y are gamma distributed with the same scale parameter. Since then many papers considered different extensions. Among others, Bolger and Harkness (1965), Wesolowski (1990) and Li *et al.* (1994) replaced the independence condition of $X/(X+Y)$ and $X+Y$ by the following regression assumptions:

$$E(X^u|X+Y) = a(X+Y)^u,$$

and

$$E(X^v|X+Y) = b(X+Y)^v,$$

where $(u,v) = (1,2),(1,-1)$ or $(-1,-2)$. Huang and Su (1997) generalized the above results, and obtained similar characterization, under the weaker conditions:

$$E(X^{r+1}|X+Y) = a(X+Y)E(X^r|X+Y),$$

and

$$E(X^{r+s+1}|X+Y) = b(X+Y)E(X^{r+s}|X+Y),$$

with $s=1$, r being some fixed integer, and a and b being some constants. Note that $(u,v) = (1,2),(1,-1)$ and $(-1,-2)$ corresponds to $r = 0,-1$ and -2, respectively. In this work we prove the case of $s = 2$.

There are also parallel characterizations for Poisson process. Let $A \equiv \{A(t), t \geq 0\}$ be a renewal process, with $\{S_k, k \geq 1\}$ being the sequence of arrival times, and F being the common distribution function of the inter-arrival times. Li *et al.* (1994) characterized A to be a Poisson process by the assumptions:

$$E(S_k^u|A(t)=n) = at^u,$$

and

$$E(S_k^v|A(t)=n) = bt^v,$$

for some fixed integers k and $n, 1 \leq k \leq n$, and constants a and b, where $(u,v) = (1,2),(1,-1)$ or $(-1,-2)$. Huang and Su (1997) characterized A to be Poisson under the weaker conditions:

$$E(S_k^{r+1}|A(t)=n) = atE(S_k^r|A(t)=n),$$

and

$$E(S_k^{r+s+1}|A(t)=n) = btE(S_k^{r+s}|A(t)=n),$$

with $s=1$, r,k,n being some integers, where $1 \le k \le n$, and a,b being some constants. Again, we prove the case of $s=2$.

On the other hand, Hall and Simons (1969) and Huang and Su (1997), characterized gamma distributions by using

$$E(X^u|X+Y) = a(X+Y)^u,$$

and

$$E(Y^u|X+Y) = b(X+Y)^u,$$

for $u=2$ or -1.

Recently, Bobecka and Wesolowski (2001) generalized the Lukacs theorem in another direction, namely, under the so-called dual regression schemes. That is, they assumed the constancy of regressions for X and Y, while independence of $U = X/(X+Y)$ and $V = X+Y$ is assumed instead of independence of X and Y. More precisely, they characterized X and Y to be gamma distributed by the assumptions:

$$E(Y^u|X) = c, \qquad \text{and} \qquad E(Y^v|X) = d,$$

or

$$E(X^u|Y) = c, \qquad \text{and} \qquad E(X^v|Y) = d,$$

where $(u,v) = (1,2), (1,-1)$ or $(-1,-2)$. In Section 3, we extend the above results by using

$$E(Y^{r+1}|X) = cE(Y^r|X), \qquad \text{and} \qquad E(Y^{r+2}|X) = dE(Y^{r+1}|X),$$

or

$$E(X^{r+1}|Y) = cE(X^r|Y), \qquad \text{and} \qquad E(X^{r+2}|Y) = dE(X^{r+1}|Y),$$

for some fixed integer r and constants c and d, to characterize gamma distributions of X and Y. Again $(u,v) = (1,2), (1,-1)$ and $(-1,-2)$ corresponds to $r = 0, -1$ and -2, respectively.

2. Characterizations of Gamma Distribution and Poisson Process by Conditional Moments

Introduce first the notation for two distributions which play the important role in this paper. Denote by $\Gamma(a, b)$ the gamma distribution defined by the density

$$f(x) = \frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)} I_{(0,\infty)}(x),$$

where a, b are positive numbers with a being the shape parameter and b being the scale parameter,

$Be(p, q)$ the beta distribution defined by the density

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} I_{(0,1)}(x),$$

where p, q are positive numbers.

We now state and prove a lemma, which will be used to prove the main results of this section.

LEMMA 1. *Suppose that Q is a twice differentiable positive function of θ on $[0, \infty)$, with $Q^{'}(\theta) < 0, \forall \theta \geq 0$. Furthermore, assume that $\lim_{\theta\to\infty} Q(\theta) = \lim_{\theta\to\infty} Q^{'}(\theta) = 0$ and $Q^{''}(\theta) = cQ^l(\theta)$, where c and l are two constants with $l \neq 1$. Then*

(i) $c > 0$ and $l > 1$;

(ii) $Q(\theta) = (c_1\theta + c_2)^{-\frac{2}{l-1}}$, where $c_1 = \frac{l-1}{2}\sqrt{\frac{2c}{l+1}}$ and c_2 is some positive constant.

PROOF. From the assumption $Q^{''}(\theta) = cQ^l(\theta)$, it yields

$$\frac{d}{d\theta}[Q^{'}(\theta)]^2 = 2Q^{'}(\theta)Q^{''}(\theta) = 2cQ^l(\theta)Q^{'}(\theta).$$

This in turn implies

$$[Q^{'}(\theta)]^2 = \begin{cases} \frac{2c}{l+1}Q^{l+1}(\theta) + C_1, & l \neq -1, \\ 2c \log Q(\theta) + C_2, & l = -1, \end{cases} \tag{1}$$

where C_1, C_2 are constants. Letting $\theta \to \infty$ in (1) and using the assumption $\lim_{\theta\to\infty} Q(\theta) = \lim_{\theta\to\infty} Q'(\theta) = 0$, we obtain $l \neq -1$ and $C_1 = 0$. Consequently, we have $\frac{2c}{l+1} > 0$, and

$$Q'(\theta) = -\left(\frac{2c}{l+1}\right)^{1/2} Q^{\frac{l+1}{2}}(\theta). \tag{2}$$

Solving (2), yields

$$[Q(\theta)]^{-\frac{l-1}{2}} = \frac{l-1}{2}\sqrt{\frac{2c}{l+1}}\theta + c_2, \tag{3}$$

where c_2 is a constant. As $Q(\theta) > 0, \forall\theta > 0$, we have $l > 1, c > 0$ and $c_2 > 0$. By letting $c_1 = \frac{l-1}{2}\sqrt{\frac{2c}{l+1}}$, it follows

$$Q(\theta) = (c_1\theta + c_2)^{-\frac{2}{l-1}}. \tag{4}$$

This completes the proof.

The following lemma is due to Huang and Su(1997).

LEMMA 2. *Let the common distribution function of the inter-arrival times of the renewal process A be $\Gamma(\alpha, \beta)$ distributed. Given integers s, r, k, n, where $s > 0, r > -k\alpha$ and $1 \le k \le n$, if for some constant $a > 0$,*

$$E(S_k^{r+s}|A(t) = n) = at^s E(S_k^r|A(t) = n), \forall t > 0, \tag{5}$$

then $\alpha = 1$, namely A becomes a Poisson process, and $a = \frac{c_{r+s,k}}{c_{r,k}}\prod_{j=1}^{s}(n + r + j)^{-1}$, where for $u = r$ or $r + s$,

$$c_{u,k} = \begin{cases} \prod_{j=0}^{u-1}(k+j) & , u \ge 1, \\ 1 & , u = 0, \\ \prod_{j=1}^{-u}(k-j)^{-1} & , -1 \ge u > -k. \end{cases}$$

THEOREM 1. *Let X and Y be two independent non-degenerate positive random variables with $E(X^{r+3}) < \infty$ and $E(X^r) < \infty$ for some integer r. If the conditions*

$$E(X^{r+1}|X + Y) = a(X + Y)E(X^r|X + Y), \tag{6}$$

and

$$E(X^{r+3}|X + Y) = b(X + Y)E(X^{r+2}|X + Y) \tag{7}$$

hold for some constants $a \neq b$, then

(i) $0 < a, b < 1$;

(ii) X and Y have gamma distributions with the same scale parameter.

PROOF. From (6) and (7), the assertion (i) is obtained immediately. We now prove the assertion (ii). First (6) and (7) imply

$$E(X^{r+1}e^{-\theta(X+Y)}) = aE((X^{r+1} + X^rY)e^{-\theta(X+Y)}), \tag{8}$$

and

$$E(X^{r+3}e^{-\theta(X+Y)}) = bE((X^{r+3} + X^{r+2}Y)e^{-\theta(X+Y)}). \tag{9}$$

Let

$$H(\theta) = E(e^{-\theta X}), Q(\theta) = E(X^r e^{-\theta X}), \text{and } I(\theta) = E(e^{-\theta Y}), \theta > 0.$$

Note that

$$Q(\theta) = (-1)^r (H(\theta))^{(r)}, r > 0, \tag{10}$$

and

$$(Q(\theta))^{(-r)} = (-1)^r H(\theta), r \leq 0. \tag{11}$$

After some simple computations, (8) and (9) imply

$$(a^{-1} - 1)\frac{Q^{'}(\theta)}{Q(\theta)} = \frac{I^{'}(\theta)}{I(\theta)}, \tag{12}$$

and

$$(b^{-1} - 1)\frac{Q^{'''}(\theta)}{Q^{''}(\theta)} = \frac{I^{'}(\theta)}{I(\theta)}. \tag{13}$$

As a and b both are less than 1, (12) and (13) in turn imply

$$\frac{Q^{'''}(\theta)}{Q^{''}(\theta)} = \frac{b - ab}{a - ab}\frac{Q^{'}(\theta)}{Q(\theta)}. \tag{14}$$

¿From this we obtain

$$Q^{''}(\theta) = cQ^{\frac{b-ab}{a-ab}}(\theta),$$

which, by Lemma 1, has the solution $Q(\theta) = (m_1\theta + m_2)^e$, where m_1, m_2 are constants and $e = 2(a - ab)/(a - b)$. This together with (10), if $r > 0$, or (11), if $r \leq 0$, imply $H(\theta) = (1 + \theta/\beta)^{-\alpha}$, where α and β are some positive constants, and the assertion(ii) follows.

REMARK 1. In Theorem 1, if $a = b$, then $0 < a < 1$ and $P(X = c) = P(Y = c(a^{-1} - 1)) = 1$, where c is a positive constant.

The following is the process version of Theorem 1.

THEOREM 2. *Assume for some fixed integers r, k, n, where $1 \leq k \leq n$,*

$$E(S_k^{r+1}|A(t) = n) = atE(S_k^r|A(t) = n), \tag{15}$$

and

$$E(S_k^{r+3}|A(t) = n) = btE(S_k^{r+2}|A(t) = n) \tag{16}$$

hold for some constants $a \neq b$, for every $t > 0$ whenever $P(A(t) = n) > 0$. Also assume $E(X_1^{r+3}) < \infty$ if $r > 0$, or $E(S_k^r) < \infty$ and $E(S_k^{r+3}) < \infty$ if $r \leq 0$. Then we have

(i) $r > -k, a = (k + r)/(n + r + 1), b = (k + r + 2)/(n + r + 3)$;

(ii) A is a Poisson process.

PROOF. From (15) and (16), it follows

$$\begin{aligned} &\int_0^t x^{r+1}(F_{n-k}(t - x) - F_{n+1-k}(t - x))dF_k(x) \\ &= at\int_0^t x^r(F_{n-k}(t - x) - F_{n+1-k}(t - x))dF_k(x), \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\int_0^t x^{r+3}(F_{n-k}(t - x) - F_{n+1-k}(t - x))dF_k(x) \\ &= bt\int_0^t x^{r+2}(F_{n-k}(t - x) - F_{n+1-k}(t - x))dF_k(x), \end{aligned} \tag{18}$$

where for $j \geq 0, F_j$ is the j-fold convolution of F with itself. Taking the Laplace transformations of both sides of (17) and (18) with respect to θ, respectively, we obtain, after some simple computations, for every $\theta > 0$,

$$\frac{((H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta)'}{(H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta} = (a^{-1} - 1)\frac{P'(\theta)}{P(\theta)}, \tag{19}$$

and

$$\frac{((H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta)'}{(H^{n-k}(\theta) - H^{n-k+1}(\theta))/\theta} = (b^{-1} - 1)\frac{P'''(\theta)}{P''(\theta)}, \tag{20}$$

where

$$P(\theta) = \int_0^\infty x^r e^{-\theta x} dF_k(x),$$

and

$$H(\theta) = \int_0^\infty e^{-\theta x} dF(x).$$

Again

$$P(\theta) = (-1)^r (H^k(\theta))^{(r)}, r > 0,$$

and

$$(P(\theta))^{(-r)} = (-1)^r H^k(\theta), r \leq 0.$$

Also, from (15) and (16) it can be seen immediately that both a and b are less than 1. Hence

$$\frac{P'''(\theta)}{P''(\theta)} = \frac{b - ab}{a - ab} \frac{P'(\theta)}{P(\theta)}, \tag{21}$$

a differential equation which exactly has the same form as (14). Along the lines of the proof of Theorem 1, we obtain the solution $H(\theta) = (1 + \theta/\beta)^{-\alpha}$, where α and β are some positive constants. The assertions (i) and (ii) now follow from Lemma 2.

3. Characterization of the Gamma Distribution by Dual Regression Schemes

Let $U = X/(X + Y)$ and $V = X + Y$, where X and Y are two non-degenerate positive random variables. In this section we consider the dual regression schemes, while independence of U and V is assumed instead of independence of X and Y. We now give an extension of Bobecka and Wesolowski (2001):

Theorem 3. *Let U and V be independent. Assume for some fixed integer $r, E(Y^r) < \infty$, and $E(U^{-r}) < \infty$. Also assume*

$$E(Y^{r+1}|X) = cE(Y^r|X), \tag{22}$$

and

$$E(Y^{r+2}|X) = dE(Y^{r+1}|X) \tag{23}$$

hold for some constants c and d. Then

(i) $d > c$;

(ii) V is $\Gamma(a,(d-c)^{-1})$ distributed, where $a = (d-c)^{-1}EV$;

(iii) U is $Be(p,q)$ distributed, and X and Y are independent and have $\Gamma(p,(d-c)^{-1})$ and $\Gamma(q,(d-c)^{-1})$ distribution, respectively, where $p = a - c(d-c)^{-1} + r > 0$, and $q = c(d-c)^{-1} - r > 0$.

Proof. First (22) and (23) imply

$$E(V^{r+1}(1-U)^{r+1}|VU) = cE(V^r(1-U)^r|VU), \tag{24}$$

and

$$E(V^{r+2}(1-U)^{r+2}|VU) = dE(V^{r+1}(1-U)^{r+1}|VU). \tag{25}$$

¿From (24) and (25) we have for every integer $k \geq 0$,

$$E(V^{k+1}(1-U)^{r+1}U^{k-r}) = cE(V^k(1-U)^rU^{k-r}), \tag{26}$$

and

$$E(V^{k+2}(1-U)^{r+2}U^{k-r}) = dE(V^{k+1}(1-U)^{r+1}U^{k-r}). \tag{27}$$

We prove that both $E(V^k)$ and $E[(\frac{1-U}{U})^rU^k]$ exist, $\forall k \geq 0$, in the following.

First the assumptions that $Y = V(1-U), U$ and V are independent, and $E(Y^r) < \infty$, imply $E(V^r) < \infty$ and $E(1-U)^r < \infty$, which together with the assumption that $E(U^{-r}) < \infty$, yield for every $k \geq 0$,

$$E\left[\left(\frac{1-U}{U}\right)^r U^k\right] < E\left[\left(\frac{1-U}{U}\right)^r\right] < \begin{cases} E(U^{-r}) < \infty, & r > 0, \\ E(1-U)^r < \infty, & r \leq 0. \end{cases} \tag{28}$$

Next by using the fact that $E(V^r) < \infty$ and $E(1-U)^r < \infty$ we will prove that $E(V^k) < \infty$, $\forall k \geq 0$, by induction. Assume $E(V^m) < \infty$ for some integer $m \geq r$, then

$$E(V^m(1-U)^rU^{m-r}) < E(V^m(1-U)^r) = E(V^m)E(1-U)^r < \infty.$$

This together with (22) imply

$$\begin{aligned} &E(V^{m+1}(1-U)^{r+1}U^{m-r}) = E(Y^{r+1}(UV)^{m-r}) \\ &= cE(Y^r(UV)^{m-r}) = cE(V^m(1-U)^rU^{m-r}) < \infty. \end{aligned} \tag{29}$$

Consequently $E(V^{m+1}) < \infty$. This proves that $E(V^k) < \infty$, $\forall k \geq r$. Now, if $r \leq 0$ then $E(V^k) < \infty, \forall k \geq 0$, and if $r > 0$ then $E(V^i) < \infty, i = 0, 1, \cdots, r-1$, which in turn yields that $E(V^k) < \infty, \forall k \geq 0$, in either case.

Now (26) and (27) can be rewritten as

$$E(V^{k+1})E\left[\left(\frac{1-U}{U}\right)^r U^k(1-U)\right] = cE(V^k)E\left[\left(\frac{1-U}{U}\right)^r U^k\right], \tag{30}$$

and

$$\begin{aligned} &E(V^{k+2})E\left[\left(\frac{1-U}{U}\right)^r U^k(1-2U+U^2)\right] \\ &= dE(V^{k+1})E\left[\left(\frac{1-U}{U}\right)^r U^k(1-U)\right], \end{aligned} \tag{31}$$

respectively. For every $k \geq 0$, let

$$h(k) = \frac{E(V^{k+1})}{E(V^k)}, \tag{32}$$

and

$$g(k) = \frac{E[(\frac{1-U}{U})^r U^{k+1}]}{E[(\frac{1-U}{U})^r U^k]}.$$

Then (30) and (31) lead to

$$c = h(k)[1-g(k)], \tag{33}$$

and

$$dh(k)[1-g(k)] = h(k)h(k+1)[1-2g(k)+g(k)g(k+1)], \forall\ k \geq 0. \tag{34}$$

Comparing (33) and (34) we have

$$cd = h(k)h(k+1)[1-g(k)-g(k)(1-g(k+1))] = ch(k+1)-ch(k)g(k),$$

$\forall\ k \geq 0$. Hence $h(k+1)-h(k) = d-c$, $\forall k \geq 0$. Consequently, $h(k) = h(0)+k(d-c)$, where $h(0) = E(V)$.

As $h(k) > 0$, $\forall k \geq 0$, we have $d \geq c$. If $d = c$, then $h(k) = h(0), \forall k \geq 0$. By (32), the constancy of $h(k)$ implies $Var(V) = E(V^2)-E^2(V) = 0$. Hence V is degenerate, namely $Y = e-X$, a.s., for some constant e. Substitute $Y = e-X$ into (22) yields $X = e$, a.s., or $X = e-c$, a.s., which contradicts to the assumption that X is non-degenerate. Therefore $d \neq c$ and the assertion (i) follows.

Let $b = (d-c)^{-1}$ and $a = bE(V) > 0$. Then by using (32) recursively we obtain

$$E(V^k) = \frac{\Gamma(a+k)}{b^k\Gamma(a)}, \forall k \geq 0. \tag{35}$$

Hence, by the uniqueness of the moments sequence for the gamma distribution, we obtain that V is $\Gamma(a,b)$ distributed. This proves the assertion (ii).

Next, we prove the assertion (iii). From (24) we have

$$E(V^{r+k+1}U^k(1-U)^{r+1}|VU) = cE(V^{r+k}U^k(1-U)^r|VU), \forall k \geq 0.$$

Thus

$$E(V^{r+k+1})E[U^k(1-U)^{r+1}] = cE(V^{r+k})E[U^k(1-U)^r], \forall k \geq 0.$$

This in turn implies

$$\frac{E((1-U)^{r+1}U^k)}{E((1-U)^rU^k)} = \frac{cE(V^{r+k})}{E(V^{r+k+1})} = \frac{bc}{a+r+k}, \forall k \geq 0. \tag{36}$$

Therefore

$$\frac{E((1-U)^rU^{k+1})}{E((1-U)^rU^k)} = 1 - \frac{bc}{a+r+k} = \frac{a+r+k-bc}{a+r+k}, \forall k \geq 0. \tag{37}$$

Since U is a random variable between 0 and 1, we have

$$0 < \frac{bc}{a+r+k} < 1, \forall k \geq 0.$$

This leads to $a+r > bc > 0$. Let F_U denote the distribution function of U. Define a new probability measure on (0,1) by

$$\eta(1-u)^r F_U(du) = G(du), \tag{38}$$

where $\eta^{-1} = E(1-U)^r < \infty$. It yields that G is a distribution function. Let Z be a random variable having a distribution function G. Then (37) and (38) yield

$$\frac{E(Z^{k+1})}{E(Z^k)} = \frac{E(\eta(1-U)^rU^{k+1})}{E(\eta(1-U)^rU^k)} = \frac{a+r+k-bc}{a+r+k} \equiv m(k), \forall k \geq 0.$$

Consequently, all the moments of Z are uniquely determined, hence its distribution is uniquely determined by the function m since Z is in (0,1), which together with the fact that $a-bc+r > 0$ and $bc > 0$ yield that Z is $Be(a-bc+r, bc)$ distributed. In view of (38), we obtain $bc > r$. This in turn implies U is $Be(a-bc+r, bc-r) = Be(p,q)$ distributed, where $p \equiv a-bc+r = a-c(d-c)^{-1}+r > 0$ and $q \equiv bc-r = c(d-c)^{-1}-r > 0$. Finally, by computing the joint density of $(X, Y) = (UV, (1-U)V)$, we conclude that X and Y are independent and have $\Gamma(p, (d-c)^{-1})$ and $\Gamma(q, (d-c)^{-1})$ distribution, respectively. This completes the proof.

We also have the following generalization of Theorems 2,4 and 6 of Bobecka and Wesolowski (2001).

THEOREM 4. *As in Theorem* 3, *assume for some integer* $r, E(X^r) < \infty$, *and* $E(1-U)^{-r} < \infty$. *Also assume*

$$E(X^{r+1}|Y) = cE(X^r|Y),$$

and

$$E(X^{r+2}|Y) = dE(X^{r+1}|Y)$$

hold for some constants c and d. Then

(i) $d > c$;

(ii) V is $\Gamma(a, (d-c)^{-1})$ distributed, where $a = (d-c)^{-1}EV$;

(iii) U is $Be(q, p)$ distributed, and X and Y are independent and have $\Gamma(q, (d-c)^{-1})$ and $\Gamma(p, (d-c)^{-1})$ distribution, respectively, where $p = a - c(d-c)^{-1} + r > 0$ and $q = c(d-c)^{-1} - r > 0$.

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