Some Skew-Symmetric Models

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Abstract: In this paper, the classes of symmetric density functions which depend on a parameter have been studied. In particular the skew normal, uniform, t, Cauchy, Laplace, and logistic distributions are given and some of their properties are explored.

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1. Introduction

The skew-normal distributions have been introduced by many authors, e.g. Azzalini (1985), Arnold et al. (1993), Aigner et al. (1977), Andel et al. (1984). This class of distributions includes the normal distribution and possesses several properties which coincide or are close to the properties of the normal family. However, this class has a skewness parameter which makes it possible to have a reasonable model for a skewed population distribution thus providing a more flexible model which represents the data as adequately as possible.

Besides being useful in modeling, they are helpful in studying the robustness, and in Bayesian analysis as priors. The construction of such models is based on the following lemma (see Azzalini, 1985).

Lemma 1. *Let Y be a random variable with density function $f(x)$ symmetric about 0, and Z a random variable with absolutely continuous distribution function $G(x)$ such that $G'(x)$ is symmetric about 0. Then*

$$g(x|\lambda) = 2f(x)G(\lambda x), \quad -\infty < x < \infty \tag{1.1}$$

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is a density function of a random variable X for any real λ.

Note that there are three special cases:

(i) $g(x|\lambda)$ tends to the density of $|Y|$ as $\lambda \to \infty$.

(ii) $g(x|\lambda)$ tends to the density of $-|Y|$ as $\lambda \to -\infty$.

(iii) $g(x|0)$ is the density of Y.

Some interesting properties of X can be proved by the following lemma.

Lemma 2. *Let X, Y and Z be three random variables as defined in Lemma 1. Then*

(i) the even moments of X are independent of λ and the same as those of Y.

(ii) X^2 and Y^2 have the same distribution function.

Proof. Let $\psi_X(t)$ denote the characteristic function of X. Then

$$\begin{aligned} \psi_X(-t) &= \int_{-\infty}^{\infty} e^{-itx}[2f(x)G(\lambda x)]\, dx \\ &= \int_{\infty}^{-\infty} -e^{ity}[2f(-y)G(-\lambda y)]\, dy \\ &= \int_{-\infty}^{\infty} e^{ity}[2f(y)(1-G(\lambda y))]\, dy. \end{aligned}$$

The second equality follows from making the change of variable $x = -y$ and the third equality from the symmetry of $f(x)$ and $G'(y)$. Therefore, it implies that $g(t) = \psi_X(t) + \psi_X(-t) = 2\int_{-\infty}^{\infty} e^{itx} f(x)\, dx = 2\psi_Y(t)$ is independent of λ. It is easy to see that if $(-1)^n g^{(2n)}(0)/2$ and $(-1)^n \psi_Y^{(2n)}(0)$ exist, then they are the $2n$th moments of X and Y, and the nth moments of X^2 and Y^2, respectively. Both are the same. Then the desired properties follow immediately. □

In the next section we summarize the results for the skew-normal distribution as given by Azzalini (1985, 1986) by taking $f(\cdot)$ and $G(\cdot)$ as the probability density (p.d.f.) and distribution function (c.d.f) of a standard normal distribution, respectively. Then we define skew-uniform, t, Cauchy, Laplace, and logistic-distribution and study some of their properties.

2. Skew-normal model

The univariate skew-normal distribution can be found in different settings in several papers. We shall follow Azzalini (1985). In econometric literature the distribution appears in the so called stochastic frontier model in papers by Aigner et al. (1977), Andel et al. (1984).

The random variable X is said to have a skew-normal distribution if it is continuous and its density function is given by

$$f_X(x) \quad = \quad 2\phi(x)\Phi(\lambda x), \quad x \in R \tag{2.1}$$

where $\lambda \in R$, $\phi(\cdot)$ is the standard normal density and $\Phi(\cdot)$ is the corresponding distribution function. Several skew-normal densities are illustrated in Figure 1 (a). It is denoted as $X \sim SN(\lambda)$ to mean that X has skew-normal density (2.1). It has been shown that $X^2 \sim \chi_1^2$ with density function $e^{-x/2}/(\sqrt{2\pi x}),\ x \geq 0$. The moment generating function of X is

$$M(t) \quad = \quad 2e^{t^2/2}\Phi(t\lambda/\sqrt{1+\lambda^2}) \tag{2.2}$$

with

$$\begin{aligned}
E(X) &= \sqrt{\frac{2}{\pi}}\frac{\lambda}{\sqrt{1+\lambda^2}}, \qquad \mathrm{Var}(X) = 1 - \frac{2}{\pi}\frac{\lambda^2}{1+\lambda^2} \\
\gamma_1 &= \frac{1}{2}(4-\pi)\ \mathrm{sign}(\lambda)\left[\frac{(E(X))^2}{\mathrm{Var}(X)}\right]^{3/2} \\
\gamma_2 &= 2(\pi-3)\left[\frac{(E(X))^2}{\mathrm{Var}(X)}\right]^2
\end{aligned}$$

where γ_1, γ_2 are the measures of skewness $\mu_3/\mu_2^{3/2}$, and kurtosis μ_4/μ_2^2, respectively ($\mu_k = E[X - E(X)]^k$). Note that $1 - 2/\pi < \mathrm{Var}(X) \leq 1$, $-\sqrt{2}(4-\pi)/(\pi-2)^{3/2} < \gamma_1 < \sqrt{2}(4-\pi)/(\pi-2)^{3/2} \approx 0.995$ and $0 \leq \gamma_2 < 8(\pi-3)/(\pi-2)^2 \approx 0.869$.

3. Skew-uniform model

We are now ready to discuss the other five univariate skewed distributions. The random variable X is said to have a skew-uniform distribution if it is continuous and its density function is given by

$$\begin{aligned}
f_X(x) &= 2f(x)F(\lambda x) \\
&= \frac{1}{\theta^2}[\max(\min(\lambda x, \theta), -\theta) + \theta], \quad -\theta < x < \theta,
\end{aligned}$$

where $\lambda \in R,\ \theta > 0,\ f(x) = 1/(2\theta),\ -\theta < x < \theta$ is the symmetric uniform density on $(-\theta, \theta)$ and $F(x) = [\max(\min(x, \theta), -\theta) + \theta]/(2\theta)$ is the corresponding distribution function. Several skew-uniform densities are illustrated in Figure 1 (b). Then X^2 has the density function $1/(2\theta\sqrt{t}),\ 0 \leq t < \theta^2$. The moment generating function of X is

$$M(t) \quad = \quad \left(\frac{\lambda}{t}\right)\ \cosh(\theta t) - \left(\frac{\lambda - \theta t}{\theta t^2}\right)\ \sinh(\theta t) \tag{3.1}$$

with

$$\begin{aligned} E(X) &= \frac{\lambda\theta}{3}, \qquad \mathrm{Var}(X) = (3-\lambda^2)\frac{\theta^2}{9} \\ \gamma_1 &= \frac{2}{5}\,\frac{\lambda(5\lambda^2-9)}{(3-\lambda^2)^{3/2}} \\ \gamma_2 &= \frac{3}{5}\,\frac{(9-5\lambda^2)(\lambda^2+3)}{(3-\lambda^2)^2} \end{aligned}$$

where γ_1, γ_2 are the measures of skewness $\mu_3/\mu_2^{3/2}$, and kurtosis μ_4/μ_2^2, respectively ($\mu_k = E[X-E(X)]^k$). Note that γ_1 is strictly decreasing from 0 to $-2\sqrt{2}/5$ whereas γ_2 is strictly increasing from 9/5 to 12/5, as λ moves from 0 to 1 .

4. Skew-t model

Let W be a random variable with skew normal distribution $SN(\lambda)$ and V a random variable with Chi squared distribution χ^2_ν. Suppose that W and V are independent. The random variable $X = W/\sqrt{V/\nu}$ is said to have a skew-t distribution with degree ν if it is continuous and its density function is given by

$$\begin{aligned} f_X(x) &= 2f_{T_\nu}(x)F(\lambda x) && (4.1) \\ &= 2\,\frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\;\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2} F_{T_{\nu+1}}\left(\lambda x\sqrt{\frac{1+\nu}{\nu(\nu+x^2)}}\right), \quad x\in R && (4.2) \end{aligned}$$

where $\lambda \in R,\ \sigma > 0,\ f_{T_\nu}(x) = [\Gamma((\nu+1)/2)/(\Gamma(\nu/2)\sqrt{\pi\nu})]\,(1+x^2/\nu)^{-(\nu+1)/2}$ is the density of a t distribution with degree ν and $F_{T_{\nu+1}}(x)$ is the distribution function to $f_{T_{\nu+1}}(x)$. Several skew-t densities are illustrated in Figure 1 (c). The density can be derived directly from the definition of X by using the following lemma.

Lemma 3. *Let Y have a standard normal distribution with distribution function $\Phi(y)$, and Z has Chi squared distribution with ν degrees of freedom and is independent of Y. Then*

$$E_Z\Phi(a\sqrt{Z}) = F_{T_\nu}(a\sqrt{\nu})$$

where F_{T_ν} is the distribution function of t distribution with ν degrees of freedom.

Some basic quantities of X are given by

$$\begin{aligned} E(X) &= b\delta\sqrt{\tfrac{\nu}{2}}\,\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)},\ \nu\geq 2 & E(X^2) &= \tfrac{\nu}{\nu-2},\ \nu\geq 3 \\ E(X^3) &= b\delta(3-\delta^2)\left(\tfrac{3}{2}\right)^{3/2}\frac{\Gamma\left(\frac{\nu-3}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)},\ \nu\geq 4 & E(X^4) &= \tfrac{3\nu^2}{(\nu-2)(\nu-4)},\ \nu\geq 5 \\ \mathrm{Var}(X) &= \tfrac{\nu}{\nu-2}-\tfrac{\nu}{\pi}\,\tfrac{\lambda^2}{1+\lambda^2}\left[\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2,\ \nu\geq 3 & & \\ \gamma_1 &= \tfrac{E(X^3)-3E(X^2)E(X)+2[E(X)]^3}{[\mathrm{Var}(X)]^{3/2}} & \gamma_2 &= \tfrac{E(X^4)-4E(X^3)E(X)+6E(X^2)[E(X)]^2-3[E(X)]^4}{[\mathrm{Var}(X)]^2} \end{aligned} \qquad (4.3)$$

where $b = \sqrt{2/\pi}$ and $\delta = \lambda/\sqrt{1+\lambda^2}$. In particular, X has a t distribution with ν degrees of freedom when $\lambda = 0$, a skew normal distribution $SN(\lambda)$ as $\nu \to \infty$, and X^2 has F distribution with degrees of freedom 1 and ν.

5. Skew-Cauchy model

The random variable X is said to have a skew-Cauchy distribution if it is continuous and its density function is given by

$$\begin{aligned} f_X(x) &= 2f(x)F(\lambda x) \\ &= \frac{\sigma[1 + 2\arctan(\lambda x/\sigma)/\pi]}{\pi(\sigma^2 + x^2)}, \qquad x \in R \end{aligned} \tag{5.1}$$

where $\lambda \in R$, $\sigma > 0$, $f(x) = \sigma/[\pi(\sigma^2 + x^2)]$ is the Cauchy density on $(-\infty, \infty)$ with $F(x) = 1/2 + \arctan(x/\sigma)/\pi$ is the corresponding distribution function. Several skew-Cauchy densities are illustrated in Figure 1 (d). Then X^2 has the density function $\sigma/[\sqrt{t}(\sigma^2 + t)]$, $t \geq 0$. The moment generating function of X does not exist and the characteristic function does not have a closed form. The kth moment of X does not exist for $k \geq 1$.

A second way of defining skew-Cauchy distribution is to take $\nu = 1$ in skew-t distribution (4.2). Then its density function is given by

$$\frac{1}{\pi(1+x^2)} \left[1 + \frac{\lambda x}{\sqrt{1 + (1+\lambda^2)x^2}}\right], \qquad x \in R. \tag{5.2}$$

Its properties are the same as those of skew-Cauchy distribution defined in (5.1).

A third way of defining skew-Cauchy distribution is as follows. Let Y and Z be independent random variables distributed as $SN(\lambda)$. Then $X = Y/Z$ is said to have a skew-Cauchy distribution. Although its density does not have a closed form, it shares the same properties of skew-Cauchy distribution defined in (5.1).

6. Skew-Laplace model

The random variable X is said to have a skew-Laplace distribution if it is continuous and its density function is given by

$$\begin{aligned} f_X(x) &= 2f(x)F(\lambda x) \\ &= \frac{e^{-|x|/\sigma}\left[1 + \text{ sign}(\lambda x)\left(1 - e^{-|\lambda x|/\sigma}\right)\right]}{2\sigma}, \quad x \in R \end{aligned} \tag{6.1}$$

where $\lambda \in R$, $\sigma > 0$, $f(x) = e^{-|x|/\sigma}/(2\sigma)$ is the Laplace density with $F(x) = [1 + \text{sign}(x)(1 - e^{-|x|/\sigma})]/2$ is the corresponding distribution function. Several skew-Laplace densities are illustrated in Figure 1 (e). Then X^2 has the density function $e^{-\sqrt{t}/\sigma}/(2\sigma\sqrt{t}),\ t \geq 0$. The characteristic function of X is

$$\psi(t) \quad = \quad \frac{\sigma t + (\lambda+1)^2 i}{(\sigma t + i)((\sigma t)^2 + (\lambda+1)^2)} \tag{6.2}$$

with

$$\begin{aligned}
E(X) &= \sigma\left[1 - \frac{1}{(\lambda+1)^2}\right], \qquad \mathrm{Var}(X) = \sigma^2\left[2 - \frac{\lambda^2(\lambda+2)^2}{(\lambda+1)^4}\right], \\
&\quad \lambda > -1 + \sqrt{\sqrt{2}-1},\ \lambda < -1 - \sqrt{\sqrt{2}-1} \\
\gamma_1 &= \frac{2\lambda(\lambda+2)(\lambda^2+\lambda+1)(\lambda^2+3\lambda+3)}{[2(\lambda+1)^4 - \lambda^2(\lambda+2)^2]^{3/2}} \\
\gamma_2 &= \frac{3\left(3\lambda^8 + 24\lambda^7 + 88\lambda^6 + 192\lambda^5 + 276\lambda^4 + 272\lambda^3 + 176\lambda^2 + 64\lambda + 8\right)}{\left(\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 2\right)^2}
\end{aligned}$$

where γ_1, γ_2 are the measures of skewness $\mu_3/\mu_2^{3/2}$, and kurtosis μ_4/μ_2^2, respectively ($\mu_k = E[X - E(X)]^k$). Note that γ_1 is strictly decreasing on $\lambda < -1 - \sqrt{\sqrt{2}-1}$ and strictly increasing on $\lambda > -1 + \sqrt{\sqrt{2}-1}$, $\lim_{\lambda\to\pm\infty}\gamma_1 = 2$, $\lim_{\lambda\to\pm-1\pm\sqrt{\sqrt{2}-1}}\gamma_1 = -\infty$, and $\lim_{\lambda\to\pm\infty}\gamma_2 = 9$.

7. Skew-logistic model

The random variable X is said to have a skew-logistic distribution if it is continuous and its density function is given by

$$\begin{aligned}
f_X(x) &= 2f(x)F(\lambda x) \\
&= \frac{2e^{-x/\sigma}}{\sigma(1+e^{-x/\sigma})^2(1+e^{-\lambda x/\sigma})}, \qquad x \in R
\end{aligned} \tag{7.1}$$

where $\lambda \in R,\ \sigma > 0,\ f(x) = e^{-x/\sigma}/[\sigma(1+e^{-x/\sigma})^2]$ is the logistic density with $F(x) = 1/(1+e^{-x/\sigma})$ the corresponding distribution function. Several skew-logistic densities are illustrated in Figure 1 (f). Then X^2 has the density function $e^{-\sqrt{t}/\sigma}/[\sigma(1+e^{-\sqrt{t}/\sigma})^2\sqrt{t}],\ t \geq 0$. Neither the moment generating function nor the characteristic function of X has a closed form. The first four moments of X are given by

$$\begin{aligned}
E(X) &= 2\sigma A_1, \qquad & E(X^2) &= \tfrac{1}{3}\,(\pi\sigma)^2 \\
E(X^3) &= 2\sigma^3 A_3, \qquad & E(X^4) &= \tfrac{7}{15}\,(\pi\sigma)^4
\end{aligned} \tag{7.2}$$

where $A_j = \int_0^\infty (\ln z)^j/[(1+z)^2(1+z^\lambda)]\,dz,\ j = 1, 3$.

8. Remarks

The density functions of skew-symmetric distributions defined in this paper are illustrated in Figure 1. Based on an intensive numerical study, it shows that all of them have a single mode m except for the skew-uniform density with $|\lambda| > \theta$ or $= 0$. Furthermore, the mode is negative if $\lambda < 0$ and positive if $\lambda > 0$, except for the skew-Laplace density, and $f_X(m)$ is an increasing function in $|\lambda|$. For the skew-Laplace density, the mode occurs at the y-axis and it has a cusp at $x = 0$ for all λ.

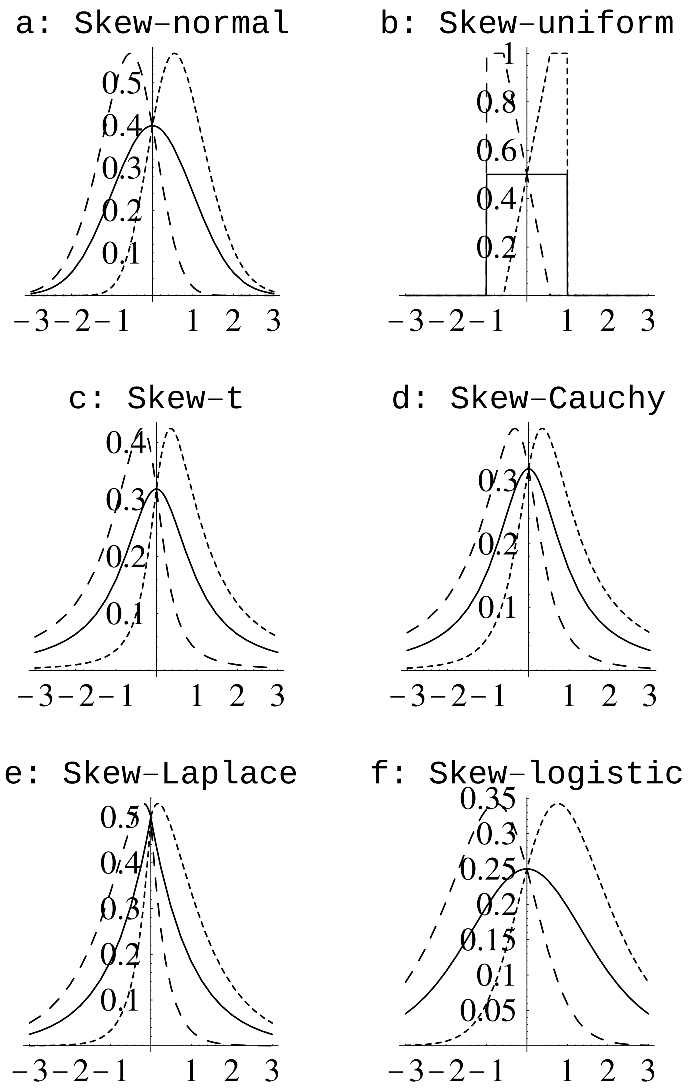

Figure 1: The density functions of skew-symmetric distributions with $\lambda = -1$ (longer dashing line), $\lambda = 1$ (shorter dashing line) and $\lambda = 0$ (solid line), and the other parameters are set to 1.

The families defined in this paper do not have a wide range of the indices of skewness and kurtosis. To include the wide range of the indices of skewness and kurtosis (see O'Hagan and Leonard, 1976; Henze, 1986) these families of distributions can also be extended.

One would like do estimate the shape parameter λ and test hypothesis about it. In the skew-normal case, Azzalini (1985) has discussed the estimation problem in the more general case. For the other cases discussed in this paper this problem is under study. The testing problem has not been discussed – not even in the normal case.

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