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Quadratic forms in skew normal variates [☆]

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Abstract

In this paper first a characterization of the multivariate skew normal distribution is given. Then the joint moment generating functions of two quadratic forms, and a linear compound and a quadratic form in skew normal variates, have been derived and conditions for their independence are given. Distribution of the ratios of quadratic forms in skew normal variates has also been studied.

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1. Introduction

The multivariate skew normal distribution has been studied by Azzalini and Dalla Valle [2] and its applications are given in Azzalini and Capitanio [1]. This class of distributions includes the normal and has some properties like the normal and yet is skew. The random vector $\boldsymbol{Z}(p \times 1)$ is said to have a multivariate skew

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normal distribution if it is continuous and its probability density function (p.d.f.) is given by

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = 2\phi_p(\boldsymbol{z}; \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}'\boldsymbol{z}), \quad \boldsymbol{z} \in \Re^p, \tag{1}$$

where $\boldsymbol{\Omega} > 0$, $\boldsymbol{\alpha} \in \Re^p$, $\phi_p(\boldsymbol{z}; \boldsymbol{\Omega})$ is the p-dimensional normal p.d.f. with zero mean vector and correlation matrix $\boldsymbol{\Omega}$ and $\Phi(\cdot)$ is the standard normal cumulative distribution function (c.d.f.). We will denote it by $\boldsymbol{Z} \sim SN_p(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, to mean that the random vector $\boldsymbol{Z}$ has p-variate skew normal p.d.f. (1). The moment generating function (m.g.f.) of $\boldsymbol{Z}$ is

$$M(\boldsymbol{t}) = 2\exp\left\{\frac{1}{2}\boldsymbol{t}'\boldsymbol{\Omega}\boldsymbol{t}\right\}\Phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{t}}{(1+\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\right), \quad \boldsymbol{t} \in \Re^p. \tag{2}$$

The mean vector and the covariance matrix of $\boldsymbol{Z}$ are given by

$$\boldsymbol{\mu} = E(\boldsymbol{Z}) = \sqrt{\frac{2}{\pi}}\boldsymbol{\delta}, \tag{3}$$

$$\mathrm{Cov}(\boldsymbol{Z}) = \boldsymbol{\Omega} - \boldsymbol{\mu}\boldsymbol{\mu}', \tag{4}$$

where $\boldsymbol{\delta} = (1+\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{-1/2}\boldsymbol{\Omega}\boldsymbol{\alpha}$. Note that the mean vector given by Azzalini and Capitanio [1] is in error. However, it is a typo and does not affect the rest of that paper.

In this paper, we first give a characterization of the multivariate skew normal distribution in Section 2. Then in Section 3 we discuss the m.g.f. of a quadratic form in the noncentral case. In Section 4, independence of a linear compound and a quadratic form, and two quadratic forms are studied, and in Section 5 an application is given.

2. Characterization

In this section a characterization of the multivariate skew normal distribution is proved which is similar to the characterization of multivariate normal distribution. Azzalini and Capitanio [1] have stated that if $\boldsymbol{Z} \sim SN_p(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, then $\boldsymbol{h}'\boldsymbol{Z}$, $\boldsymbol{h} \in \Re^p$, is univariate skew normal. The converse of this result is also true as given in the theorem below.

Theorem 1. *Let the mean of $\boldsymbol{Z}$ be $\boldsymbol{\mu}$ and the covariance matrix be $\boldsymbol{\Sigma}$. Let $\boldsymbol{\Omega} = \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}'$. Suppose that for any $\boldsymbol{h} \in \Re^p$ such that $\boldsymbol{h}'\boldsymbol{\Omega}\boldsymbol{h} = 1$, $\boldsymbol{h}'\boldsymbol{Z}$ is univariate skew normal . Then $\boldsymbol{Z}$ is multivariate skew normal.*

Proof. For any $\boldsymbol{h} \in \Re^p$,

$$E(\boldsymbol{h}'\boldsymbol{Z}) = \boldsymbol{h}'\boldsymbol{\mu}, \qquad \mathrm{Cov}(\boldsymbol{h}'\boldsymbol{Z}) = \boldsymbol{h}'\boldsymbol{\Sigma}\boldsymbol{h}.$$

If $\boldsymbol{h}'\boldsymbol{\Omega}\boldsymbol{h} = 1$ and $\boldsymbol{h}'\boldsymbol{Z}$ is univariate skew normal with m.g.f. $2\exp\{t^2/2\}\Phi(\boldsymbol{\delta_h} t)$, $t \in R$, then

$$M_{\boldsymbol{h}'\boldsymbol{Z}}(t) = E\left(e^{t\boldsymbol{h}'\boldsymbol{Z}}\right) = M_{\boldsymbol{Z}}(t\boldsymbol{h}) = 2\exp\left\{t^2/2\right\}\Phi(\boldsymbol{\delta_h} t). \tag{5}$$

Note that $E(\boldsymbol{h}'\boldsymbol{Z}) = b\boldsymbol{\delta_h} = \boldsymbol{h}'\boldsymbol{\mu}$, where $b = \sqrt{2/\pi}$.

Let

$$\boldsymbol{\alpha} = \frac{\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}{\sqrt{b^2 - \boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}}. \tag{6}$$

We now prove that

$$\frac{\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{h}}{(1 + \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}} = \boldsymbol{\delta_h}. \tag{7}$$

This can be obtained by noting that

$$1 + \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha} = 1 + \frac{\boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega}\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}{b^2 - \boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}} = \frac{b^2}{b^2 - \boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}, \tag{8}$$

and

$$\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{h} = \frac{\boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}}{\sqrt{b^2 - \boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}}\boldsymbol{\Omega}\boldsymbol{h} = \frac{\boldsymbol{\mu}'\boldsymbol{h}}{\sqrt{b^2 - \boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}} = \frac{b\boldsymbol{\delta_h}}{\sqrt{b^2 - \boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}}. \tag{9}$$

Therefore (note that $\boldsymbol{h}'\boldsymbol{\Omega}\boldsymbol{h} = 1$),

$$M_{\boldsymbol{h}'\boldsymbol{Z}}(t) = 2\exp\left\{\frac{1}{2}t^2\boldsymbol{h}'\boldsymbol{\Omega}\boldsymbol{h}\right\}\Phi\left(\frac{\boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{h}t}{(1 + \boldsymbol{\alpha}'\boldsymbol{\Omega}\boldsymbol{\alpha})^{1/2}}\right). \tag{10}$$

Now set $t = 1$. Since the right-hand side of (10) is then the m.g.f. of $SN_p(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, the result is proved. □

3. M.G.F. of $(\boldsymbol{Z} - \boldsymbol{a})'A(\boldsymbol{Z} - \boldsymbol{a})$

In this section we derive the m.g.f. of the quadratic form $Q = (\boldsymbol{Z} - \boldsymbol{a})'A(\boldsymbol{Z} - \boldsymbol{a})$, $A' = A$. For this we need the following lemma (see Zacks [10, pp. 53–59]).

Lemma 1. *Let $\boldsymbol{U} \sim N_p(\boldsymbol{0}, \boldsymbol{\Omega})$. Then, for any scalar u and $\boldsymbol{v} \in \Re^p$, we have*

$$E\left[\Phi(u + \boldsymbol{v}'\boldsymbol{U})\right] = \Phi\left\{\frac{u}{(1 + \boldsymbol{v}'\boldsymbol{\Omega}\boldsymbol{v})^{1/2}}\right\}. \tag{11}$$

Theorem 2. *The m.g.f. of $Q = (\boldsymbol{Z} - \boldsymbol{a})'A(\boldsymbol{Z} - \boldsymbol{a})$, $\boldsymbol{a} \in \Re^p$, is given by*

$$M(t) = \frac{2\exp\{\boldsymbol{a}'[tA + 2t^2A(\boldsymbol{\Omega}^{-1} - 2tA)^{-1}A]\boldsymbol{a}\}}{|I - 2tA\boldsymbol{\Omega}|^{1/2}} \times \Phi\left[-\frac{2t\boldsymbol{\alpha}'(\boldsymbol{\Omega}^{-1} - 2tA)^{-1}A\boldsymbol{a}}{(1 + \boldsymbol{\alpha}'(\boldsymbol{\Omega}^{-1} - 2tA)^{-1}\boldsymbol{\alpha})^{1/2}}\right], \quad t \in \Re^1. \tag{12}$$

Proof. For $t \in \Re^1$, the m.g.f. of Q is

$$\begin{aligned} M(t) &= 2\int_{\Re^p} \exp\{t(\boldsymbol{z} - \boldsymbol{a})'A(\boldsymbol{z} - \boldsymbol{a})\}\phi_p(\boldsymbol{z}; \boldsymbol{\Omega})\Phi(\boldsymbol{\alpha}'\boldsymbol{z})\,d\boldsymbol{z} \\ &= \frac{2}{(2\pi)^{p/2}|\boldsymbol{\Omega}|^{1/2}} \int_{\Re^p} \exp\left\{-\frac{1}{2}\left(\boldsymbol{z}'\boldsymbol{\Omega}^{-1}\boldsymbol{z} - 2t(\boldsymbol{z} - \boldsymbol{a})'A(\boldsymbol{z} - \boldsymbol{a})\right)\right\} \\ &\qquad \times \Phi(\boldsymbol{\alpha}'\boldsymbol{z})\,d\boldsymbol{z} \\ &= \frac{2\exp\{\boldsymbol{a}'[tA + 2t^2A(\boldsymbol{\Omega}^{-1} - 2tA)^{-1}A]\boldsymbol{a}}{|I - 2tA\boldsymbol{\Omega}|^{1/2}} \\ &\qquad \times E_{\boldsymbol{U}}\left\{\Phi\left(-2t\boldsymbol{\alpha}'\left(\boldsymbol{\Omega}^{-1} - 2tA\right)^{-1}A\boldsymbol{a} + \boldsymbol{\alpha}'\left(\boldsymbol{\Omega}^{-1} - 2tA\right)^{-1/2}\boldsymbol{U}\right)\right\} \\ &= \frac{2\exp\{\boldsymbol{a}'[tA + 2t^2A(\boldsymbol{\Omega}^{-1} - 2tA)^{-1}A]\boldsymbol{a}\}}{|I - 2tA\boldsymbol{\Omega}|^{1/2}} \\ &\qquad \times \Phi\left[-\frac{2t\boldsymbol{\alpha}'(\boldsymbol{\Omega}^{-1} - 2tA)^{-1}A\boldsymbol{a}}{(1 + \boldsymbol{\alpha}'(\boldsymbol{\Omega}^{-1} - 2tA)^{-1}\boldsymbol{\alpha})^{1/2}}\right], \end{aligned}$$

where $\boldsymbol{U} \sim N_p(0, I)$, and the result follows from Lemma 1. □

3.1. Special cases

Case (i). $Q_1 = \boldsymbol{Z}'\boldsymbol{\Omega}^{-1}\boldsymbol{Z}$.

Substitute $\boldsymbol{a} = \boldsymbol{0}$, and $A = \boldsymbol{\Omega}^{-1}$ in (12) to get the m.g.f. of Q_1 as

$$M_1 = (1 - 2t)^{-p/2}, \quad t \in \Re^1. \tag{13}$$

Hence $\boldsymbol{Z}'\boldsymbol{\Omega}^{-1}\boldsymbol{Z} \sim \chi_p^2$. This result is also given by Proposition 7 of Azzalini and Dalla Valle [2]. A more general result can be found in Proposition 7 of Azzalini and Capitanio [1].

Case (ii). $Q_2 = \boldsymbol{Z}'A\boldsymbol{Z}$, where $A\boldsymbol{\Omega} = \mathrm{diag}(\delta_1, \ldots, \delta_p)$.

Substitute $\boldsymbol{a} = \boldsymbol{0}$, and $A\boldsymbol{\Omega} = \mathrm{diag}(\delta_1, \ldots, \delta_p)$ in (12) to get the m.g.f. of Q_2 as

$$M_2(t) = \prod_{j=1}^{p}(1 - 2t\delta_j)^{-1/2}, \quad t \in \Re^1. \tag{14}$$

Hence $\boldsymbol{Z}'A\boldsymbol{Z} \sim \sum_{j=1}^{p} \delta_j X_j$, where $X_j \sim \chi_1^2$, $j = 1, 2, \ldots, p$, are independently distributed (e.g. see Press [9]).

Case (iii). $Q_3 = (\boldsymbol{Z} - \boldsymbol{a})'\boldsymbol{\Omega}^{-1}(\boldsymbol{Z} - \boldsymbol{a})$.

Substitute $A = \boldsymbol{\Omega}^{-1}$ in (12) to get the m.g.f. of Q_3 as

$$M_3(t) = \frac{2\exp\{[t + 2t^2/(1-t^2)^p]\boldsymbol{a}'\boldsymbol{\Omega}^{-1}\boldsymbol{a}\}}{(1-2t)^{p/2}} \times \Phi\left[\frac{-2t}{(1-2t)^p} \frac{\boldsymbol{\alpha}'\boldsymbol{\alpha}}{(1 + \boldsymbol{a}'\boldsymbol{\Omega}\boldsymbol{a}/(1-2t)^p)^{1/2}}\right], \quad t \in \Re^1. \tag{15}$$

Case (iv). $Q_4 = \boldsymbol{Z}'A\boldsymbol{Z}$.

Substitute $\boldsymbol{a} = \boldsymbol{0}$ in (12) to get the m.g.f. of Q_4 as

$$M_4(t) = |I - 2tA\boldsymbol{\Omega}|^{-1/2}, \quad \boldsymbol{\Omega}^{-1} - 2tA > 0, \quad t \in \Re^1. \tag{16}$$

As $M_4(t)$ does not depend on $\boldsymbol{\alpha}$, hence the distribution of Q_4 is the same as in the usual multivariate normal case. Consequently, properties of Q_4 can be obtained by using known results for the usual multivariate normal case; see for instance Box [3] and the more general account in Chapter 29 of Johnson and Kotz [7].

From $M_4(t)$ we find the rth cumulant of $\boldsymbol{Z}'A\boldsymbol{Z}$ as

$$K_r = 2^{r-1}(r-1)!\,\mathrm{tr}(A\boldsymbol{\Omega})^r, \quad r = 1, 2, \ldots.$$

In particular $E(\boldsymbol{Z}'A\boldsymbol{Z}) = \mathrm{tr}\,A\boldsymbol{\Omega}$, and $\mathrm{Var}(\boldsymbol{Z}'A\boldsymbol{Z}) = 2\,\mathrm{tr}(A\boldsymbol{\Omega})^2$. These two moments are also given by Proposition 2 of Genton et al. [4].

4. Independence of linear forms and quadratic forms

In this section we study the conditions for determining when linear functions of skew normal variable are independent of a quadratic form of skew normal variables. We also give conditions when two quadratic forms are independent. Here $\boldsymbol{Z} \sim SN_p(\boldsymbol{\Omega}, \boldsymbol{\alpha})$, and we derive the joint m.g.f.'s for this purpose.

Theorem 3. *For $\boldsymbol{h} \in \Re^p$, the linear form $\boldsymbol{h}'\boldsymbol{Z}$ and the quadratic form $\boldsymbol{Z}'A\boldsymbol{Z}$ are independent if and only if $A\boldsymbol{\Omega}\boldsymbol{h} = \boldsymbol{0}$, and $A\boldsymbol{\Omega}\boldsymbol{\alpha} = \boldsymbol{0}$.*

Proof. The joint m.g.f. of $\boldsymbol{h}'\boldsymbol{Z}$ and $\boldsymbol{Z}'A\boldsymbol{Z}$ is

$$M(t_1, t_2) = 2\int_{\Re^p} \frac{\exp\left\{-\frac{1}{2}[z'\boldsymbol{\Omega}^{-1}z - 2t_1\boldsymbol{h}'z - 2t_2 z'Az]\right\}}{(2\pi)^{p/2}|\boldsymbol{\Omega}|^{1/2}} \Phi(\boldsymbol{\alpha}'z)\,dz$$

$$= \frac{2\exp\{\frac{1}{2}t_1^2\boldsymbol{h}'(\boldsymbol{\Omega}^{-1}-2t_2A)^{-1}\boldsymbol{h}\}}{|I-2t_2A\boldsymbol{\Omega}|^{1/2}} \times E_{\boldsymbol{U}}\Phi\big[t_1\boldsymbol{\alpha}'\big(\boldsymbol{\Omega}^{-1}-2t_2A\big)^{-1}\boldsymbol{h}+\boldsymbol{\alpha}'\big(\boldsymbol{\Omega}^{-1}-2t_2A\big)^{-1/2}\boldsymbol{U}\big]$$
$$= \frac{2\exp\{\frac{1}{2}t_1^2\boldsymbol{h}'(\boldsymbol{\Omega}^{-1}-2t_2A)^{-1}\boldsymbol{h}\}}{|I-2t_2A\boldsymbol{\Omega}|^{1/2}} \times \Phi\left[t_1\frac{\boldsymbol{\alpha}'(\boldsymbol{\Omega}^{-1}-2t_2A)^{-1}\boldsymbol{h}}{(1+\boldsymbol{\alpha}'(\boldsymbol{\Omega}^{-1}-2t_2A)^{-1}\boldsymbol{\alpha})^{1/2}}\right], \quad t_1, t_2 \in \Re^1, \tag{17}$$

where $\boldsymbol{U} \sim N_p(\mathbf{0}, I)$ and the last step is obtained using Lemma 1.

Now note that

$$\big(\boldsymbol{\Omega}^{-1}-2t_2A\big)^{-1} = \boldsymbol{\Omega}\sum_{j=0}^{\infty}(2t_2)^j(A\boldsymbol{\Omega})^j, \tag{18}$$

for $\|2\operatorname{tr} A\boldsymbol{\Omega}\| < 1$ where $\|\cdot\|$ is a matrix norm. Hence the expansion (18) is always valid in the neighborhood of $t_2 = 0$ (see Horn and Johnson [6, p. 301]). Finally from (17) and (18) it follows that the necessary and sufficient conditions for independence are $A\boldsymbol{\Omega}\boldsymbol{h} = \mathbf{0}$ and $A\boldsymbol{\Omega}\boldsymbol{\alpha} = \mathbf{0}$. This completes the proof. □

Theorem 4. *The quadratic forms $\boldsymbol{Z}'A\boldsymbol{Z}$ and $\boldsymbol{Z}'B\boldsymbol{Z}$ are independent if and only if $A\boldsymbol{\Omega}B = \mathbf{0}$.*

Proof. The joint m.g.f. of $\boldsymbol{Z}'A\boldsymbol{Z}$ and $\boldsymbol{Z}'B\boldsymbol{Z}$ is

$$M(t_1,t_2) = 2\int_{\Re^p}\frac{\exp\{-\frac{1}{2}[z'\boldsymbol{\Omega}^{-1}z-2t_1z'Az-2t_2z'Bz]\}}{(2\pi)^{p/2}|\boldsymbol{\Omega}|^{1/2}}\Phi(\boldsymbol{\alpha}'z)\,dz$$
$$= \frac{2}{|I-2t_1A\boldsymbol{\Omega}-2t_2B\boldsymbol{\Omega}|^{1/2}} \times E_{\boldsymbol{Y}}\Phi\big[\boldsymbol{\alpha}'\big(\boldsymbol{\Omega}^{-1}-2t_1A-2t_2B\big)^{-1/2}\boldsymbol{Y}\big]$$
$$= \big|I-2(t_1A+t_2B)\boldsymbol{\Omega}\big|^{-1/2}, \quad t_1,t_2 \in \Re^1, \tag{19}$$

where $\boldsymbol{Y} \sim N_p(\mathbf{0}, I)$. The last step is obtained by using Lemma 1. Now from (19) for independence we get the condition $A\boldsymbol{\Omega}B = \mathbf{0}$. □

Corollary 1. *The quadratic forms $\boldsymbol{Z}'A_i\boldsymbol{Z}$, $i = 1,\ldots,n$, are mutually independent if $A_i\boldsymbol{\Omega}A_j = \mathbf{0}$, $i \neq j$.*

Note that the conditions for independence in Corollary 1 weaken those required by Proposition 8 of Azzalini and Capitanio [1].

5. An application

Applications of quadratic forms in normal random variables are given in Johnson and Kotz [8], and Gupta and Nagar [5]. Recently an application comes from the time series context and is given by Genton et al. [4]. Let $X_1, X_2, \ldots, X_n$ denote a series of observations from $SN_k(\boldsymbol{\Omega}, \boldsymbol{\alpha})$. Then the sample serial covariance of lag-k is

$$c_k^{(n)} = \frac{1}{n} \sum_{i=1}^{n-k} \bigl(X_i - \bar{X}\bigr)\bigl(X_{i+k} - \bar{X}\bigr), \quad k = 1, 2, \ldots, n-1,$$

where $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ and n denotes the length of the series under observation. Further letting $\boldsymbol{X} = (X_1, \ldots, X_n)'$, $\boldsymbol{\varepsilon} = (1, \ldots, 1)'$, $V = (I_n - \frac{1}{n}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}')$, $A_k = V C_k V$, where $C_k(n \times n)$ is a null matrix except for values $1/(2n)$ everywhere on the kth upper and lower diagonal, we can write

$$c_k^{(n)} = \boldsymbol{X}' A_k \boldsymbol{X}.$$

The m.g.f. of $c_k^{(n)}$ is given as a special case by (16).

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