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ON CERTAIN PROBLEMS INVOLVING ORDER STATISTICS - A UNIFIED APPROACH THROUGH ORDER STATISTICS PROPERTY OF POINT PROCESSES*

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SUMMARY. It is known there are some similarities between order statistics and record values, moreover, when viewed as point processes, the two processes both share the order statistics property. These motivated us to investigate some intrinsic properties within the class of order statistics point processes, which in turn lead to some useful characterizations by using certain conditional moment of the spacings of the jump times. Our results explain not only why order statistics and record values have parallel characterizations when using the backward conditional expectations, but also generalize several existing characterization results.

1. **Introduction**

It is known that record values and order statistics are closely related. As mentioned in Nagaraja (1988a), record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations. On the other hand, based on the fact that exponential distribution can be characterized respectively by using the independence of certain functions of record values and similar functions of order statistics, Deheuvels (1984) and Gupta (1984) established some relationships between the conditional distribution of forward record values and that of forward order statistics. Their results enable us to understand why there are many parallel characterizations of

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exponential distribution, by using some properties of constant regression of the spacings of forward record values and the spacings of forward order statistics. Excellent reviews of the properties and characterizations related to record values and order statistics can be found in books by Arnold *et al.* (1992) and Rao and Shanbhag (1994). In this work, we will look at the problem from another point of view. Note that when record values and order statistics are viewed as point processes, the two processes both share the order statistics property (defined in the next section). These motivated us to investigate some instrinsic properties within the class of order statistics point processes in order to see why record values and order statistics have similar characterizations.

Now we introduce the formal definitions of record values and sample processes. Let $\{W_i, i \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables having a continuous distribution function H with $H(0) = 0$. Define the sequence of record times $\{L(n), n \geq 1\}$ by $L(1) = 1$ and $L(n) = \min\{j | W_j > W_{L(n-1)}\}$, $n \geq 2$. Let $Y_n = W_{L(n)}$, then $\{Y_n, n \geq 1\}$ is called the sequence of record values corresponding to $\{W_i, i \geq 1\}$. Also denote $N(t)$ as the number of $Y_n \leq t, t \geq 0$. Shorrock (1972) proved that the point process $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with mean value function $E(N(t)) = -\ln(1 - H(t))$. On the other hand, for every $n \geq 1$, denote $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$, as the order statistics from a random sample $X_1, X_2, ..., X_n$ having distribution function F. Then $\{M_n(t), t \geq 0\}$, where $M_n(t)$ is the number of $X_{k,n} \leq t$, $1 \leq k \leq n$, is called the sample process generated by n and F.

Huang and Li (1993) gave the following result. Let G be a monotone function and $j \geq 1, k \geq 0$, be two fixed integers. Under certain conditions, if

$$E(G(Y_{j+k+1} - Y_{j+k}) | Y_j = x) = \text{constant}, \qquad \ldots (1.1)$$

or

$$E(G(Y_{j+k+2} - Y_{j+k}) | Y_j = x) = \text{constant}, \qquad \ldots (1.2)$$

then H is exponential. On the other hand, Gupta (1984) showed

$$E((X_{j+1,n} - X_{j,n})^r | X_{j,n} = x) = \text{constant}, \qquad \ldots (1.3)$$

if and only if F is exponential. It also can be shown that for the order statistics, we can use conditions similar to (1.1) or (1.2) to characterize F to be exponential.

For a sequence of order statistics, if F is exponential, then the sequence of spacings $\{X_{i+1,n} - X_{i,n}\}$ are independent yet are not identically distributed. In fact independence of the spacings is a characteristic property of the exponential distribution (see Ferguson (1967)), while identically distributed property of the spacings implies F is uniform (see Huang *et al.* (1979)). It can be seen that for the order statistics case, conditions such as (1.3) or similar to (1.1) or (1.2) are weaker than the independence assumption of the spacings. In Theorem 7, we will give a result based on the forward conditional expectations, yet the condition used there originates from the identically distributed property of the spacings.

Regarding the backward conditional expectations, for record values, there are some results which also use conditions in the spirit of identically distributed property. For example, Huang and Li (1993) characterized H by some properties based on the conditional expectations of $Y_i - Y_{i-1}$ given Y_j, where $i \leq j$, or $N(t)$. On the other hand, the following are results based on the conditional expectations of a sequence of order statistics. Ferguson (1967) characterized F using the condition that

$$E(X_{i,n}|X_{i+1,n} = x) = ax + b, \qquad \ldots (1.4)$$

where a and b are constants and $i+1 \leq n$. Beg and Kirmani (1974) characterized F through the condition

$$E(X_1|X_{n,n} = x) = ax + b. \qquad \ldots (1.5)$$

Beg and Balasubramanian (1990) studied a similar problem through the following property

$$E\left(\frac{1}{j-1}\sum_{i=1}^{j-1} g(X_{i,n})|X_{j,n} = x\right) = \frac{g(x) + g(\alpha+)}{2}, \quad \forall\, x \in (\alpha, \beta), \qquad \ldots (1.6)$$

where g is a suitable function and $2 \leq j \leq n$. Das Gupta *et al.* (1993) characterized F to be uniformly distributed by

$$E(X_1|X_{1,n} = x, X_{n,n} = y) = \frac{1}{2}(x + y). \qquad \ldots (1.7)$$

In contrast to the case of forward conditional expectations, the similarity between order statistics and record values regarding backward conditional expectations, has seldom been considered. Nagaraja (1988b) pointed out that for every $n \geq 1$, given Y_{n+1}, Y_n behaves like the sample maximum from certain distribution. Using this, Nagaraja (1988b) gave parallel results for characterizations based on some properties of regressions of adjacent order statistics and record values. Actually it is known when H is exponential, $\{N(t), t \geq 0\}$ becomes a homogeneous Poisson process. Hence given $N(t) = n, Y_1, \cdots, Y_n$ are distributed as the order statistics of n i.i.d. random variables with the common $\mathcal{U}[0,t]$ distribution. From this it can be seen that if there is a characterization for H to be exponentially distributed in the case of record values, there will be a corresponding characterization for F to be uniformly distributed in the case of order statistics. Furthermore, inspired by the fact that both sample processes and nonhomogeneous Poisson processes have the order statistics property, in this paper first we will investigate properties for the order statistics point processes. Then we establish some characterizations using certain conditional moment of the spacings of the jump times within the class of order statistics point processes. Many of the characterization results in the literature for the order statistics or record values are immediate consequences of our theorems. In particular, our results can be applied to characterize uniform distribution, for the sequence of order statistics, and exponential distribution, for the sequence of record values, respectively.

2. **Order Statistics Property**

Let $\{A(t), t \geq 0\}$ with $A(0) = 0, A(t) < \infty, \forall\ t \geq 0$, be a point process with right continuous sample paths having successive unit steps at times $S_1, S_2, ...$ For convenience, let $S_0 = 0$. The process $\{A(t), t \geq 0\}$ is said to have the order statistics property (and $\{A(t), t \geq 0\}$ is called an order statistics point process) if for every $t > 0$ and integer $k \geq 1$, whenever $P(A(t) = k) > 0$, given $A(t) = k$, the successive jump times $(S_1, ..., S_k)$ are distributed as the order statistics of k i.i.d. random variables with distribution function $F_t(\cdot)$ supported on $[0, t]$.

Properties and characterizations of point processes with the order statistics property have been studied by Nawrotzki (1962), Westcott (1973), Crump (1975), Kallenberg (1976), Feigin (1979) and Puri (1982). Among other results, it was proved by the above authors that the order statistics point processes are Markovian, and for every $t > 0$, the associated distribution function $F_t(\cdot)$ is continuous, $F_t(x) = E(A(x))/E(A(t)), \forall\ 0 \leq x \leq t$, if $E(A(t)) < \infty$. Also the class of order statistics point processes with $E(A(t)) < \infty$, $\forall t > 0$, is characterized by mixed Poisson processes (up to a time-scale transformation) if $\lim_{t\to\infty} E(A(t)) = \infty$, or mixed sample processes if $\lim_{t\to\infty} E(A(t))$ is finite. Here $\{A(t), t \geq 0\}$ is called the mixed sample process based on Z and F, if Z is a random variable taking values in the non-negative integers, $F(\cdot)$ is a continuous distribution function with $F(0) = 0$, and

$$A(t) = \begin{cases} 0\ , & \text{if}\ \ Z = 0\ , \\ \#\{j | T_j \leq t, j = 1, ..., k\}, & \text{if}\ \ Z = k \geq 1, \end{cases}$$

where $T_j, j = 1, ..., k$, are i.i.d. random variables with F being their common distribution function.

Note that when $\{A(t), t \geq 0\}$ has the order statistics property and associated with the distribution function $F_t(\cdot)$, it is easy to show that for any $0 < t_1 < t_2$ and integer $k \geq 1$, whenever $P(A(t_2) - A(t_1) = k) > 0$, given $A(t_2) - A(t_1) = k$, the successive k jump times in the interval $(t_1, t_2]$ are also distributed as the order statistics of k i.i.d. random variables with distribution function $(F_{t_2}(\cdot) - F_{t_2}(t_1))/(1 - F_{t_2}(t_1))$ supported on $(t_1, t_2]$. Using this, the concept of the order statistics property can be extended to the point processes defined in $(-\infty, \infty)$. Yet as the number of points in $(-\infty, t]$ may not be finite, instead of considering the jumps happened before time t, we restrict our attention to the consideration of jumps in the interval $(t_1, t_2], t_1 < t_2$ (and denoting this by $A(t_1, t_2]$).

We now give a simple lemma which indicates that for an order statistics point process, the sequence of jump times $\{S_n, n \geq 1\}$ also forms a sequence with order statistics property. In the following for any two random vectors $\mathbf{X}$ and $\mathbf{Y}$, let $P_{\mathbf{X}|\mathbf{Y}}$ denote the conditional distribution of $\mathbf{X}$ given $\mathbf{Y}$. Part (i) of the following lemma is a consequence of part (iii), as it will be used often later, we still state it.

Lemma 1. Assume $\{A(t), t \geq 0\}$ has the order statistics property. Then

(i) for every $t > 0$ and integer $n \geq 1$, whenever $P(t - \delta < S_{n+1} \leq t + \delta) > 0$, $\forall \delta > 0$,

$$P_{S_1,\ldots,S_n|S_{n+1}=t} = P_{S_1,\ldots,S_n|A(t)=n},$$

(ii) for every $s > 0$ and integers $k \geq 1$, $n \geq k + 1$, whenever $P(s - \delta < S_k \leq s + \delta) > 0$, $\forall \delta > 0$,

$$P_{S_{k+1},\ldots,S_n|S_k=s} = P_{S_{k+1},\ldots,S_n|A(s)=k},$$

(iii) for every $0 < s < t$ and integers $k \geq 1$, $n \geq k + 1$, whenever $P(s - \delta_1 < S_k \leq s + \delta_1, t - \delta_2 < S_{n+1} \leq t + \delta_2) > 0$, $\forall \delta_1, \delta_2 > 0$,

$$P_{S_{k+1},\ldots,S_n|S_k=s,S_{n+1}=t} = P_{S_{k+1},\ldots,S_n|A(s)=k,A(t)=n} = P_{S_1,\ldots,S_{n-k}|A(s)=0,A(t)=n-k}.$$

Proof. We only prove part (i), the other two parts can be proved similarly. For every $0 < t_1 < t_2 < \cdots < t_n < t$, let C be the event

$$C = \{t_i < S_i \leq t_i + dt_i, i = 1, 2, \ldots, n\}.$$

Then

$$\begin{aligned} P(C|S_{n+1} = t) &= \lim_{\delta \downarrow 0} P(C|t - \delta < S_{n+1} \leq t + \delta) \\ &= \lim_{\delta \downarrow 0} P(C|A(t - \delta) = n, A(t + \delta) = n + 1) \\ &= \lim_{\delta \downarrow 0} P(C|A(t - \delta) = n) \\ &= P(C|A(t) = n), \end{aligned} \qquad \ldots (2.1)$$

where the first equality is due to the definition of the conditional probability, the second equality is obvious, the third equality is because $\{A(t), t \geq 0\}$ is Markovian, and the last equality is due to the fact that the associated distribution function $F_t(\cdot)$ is continuous.

From part (iii) of the above lemma, we find that given $A(s) = k$ and $A(t) = n$, the conditional distribution of $S_{k+1}, \ldots, S_n$ depends on k and n only through $n - k$, namely the difference of $A(t)$ and $A(s)$.

3. Characterizations by the Conditional Expectations of the Spacings for the Order Statistics Point Processes

Suppose that $\{Y_i, i \geq 1\}$ forms the sequence of upper record values from a sequence of i.i.d. random variables $\{W_i, i \geq 1\}$ having a continuous distribution function H with $H(0) = 0$ and $H(x) < 1, \forall\, x > 0$. Then as mentioned it before, $N(t) = \#\{i | Y_i \leq t\}$ is a nonhomogeneous Poisson process with $E(N(t)) = -\ln(1 - H(t))$. By convention Y_0 is defined to be 0.

Kirmani and Gupta (1989) and Huang and Li (1993) proved that $E(N(t))$ is a linear function of t, or equivalently H is an exponential distribution function, by using the equality of the conditional expectations of the spacings such as

$$E(G(Y_i - Y_{i-1}) | N(t) = n) = E(G(Y_{i-1} - Y_{i-2}) | N(t) = n),\ \forall\, t > 0, \quad \ldots (3.1)$$

for some integers $2 \leq i \leq n$, or

$$E(G(Y_1) | N(t) = n) = E(G(t - Y_n) | N(t) = n),\ \forall\, t > 0, \quad \ldots (3.2)$$

for some integer $n \geq 1$, where G is a non-decreasing function such that for any $x > 0$, G has a point of increase in $(0, x)$.

In this section, we will extend the above results. Within the more general order statistics point processes class (recall that nonhomogeneous Poisson process has the order statistics property), we present some similar characterizations. Throughout this section, let $\{A(t), t \geq 0\}$ be an order statistics point process with $E(A(t)) = m(t) < \infty, t \geq 0$. Also let $\{S_i, i \geq 1\}$ denote the sequence of successive jump times of $\{A(t), t \geq 0\}$. Again S_0 is defined to be 0.

Theorem 1. *Let G be a non-decreasing function such that for any $x > 0$, G has a point of increase in $(0, x), 0 \leq \eta \leq \infty$.*

(i) *Assume $m'(t) > 0$, $\forall\, 0 < t < \eta$ and $m'(0+)$ exists. Also assume for some fixed integers $2 \leq j \leq n$,*

$$E(G(S_j - S_{j-1}) | A(t) = n) = E(G(S_1) | A(t) = n), \quad \forall\, 0 < t < \eta. \quad \ldots (3.3)$$

Then $m(t) = \lambda t$, $\forall\, 0 < t < \eta$, where $\lambda = m'(0+)$.

(ii) *Assume $m'(\cdot)$ is positive and continuous in $(0, \eta)$, and for some fixed integers $2 \leq j \leq n$,*

$$E(G(S_j - S_{j-1}) | A(t) = n) = E(G(S_{j-1} - S_{j-2}) | A(t) = n), \quad \forall\, 0 < t < \eta. \quad \ldots (3.4)$$

Then $m(t) = \lambda t$, $\forall\, 0 < t < \eta$, where $\lambda = m'(0+)$.

(iii) *Assume $m'(0+)$ exists and for some fixed integer $n \geq 1$,*

$$E(G(t - S_n) | A(t) = n) = E(G(S_1) | A(t) = n), \quad \forall\, 0 < t < \eta, \quad \ldots (3.5)$$

whenever $P(A(t) = n) > 0$. Then $m(t) = \lambda t$, $\forall\, 0 < t < \eta$, where $\lambda = m'(0+)$.

(iv) *Assume* $m(\cdot)$ *is positive with a continuous derivative in* $(0,\eta)$, *and for some fixed integer* $n \geq 1$,

$$E(G(t-S_n)|A(t)=n) = E(G(S_n-S_{n-1})|A(t)=n), \quad \forall\ 0<t<\eta, \quad \ldots (3.6)$$

whenever $P(A(t)=n)>0$. *Then* $m(t)=\lambda t$, $\forall\ 0<t<\eta$, *where* $\lambda = m^{'}(0+)$.

Proof. As the other parts can be proved similarly we only prove part (i). By the order statistics property, we have

$$\begin{aligned} &E(G(S_j-S_{j-1})|A(t)=n) \\ &= \textstyle\int_0^t P(S_j-S_{j-1}>u|A(t)=n)dG(u) \\ &= \tfrac{n!}{(j-2)!(n-j)!}(m(t))^{-n} \\ &\quad \cdot \textstyle\int_0^t \int_u^t \int_0^{v-u} m^{j-2}(w)(m(t)-m(v))^{n-j}m'(v)m'(w)dwdvdG(u). \end{aligned} \quad \ldots (3.7)$$

Similarly,

$$E(G(S_1)|A(t)=n) = n(m(t))^{-n}\int_0^t\int_u^t (m(t)-m(v))^{n-1}m'(v)dvdG(u). \quad \ldots (3.8)$$

Therefore, (3.3) is equivalent to

$$\begin{aligned} &\tfrac{(n-1)!}{(j-2)!(n-j)!}\textstyle\int_0^t\int_u^t\int_0^{v-u} m^{j-2}(w)(m(t)-m(v))^{n-j}m'(v)m'(w)dwdvdG(u) \\ &= \textstyle\int_0^t\int_u^t (m(t)-m(v))^{n-1}m'(v)dvdG(u)\ . \end{aligned} \quad \ldots (3.9)$$

Differentiating both sides of (3.9) $(n-j+1)$ times with respect to t, yields

$$\int_0^t\int_0^{t-u} m^{j-2}(w)m'(w)dwdG(v) = \int_0^t\int_u^t (m(t)-m(v))^{j-2}m'(v)dvdG(u)\ , \quad \ldots (3.10)$$

which is equivalent to

$$\int_0^t m^{j-1}(t-u)dG(u) = \int_0^t (m(t)-m(u))^{j-1}dG(u)\ . \quad \ldots (3.11)$$

Now (3.11) has the same form as (2.5) of Huang and Li (1993). Hence as in the proof of Theorem 1 of Huang and Li (1993), we obtain $m(t)=\lambda t$, $\forall\ 0<t<\eta$, where $\lambda = m'(0+)$. This completes the proof.

Note that by (i) of Lemma 1, the conditions (3.3)-(3.6) of Theorem 1 can be replaced by

$$E(G(S_j-S_{j-1})|S_{n+1}=t) = E(G(S_1)|S_{n+1}=t),\ \forall\ 0<t<\eta, \quad \ldots (3.12)$$

$$E(G(S_j - S_{j-1})|S_{n+1} = t) = E(G(S_{j-1} - S_{j-2})|S_{n+1} = t),\ \forall\ 0 < t < \eta, \quad \ldots (3.13)$$

$$E(G(S_{n+1} - S_n)|S_{n+1} = t) = E(G(S_1)|S_{n+1} = t),\ \forall\ 0 < t < \eta, \quad \ldots (3.14)$$

and

$$E(G(S_{n+1} - S_n)|S_{n+1} = t) = E(G(S_n - S_{n-1})|S_{n+1} = t),\ \forall\ 0 < t < \eta, \quad \ldots (3.15)$$

respectively. Thus, by using of Lemma 1, most of the results in this paper have two versions. One is conditional on the number of jumps at some time (such as given $A(t) = n$), the other is conditional on some jump time (such as given $S_{n+1} = t$). In the following we will only state the first version of the results.

If $\{A(t), t \geq 0\}$ is a nonhomogeneous Poisson process which satisfies any one of the conditions (i)-(iv) of Theorem 1 with $\eta = \infty$, then $E(A(t)) = \lambda t, \forall\ t \geq 0$, for some $\lambda > 0$. Consequently, $\{A(t), t \geq 0\}$ is a homogeneous Poisson process (this corresponds to H is exponentially distributed for the case of record values). Hence Theorem 1 is indeed an extension of Theorems 1, 7 and 8 of Huang and Li (1993). On the other hand, let $\{M_n(t), t \geq 0\}$ be the (non-mixed) sample process generated by a distribution function F ($Z \equiv n$ in this case), where F is assumed to be absolutely continuous with $F(0) = 0$ and $F(\eta) = 1$, where $\eta \leq \infty$. For this order statistics point process, Puri (1982, p.42) showed that the associated distribution function is given by

$$F_t(x) = \frac{m(x)}{m(t)} = \frac{F(x)}{F(t)}\ , 0 \leq x \leq t\ , \quad \ldots (3.16)$$

for $0 < t < \eta$. Now since $m(t) = nF(t), \forall\ 0 < t < \eta, m(t) = \lambda t, \forall\ 0 < t < \eta$, implies $\eta < \infty$ and $F(t)$ is also linear in $(0, \eta)$. Therefore, for the sequence of order statistics $\{X_{j,n}, 1 \leq j \leq n\}$, under suitable conditions, one of the equalities (3.3)-(3.6) implies F has a uniform distribution in $(0, \eta)$. In fact some results related to order statistics in the literature (e.g. Huang *et al.* (1979), Shimizu and Huang (1983), Nagaraja (1988b)) will become the immediate consequences of Theorem 1 or some theorems given later.

Theorem 2 states that given $A(t) = n$, the conditional mean function of S_n can determine the mean function of $A(t)$. The proof is standard and is omitted.

THEOREM 2. *Assume for some fixed integer* $n \geq 1$,

$$E(S_n|A(t) = n) = g(t)\ , \forall\ 0 < t < \eta, \quad \ldots (3.17)$$

whenever $P(A(t) = n) > 0$, *where* $0 < \eta \leq \infty$. *Also assume* $(t - g(t))^{-1}$ *is integrable. Then* $m(t) = (t - g(t))^{-\frac{1}{n}} e^{\int \frac{1}{n}(t-g(t))^{-1} dt}$, $\forall\ 0 < t < \eta$.

Of course, in the above theorem, in order that there are solutions for $\{A(t), t \geq 0\}$, g must be a function such that $m(\cdot)$ is non-decreasing and non-negative in $(0, \infty)$.

On the other hand, the following Theorem 3 is parallel to Theorem 4 of Huang and Li (1993), where nonhomogeneous Poisson is considered; Theorem 4 uses more general condition (3.19) to determine the mean function $m(t)$. Both theorems can be proved along the lines of Theorems 4 and 5 of Huang and Li (1993), respectively.

Theorem 3. *Let G be a non-constant and non-decreasing differentiable function with $G(t) > 0, \forall\ 0 < t < \eta$, where $0 < \eta \le \infty$. Also assume $m(\cdot)$ is positive and differentiable in $(0, \eta)$. If for some fixed integer $n \ge 1$,*

$$E(G(S_n)|A(t) = n) = cG(t),\ \forall\ 0 < t < \eta, \qquad \ldots(3.18)$$

whenever $P(A(t) = n) > 0, c > 0$ is a constant, then
(i) $0 < c < 1$;
(ii) $m(t) = \lambda(G(t))^{c/[n(1-c)]}, 0 < t < \eta$, *for some constant* $\lambda > 0$.

Theorem 4. *Let G and m satisfy the conditions in Theorem 3. If for some fixed integers $0 \le l < m \le n-1$ and $k = n, n-1$,*

$$E(G(S_m - S_l)|A(t) = k) = c_k G(t),\ \forall\ 0 < t < \eta, \qquad \ldots(3.19)$$

where c_n and c_{n-1} are positive constants, then
(i) $c_{n-1} > c_n$,
(ii) $m(t) = \lambda(G(t))^{c_n/[n(c_{n-1}-c_n)]}$, $\forall\ 0 < t < \eta$, *for some constant* λ.

For the current life $t - S_{A(t)}$, we do not have result which is as general as Theorem 3. Yet we have the following partial result.

Theorem 5. *Assume $m(\cdot)$ is differentiable in $(0, \eta)$, where $0 < \eta \le \infty$. If for some fixed integer $n \ge 1$,*

$$E((t - S_n)^2|A(t) = n) = at^2,\ \forall\ 0 < t < \eta, \qquad \ldots(3.20)$$

where $0 < a < 1$ is a constant, then $m(t) = \lambda t^{h(a)}, \forall\ 0 < t < \eta$, where $h(a) = (-3a + \sqrt{a^2 + 8a})/(2an)$ and λ is a constant. In particular, if $a = 2/[(n+1)(n+2)]$, then $m(t) = \lambda t,\ \forall\ 0 < t < \eta$.

4. Characterizations by the Conditional Expectations based on Order Statistics

Let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the order statistics based on n (fixed) samples $X_1, ..., X_n$ having distribution F. Here we do not assume that X_i is non-negative, that is $F(0)$ may not be equal to zero. Under certain conditions, Das Gupta *et al.* (1993) proved that for some $n \ge 1$, the property

$$E(X_1|X_{1,n}, X_{n,n}) = \frac{1}{2}(X_{1,n} + X_{n,n}) \qquad \ldots(4.1)$$

implies F has a uniform distribution on some interval. Das Gupta *et al.* (1993) also pointed out that (4.1) is equivalent to

$$E(\frac{1}{n-2}\sum_{i=2}^{n-1} X_{i,n}|X_{1,n}, X_{n,n}) = \frac{1}{2}(X_{1,n} + X_{n,n}) \ . \qquad \ldots (4.2)$$

In this section we will discuss some similar characterizations within the class of order statistics point processes defined in the real line.

As mentioned in Section 1, for each interval $(t_1, t_2]$ in the real line, let $A(t_1, t_2]$ denote the number of jumps in $(t_1, t_2]$, and $m(t_1, t_2] = E(A(t_1, t_2])$. The point process $A = \{A(t_1, t_2], t_1 < t_2\}$ is said to have the order statistics property if for every $t_1 < t_2$ and integer $n \geq 1$,whenever $P(A(t_1, t_2] = n) > 0$, given $A(t_1, t_2] = n$, the successive jump times $S_{(t_1,t_2],i}, i = 1, ..., n$, are distributed as the order statistics of n i.i.d. random variables with distribution function $F_{(t_1,t_2]}(\cdot)$ supported on $(t_1, t_2]$. It can be shown easily that

$$F_{(t_1,t_2]}(x) = m(t_1, x]/m(t_1, t_2] \ , t_1 < x < t_2 \ . \qquad \ldots (4.3)$$

Let $K(m)$ denote the support of m, that is

$$K(m) = \{x|x \in R, m(x - \epsilon, x + \epsilon] > 0 \ , \forall \ \epsilon > 0\} \ . \qquad \ldots (4.4)$$

We now give the following theorem which is a generalization of Theorem 3.2 of Das Gupta *et al.* (1993).

THEOREM 6. *Assume A has the order statistics property in R, and for any $t_1 < t_2$, $m(t_1, t_2]$ and $\int_{t_1}^{t_2} udm(t_1, u]$ both are finite. For some fixed positive integer n, let*

$$H(t_1, t_2) = E\left(\frac{1}{n}\sum_{i=1}^{n} S_{(t_1,t_2],i}\middle| A(t_1, t_2] = n\right), \forall \ t_1, t_2 \in K(m) \text{ and } t_1 < t_2 \ . \qquad \ldots (4.5)$$

Then $m(t_1, t_2] = (t_2 - H(t_1, t_2))^{-1} e^{\int (t_2 - H(t_1,t_2))^{-1} dt_2}, \forall \ t_1, t_2 \in K(m), \ t_1 < t_2$. Also if t_1 or $t_2 \in R \backslash K(m)$, then $m(t_1, t_2]$ can be determined by using that $m(t_1, t]$ is continuous and non-decreasing in t.

PROOF. For any $t_1, t_2 \in K(m), t_1 < t_2$, given $A(t_1, t_2] = n$, the successive jump times $S_{(t_1,t_2],i}, i = 1, ..., n$, are distributed as the order statistics of n i.i.d. random variables with distribution function $m(t_1, x]/m(t_1, t_2], t_1 < x < t_2$. Then

$$H(t_1, t_2) = \int_{t_1}^{t_2} \frac{u}{m(t_1, t_2]} \ dm(t_1, u] \ . \qquad \ldots (4.6)$$

By the integration by parts, (4.6) implies

$$\frac{\int_{t_1}^{t_2} m(t_1, u]du}{m(t_1, t_2]} = t_2 - H(t_1, t_2) \ . \qquad \ldots (4.7)$$

Solving (4.7), yields

$$m(t_1, t_2] = (t_2 - H(t_1, t_2))^{-1} e^{\int (t_1 - H(t_1,t_2))^{-1} dt_2} . \qquad \ldots (4.8)$$

This completes the proof.

In the above theorem, if $m(-\infty, t]$ or $m(t, \infty)$ is finite, then (4.5) can be replaced by

$$H_1(t) = E\left(\frac{1}{n} \sum_{i=1}^{n} S_{(-\infty,t],i} \middle| A(-\infty, t] = n \right), \qquad \ldots (4.9)$$

$\forall\ t \in K(m)$, or

$$H_2(t) = E\left(\frac{1}{n} \sum_{i=1}^{n} S_{(t,\infty),i} \middle| A(t, \infty) = n \right), \qquad \ldots (4.10)$$

$\forall\ t \in K(m)$, respectively. That is $m(-\infty, t]$ and $m(t, \infty)$ can be determined by using $H_1(\cdot)$ and $H_2(\cdot)$, respectively. As an application of Theorem 6, consider the order statistics as defined in the beginning of this section and let $S(F)$ denote the support of F. Then we have

Corollary 1. (i) *Assume* $\int_{x_1}^{x_2} u dF(u) < \infty$, $\forall x_1, x_2 \in S(F)$ *and* $x_1 < x_2$. *Then for any integers* $0 \le m_1 < m_2 \le n$ *and* $m_2 - m_1 \ge 2$, *the function*

$$\begin{aligned} P(x_1, x_2) = \ & E\left(\tfrac{1}{m_2 - m_1 - 1} \textstyle\sum_{i=m_1+1}^{m_2-1} X_{i,n} \middle| X_{m_1,n} = x_1, X_{m_2,n} = x_2 \right) , \\ & \forall x_1, x_2 \in S(F),\ x_1 < x_2, \end{aligned} \qquad \ldots (4.11)$$

can determine F, *and*

$$F(x_2) - F(x_1) = (x_2 - P(x_1, x_2))^{-1} e^{\int (x_2 - P(x_1,x_2))^{-1} dx_2},\ \forall\ x_1, x_2 \in S(F), x_1 < x_2.$$

(ii) *Assume* $EX_i^- < \infty$. *Then for any integers* $2 \le m \le n$, *the function*

$$P_1(x) = E\left(\frac{1}{m-1} \sum_{i=1}^{m-1} X_{i,n} \middle| X_{m,n} = x \right),\ \forall\ x \in S(F), \qquad \ldots (4.12)$$

can determine F, *and*

$$F(x) = (x - P_1(x))^{-1} e^{\int (x - P_1(x))^{-1} dx},\ \forall x \in S(F).$$

(iii) *Assume* $EX_i^+ < \infty$. *Then for any integers* $1 \le m \le n - 1$, *the function*

$$P_2(x) = E\left(\frac{1}{n-m} \sum_{i=m+1}^{n} X_{i,n} \middle| X_{m,n} = x \right),\ \forall x \in S(F), \qquad \ldots (4.13)$$

can determine F, and

$$F(x) = 1 - (P_2(x) - x)^{-1} e^{-\int (P_2(x)-x)^{-1} dx}, \ \forall x \in S(F).$$

Also for $x \in R \backslash S(F)$, $F(x)$ can be determined by the continuity and monotonocity of F.

Corollary 1 is a generalization of Ferguson (1967) and Nagaraja (1988b), as can be seen by letting $m = n - 1$ in (4.13), then $E(X_{m-1,n}|X_{m,n} = x)$, as a function of x, can determine F. The following corollary is also a generalization of Das Gupta *et al.* (1993) and Beg and Kirmani (1974), the latter used the function $E(X_1|X_{n,n} = x) = ax - b$, for some constants a and b, to determine F.

COROLLARY 2. *Each of the following statements can determine F.*

(i) $Q(x_1, x_2) = E(X_1|X_{1,n} = x_1, X_{n,n} = x_2), \forall x_1, x_2 \in S_F$ *and* $x_1 < x_2$, *for some fixed integer* $n \geq 3$*;*

(ii) $Q_1(x) = E(X_1|X_{n,n} = x), \forall x \in S_F$, *for some fixed integer* $n \geq 2$*;*

(iii) $Q_2(x) = E(X_1|X_{1,n} = x), \forall x \in S_F$, *for some fixed integer* $n \geq 2$.

EXAMPLE 1. If $E(X_1 - X_{1,n}|X_{1,n} = x) = c$, for some fixed integer $n \geq 2$, where $c > 0$ is a constant, then by Corollary 2, we have that $F(x) = 1 - e^{-\lambda x}$, where $\lambda = (n-1)/(nc)$. From this fact, we find that the independence of $X_{1,n}$ and $X_1 - X_{1,n}$ implies F has an exponential distribution.

5. Characterization not Related to Order Statistics Property

Let $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ be defined as in Section 4, with $F(0) = 0$. Also let $B(t) = \# \{i|X_{i,n} \leq t\}$. For the forward order statistics, Gupta (1984) proved that for some fixed integers $j \leq n$ and $r \geq 1, E((X_{j+1,n} - X_{j,n})^r|X_{j,n}) =$ constant, implies that F is exponential. We also have the following result, which can be compared with part (ii) of Theorem 1.

THEOREM 7. *Assume for some $0 < \eta \leq \infty, F(\eta) = 1$, and F has a positive derivative in $(0, \eta)$. Also let G be a strictly monotone function. If for some fixed integers $k \geq 0, m \geq 1$ and $m + k + 2 \leq n$,*

$$\begin{aligned} &E(G(X_{m+k+1,n} - X_{m+k,n})|X_{m,n} = x) \\ &= E(G(X_{m+k+2,n} - X_{m+k+1,n})|X_{m,n} = x), \ \forall\ 0 < x < \eta\ , \end{aligned} \qquad \ldots (5.1)$$

or, equivalently,

$$\begin{aligned} &E(G(X_{m+k+1,n} - X_{m+k,n})|B(x) = m) \\ &= E(G(X_{m+k+2,n} - X_{m+k+1,n})|B(x) = m), \ \forall\ 0 < x < \eta\ , \end{aligned} \qquad \ldots (5.2)$$

then $\eta < \infty$ and F has a uniform distribution.

The above result can not be generalized to the class of mixed sample processes. Yet for a nonhomogeneous Poisson process $\{L(t), t \geq 0\}$, if the mean function $m(t) = E(L(t)), t \geq 0$, is differentiable with m' being monotone, then it can be shown that parallel to Theorem 7, under similar conditions,

$$E(G(S_{n+k+1} - S_{n+k})|A(t) = n) = E(G(S_{n+k+2} - S_{n+k+1})|A(t) = n), \ \forall \ t \geq 0,$$

implies $m(t) = \lambda t, \ \forall t \geq 0$, for some $\lambda > 0$, hence $\{L(t), t \geq 0\}$ is a homogeneous Poisson process.

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