TAIWANESE JOURNAL OF MATHEMATICS
Vol. 3, No. 2, pp. 211-223, June 1999

ON THE CONVERGENCE OF SUPERPOSITIONS OF POINT PROCESSES

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Abstract. In this work, we give conditions for the superposition of independent point processes to converge to a nonhomogeneous two-dimensional Poisson process. Using our results, not only that of Grigelionis (1963) and Weissman (1975), but also some limiting results related to thinned point processes can be obtained. Hence this establishes the connection between thinnings and superpositions of point processes.

1. Introduction

For each $n \geq 1$, if $X_{n1}, \cdots, X_{nk_n}$ are independent and identically distributed (i.i.d.) Bernoulli random variables with $P(X_{ni} = 1) = p_n = 1 - P(X_{ni} = 0)$, where $p_n \to 0$ and $k_n p_n \to \lambda > 0$ as $n \to \infty$, then $\sum_{i=1}^{k_n} X_{ni}$ converges weakly to Poisson with parameter λ. This is known as Poisson convergence theorem. There have been many attempts to generalize the above result in different directions. Among others, the convergence of superpositions and thinnings of point processes are two that have been investigated by many authors, where instead of converging to a single random variable, the limits are one-dimensional point processes.

About the superpositions of point processes, a famous result is given as follows. Let N_{k1}, N_{k2}, $\cdots$, $N_{kl_k}, k \geq 1$, be a double array of point processes and let N_k be the superposition $N_{k1} + \cdots + N_{kl_k}$. Grigelionis (1963) gave necessary and sufficient conditions for N_k to converge weakly to a (possibly

Received March 26, 1997; revised November 4, 1997.
Communicated by Y.-J. Lee.
1991 *Mathematics Subject Classification*: Primary 60G55; Secondary 60G50.
Key words and phrases: Compound Poisson process, random sum of random variables, superposition of point processes, thinning of point processes, extreme-value distribution, order statistics, regression model.
* Support for this research was provided in part by the National Science Council of the Republic of China, Grant No. NSC 85-2121-M110-016.

nonhomogeneous) Poisson process. Next, an interesting result by Isham (1980) concerning the thinning of point processes is stated below. Assume a point process N is thinned by a stationary k-dependent Markov Bernoulli sequence $\{\chi_{pi}, i \geq 1\}$ with $P(\chi_{pi} = 1) = p = 1 - P(\chi_{pi} = 0)$, $0 < p < 1$, and then is rescaled by the constant factor of p. Isham (1980) proved that the thinned-rescaled process $N_p(t) = \sum_{i=1}^{N(t/p)} \chi_{pi}$, $t \geq 0$, converges weakly to a compound Poisson process as p tends to 0. Note that $N_p(t)$ can be viewed as a random sum of Bernoulli random variables. Motivated by Isham (1980), Su and Huang (1995) considered a model which allows $\{\chi_{pi}, i \geq 1\}$ to be a fairly arbitrary k-dependent Markov chain with general state space (hence the concept of thinning has been dropped). They also indicated that some known results in the literature about Poisson convergence can be expressed as special cases of their theorems. On the other hand, Weissman (1975) studied the multivariate extremal processes generated by independent but nonidentically distributed random variables. He proved that the limiting process can be represented in terms of a two-dimensional nonhomogeneous Poisson process. This extends further the classical Poisson convergence theorem such that the limit becomes a two-dimensional point process.

Inspired by Weissman (1975), we will prove in this paper that under certain assumptions, the random sum of point processes will converge weakly to a two-dimensional compound Poisson process. Using it, not only the results in Grigelionis (1963) and Weissman (1975), but also some limiting results related to thinned point processes can be obtained. Hence this establishes the connection between thinnings and superpositions of point processes, although they seem to be two different operations.

2. The Results

Before proving the main theorem, we prove the following simple yet useful lemma.

Lemma 1. *Let* $p_{ni} \geq 0$, $\forall n \geq 1, i \geq 1$, *and* h_1, h_2 *be two nonnegative functions defined on* $[0, \infty)$. *Assume*

$(*)$ *for any* $u < t < v$, *as* n *becomes large enough,*

$$h_1(nu) \leq h_2(nt) \leq h_1(nv).$$

Also assume

$$\lim_{n\to\infty} \sum_{i=1}^{[h_1(nt)]} p_{ni} = a(t), t \geq 0, \tag{1}$$

where a is a continuous and nondecreasing function with $a(0) = 0$ and $[x]$ denotes the integral part of x. Then

$$\lim_{n\to\infty} \sum_{i=1}^{[h_2(nt)]} p_{ni} = a(t), t \geq 0. \tag{2}$$

Proof. Since a is continuous, for every $t > 0$ and $\varepsilon > 0$, there exist t_1, t_2, $t_2 < t < t_1$, such that

$$a(t) - \frac{\varepsilon}{2} < a(t_2) \leq a(t_1) < a(t) + \frac{\varepsilon}{2}. \tag{3}$$

Using (1), (3) and condition $(*)$, we have for sufficiently large n,

$$\begin{aligned} a(t) - \varepsilon < a(t_2) - \frac{\varepsilon}{2} &\leq \sum_{i=1}^{[h_1(nt_2)]} p_{ni} \\ &\leq \sum_{i=1}^{[h_2(nt)]} p_{ni} \\ &\leq \sum_{i=1}^{[h_1(nt_1)]} p_{ni} \leq a(t_1) + \frac{\varepsilon}{2} < a(t) + \varepsilon. \end{aligned}$$

Thus (2) follows. ∎

It is easy to see that if for some $b, c > 0$, $h_1(t) = bt^c + o_1(t^c)$ and $h_2(t) = bt^c + o_2(t^c)$ as $t \to \infty$, where both o_1 and o_2 are the usual little-oh notation, then the condition $(*)$ holds. The following corollary is immediate.

Corollary 1. *Let $\{N(t), t \geq 0\}$ be an orderly point process (i.e. for $\forall t > 0$, $P(N([t, t+\delta)) > 1) = o(\delta)$ as $\delta \to 0$), $h_1(t) = bt^c + o(t^c)$ as $t \to \infty$, where $b, c > 0$, such that (1) holds, where $\{p_{ni}\}$ and a are defined as in Lemma 1. If $N(t)/bt^c \to 1$, a.s., as $t \to \infty$, then $\lim_{n\to\infty} \sum_{i=1}^{N(nt)} p_{ni} = a(t)$, a.s., $\forall t \geq 0$.*

We now give the definition of compound Poisson process.

Definition 1. Let μ be a non-atomic measure in $\mathbb{R}^n$, N be a point process defined in $\mathbb{R}^n$. Assume for any pairwise disjoint Borel sets $A_1, A_2, \cdots, A_m$ in $\mathbb{R}^n$, $N(A_1), N(A_2), \cdots, N(A_m)$ are independent, such that

$$N(A_i) \stackrel{d}{=} \sum_{j=1}^{Z(A_i)} X_j, i = 1, \cdots, m,$$

where $Z(A_i)$ is a Poisson random variable with parameter $\mu(A_i)$, and $\{X_j, j \geq 1\}$ are i.i.d. random variables with $P(X_j = k) = p_k$, where $\sum_{k=1}^{\infty} p_k = 1$. Then N is said to be an (n-dimensional) compound Poisson process with parameter measure μ and compounding distribution $\{p_k, k \geq 1\}$.

Now for each $n \geq 1$ and $i \geq 1$, let $X_{nij}, j = 0, \pm 1, \pm 2, \cdots$, be i.i.d. random variables with $P(X_{nij} = l) = p_{nil}$, where $\sum_{l=1}^{\infty} p_{nil} = 1$. Next, for each $n \geq 1$, let $\{A_{ni}, i \geq 1\}$ be a sequence of independent orderly point processes each with index set $\mathbb{R}$ and independent of the random variables $\{X_{nij}\}$. Let $A_{ni}(x) = A_{ni}((0, x])$, if $x \geq 0$; $= -A_{ni}((x, 0])$, if $x < 0$. Also for each $n \geq 1$ and $i \geq 1$, define the point process N_{ni} by

$$N_{ni}((x, x']) = \sum_{j=A_{ni}(x)+1}^{A_{ni}(x')} X_{nij}, x < x'. \tag{4}$$

Finally, let $M \equiv \{M(t), t \geq 0\}$ be an orderly point process with $M(0) = 0$, defined on some probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ and independent of everything else. Now we define the two-dimensional point process N_n by

$$N_n((0, t] \times (x, x']) = \sum_{i=1}^{M(nt)} N_{ni}((x, x']), t > 0, x < x'. \tag{5}$$

We have a generalization of Grigelionis (1963).

Theorem 1. *Assume there exists a function $f(t) = bt^c + o(t^c)$, as $t \to \infty$, where $b, c > 0$, such that for any $t > 0$ and $x < x'$,*

(i) $\lim_{n\to\infty} \max_{1 \leq i \leq [f(nt)]} P(A_{ni}((x, x']) \geq 1) = 0,$

(ii) $\lim_{n\to\infty} \sum_{i=1}^{[f(nt)]} P(A_{ni}((x, x']) \geq 2) = 0,$ *and*

(iii) $\lim_{n\to\infty} \sum_{i=1}^{[f(nt)]} p_{nil} P(A_{ni}((x, x']) = 1) = p_l \mu((0, t] \times (x, x']), \forall l \geq 1,$ *where $p_l \geq 0$, $\sum_{l=1}^{\infty} p_l = 1$ and μ is a non-atomic measure defined in $(0, \infty) \times (-\infty, \infty)$.*

If

$$M(t)/t^c \to \lambda, \ a.s., \ as \ t \to \infty, \tag{6}$$

then N_n converges weakly to a compound Poisson process with parameter measure μ_1 and compounding distribution $\{p_l, l \geq 1\}$, where μ_1 satisfies

$$\mu_1((0, t] \times (x, x']) = \mu((0, (\lambda/b)^{\frac{1}{c}} t] \times (x, x']), \forall t > 0, x < x'.$$

Proof. For $k=1,2,\cdots,r$, $m=1,2,\cdots,s$, $u_{km}>0$, and $0\le t_0<t_1<\cdots<t_r$, $x_0<x_1<\cdots<x_s$,

$$
\begin{aligned}
&E(\exp\{-\sum_{k=1}^{r}\sum_{m=1}^{s}u_{km}N_n((t_{k-1},t_k]\times(x_{m-1},x_m])\})\\
&=E(E(\exp\{-\sum_{k=1}^{r}\sum_{m=1}^{s}u_{km}\sum_{i=M(nt_{k-1})+1}^{M(nt_k)}N_{ni}((x_{m-1},x_m])\}\\
&\quad |M(nt_k),k=1,2,\cdots,r))\\
&=E(\prod_{k=1}^{r}\prod_{i=M(nt_{k-1})+1}^{M(nt_k)}E(\exp\{-\sum_{m=1}^{s}u_{km}N_{ni}((x_{m-1},x_m])\}).
\end{aligned}
\tag{7}
$$

If we can show that

$$
\begin{aligned}
&\lim_{n\to\infty}\prod_{i=M(nt_{k-1})+1}^{M(nt_k)}E(\exp\{-\sum_{m=1}^{s}u_{km}N_{ni}((x_{m-1},x_m])\})\\
&=\exp\{-\sum_{m=1}^{s}\mu_1((t_{k-1},t_k]\times(x_{m-1},x_m])\sum_{l=1}^{\infty}p_l(1-e^{-lu_{km}})\},\ \text{a.s.},
\end{aligned}
\tag{8}
$$

then, by the Dominated Convergence Theorem,

$$
\begin{aligned}
&\lim_{n\to\infty}E(\exp\{-\sum_{k=1}^{r}\sum_{m=1}^{s}u_{km}N_n((t_{k-1},t_k]\times(x_{m-1},x_m])\})\\
&=\exp\{-\sum_{k=1}^{r}\sum_{m=1}^{s}\mu_1((t_{k-1},t_k]\times(x_{m-1},x_m])\sum_{l=1}^{\infty}p_l(1-e^{-lu_{km}})\}\\
&=\prod_{k=1}^{r}\prod_{m=1}^{s}\lim_{n\to\infty}E(\exp\{-u_{km}N_n((t_{k-1},t_k]\times(x_{m-1},x_m])\}).
\end{aligned}
\tag{9}
$$

Thus $N_n((t_{k-1},t_k]\times(x_{m-1},x_m])$ converges weakly to a compound Poisson random variable with parameter $\mu_1((t_{k-1},t_k]\times(x_{m-1},x_m])$ and compounding distribution $\{p_l,l\ge1\}$, and the random variables $N_n((t_{k-1},t_k]\times(x_{m-1},x_m])$, $k=1,2,\cdots,r$, $m=1,2,\cdots,s$, are asymptotically independent as $n\to\infty$. Hence the conclusion of the theorem follows.

Now, we begin to prove (8). First for each vector $\mathcal{X}=(x_0,x_1,\cdots,x_s)'$, we denote the random vectors $N_{ni}(\mathcal{X}),A_{ni}(\mathcal{X})$, $n,i\ge1$, as

$$
\begin{aligned}
N_{ni}(\mathcal{X})&=(N_{ni}((x_0,x_1]),N_{ni}((x_1,x_2]),\cdots,N_{ni}((x_{s-1},x_s]))',\\
A_{ni}(\mathcal{X})&=(A_{ni}((x_0,x_1]),A_{ni}((x_1,x_2]),\cdots,A_{ni}((x_{s-1},x_s]))',
\end{aligned}
$$

and for $k = 1, \cdots, r$, denote

$$U_k = (u_{k1}, u_{k2}, \cdots, u_{ks})'.$$

Then the general term of the product on the left side of (8) becomes

$$\begin{aligned}
&E(\exp\{-\sum_{m=1}^{s} u_{km} N_{ni}((x_{m-1}, x_m])\}) \\
&= E(\exp\{-U_k' N_{ni}(\mathcal{X})\}) \\
&= P(N_{ni}(\mathcal{X}) = (0,0,\cdots,0)') + \sum_{V \neq (0,0,\cdots,0)'} e^{-U_k' V} P(N_{ni}(\mathcal{X}) = V) \qquad (10)\\
&= 1 - \sum_{V \neq (0,0,\cdots,0)'} (1 - e^{-U_k' V}) P(N_{ni}(\mathcal{X}) = V) \\
&= 1 - \theta_{ni},
\end{aligned}$$

where $\theta_{ni} = \sum_{V \neq (0,0,\cdots,0)'} (1 - e^{-U_k' V}) P(N_{ni}(\mathcal{X}) = V)$. From (6) we have

$$\frac{M(nt)}{b(n(\lambda/b)^{\frac{1}{c}} t)^c} \longrightarrow 1, \text{ a.s., as } t \to \infty. \qquad (11)$$

Since

$$\begin{aligned}
0 \leq \theta_{ni} &\leq \sum_{V \neq (0,0,\cdots,0)'} P(N_{ni}(\mathcal{X}) = V) \\
&= 1 - P(N_{ni}(\mathcal{X}) = (0,0,\cdots,0)') \qquad (12)\\
&= 1 - P(A_{ni}(\mathcal{X}) = (0,0,\cdots,0)') \\
&= P(A_{ni}((x_0, x_s]) \geq 1),
\end{aligned}$$

by condition (i), we obtain

$$\lim_{n \to \infty} \max_{M(nt_{k-1}) < i \leq M(nt_k)} \theta_{ni} = 0, \text{ a.s., } \forall k = 1, 2, \cdots, r. \qquad (13)$$

It also can be shown that

$$\begin{aligned}
&\sum_{m=1}^{s} \sum_{l=1}^{\infty} (1 - e^{-l u_{km}}) P(X_{niA_{ni}(x_m)} = l, A_{ni}((x_{m-1}, x_m]) = 1, \\
&\quad A_{ni}((x_{v-1}, x_v]) = 0, \forall 1 \leq v \leq s, v \neq m) \\
&\leq \sum_{m=1}^{s} \sum_{l=1}^{\infty} (1 - e^{-l u_{km}}) P(N_{ni}((x_{m-1}, x_m]) = l, N_{ni}((x_{v-1}, x_v]) = 0, \qquad (14)\\
&\quad \forall 1 \leq v \leq s, v \neq m) \leq \theta_{ni} \\
&\leq \sum_{m=1}^{s} \sum_{l=1}^{\infty} (1 - e^{-l u_{km}}) P(X_{niA_{ni}(x_m)} = l, A_{ni}((x_{m-1}, x_m]) = 1, \\
&\quad A_{ni}((x_{v-1}, x_v]) = 0, \forall 1 \leq v \leq s, v \neq m) + P(A_{ni}((x_0, x_s]) \geq 2).
\end{aligned}$$

In the following we will use the squeezing principle to find the limit of $\sum_{i=M(t_{k-1})+1}^{M(t_k)} \theta_{ni}$. First by condition (ii) and Corollary 1, we have

$$\lim_{n\to\infty} \sum_{i=M(nt_{k-1})+1}^{M(nt_k)} P(A_{ni}((x_0, x_s]) \geq 2) = 0, \text{ a.s.} \tag{15}$$

Also

$$\begin{aligned} P(X_{niA_{ni}(x_m)} &= l, A_{ni}((x_{m-1}, x_m]) = 1, A_{ni}((x_{v-1}, x_v]) = 0, \\ &\qquad \forall 1 \leq v \leq s, v \neq m) \\ &= p_{nil} P(A_{ni}((x_{m-1}, x_m]) = 1, A_{ni}((x_{v-1}, x_v]) = 0, \\ &\qquad \forall 1 \leq v \leq s, v \neq m). \end{aligned} \tag{16}$$

Hence it is equivalent to finding the limit of

$$\begin{aligned} &\sum_{i=M(nt_{k-1})+1}^{M(nt_k)} \sum_{m=1}^{s} \sum_{l=1}^{\infty} (1 - e^{-lu_{km}}) p_{nil} P(A_{ni}((x_{m-1}, x_m]) = 1, \\ &\qquad A_{ni}((x_{v-1}, x_v]) = 0, \forall 1 \leq v \leq s, v \neq m). \end{aligned}$$

The following inequality is obvious:

$$\begin{aligned} &P(A_{ni}((x_{m-1}, x_m]) = 1) - P(A_{ni}((x_0, x_s]) \geq 2) \\ &\leq P(A_{ni}((x_{m-1}, x_m]) = 1, A_{ni}((x_{v-1}, x_v]) = 0, \forall 1 \leq v \leq s, v \neq m) \\ &\leq P(A_{ni}((x_{m-1}, x_m]) = 1). \end{aligned} \tag{17}$$

Now condition (ii) implies

$$\begin{aligned} &\left| \sum_{i=M(nt_{k-1})+1}^{M(nt_k)} \sum_{m=1}^{s} \sum_{l=1}^{\infty} (1 - e^{-lu_{km}}) p_{nil} P(A_{ni}((x_0, x_s]) \geq 2) \right| \\ &\leq \sum_{i=M(nt_{k-1})+1}^{M(nt_k)} \sum_{m=1}^{s} \sum_{l=1}^{\infty} p_{nil} P(A_{ni}((x_0, x_s]) \geq 2) \\ &= s \sum_{i=M(nt_{k-1})+1}^{M(nt_k)} P(A_{ni}((x_0, x_s]) \geq 2) \to 0, \text{ a.s., as } n \to \infty, \end{aligned} \tag{18}$$

and condition (iii) implies

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{i=M(nt_{k-1})+1}^{M(nt_k)}\sum_{m=1}^{s}\sum_{l=1}^{\infty}(1-e^{-lu_{km}})p_{nil}P(A_{ni}((x_{m-1},x_m])=1)\\
&=\sum_{m=1}^{s}\sum_{l=1}^{\infty}(1-e^{-lu_{km}})\lim_{n\to\infty}\sum_{i=M(nt_{k-1})+1}^{M(nt_k)}p_{nil}P(A_{ni}((x_{m-1},x_m])=1)\\
&=\sum_{m=1}^{s}\sum_{l=1}^{\infty}(1-e^{-lu_{km}})p_l\mu(((\lambda/b)^{\frac{1}{c}}t_{k-1},(\lambda/b)^{\frac{1}{c}}t_k]\times(x_{m-1},x_m])\\
&=\sum_{m=1}^{s}\mu_1((t_{k-1},t_k]\times(x_{m-1},x_m])\sum_{l=1}^{\infty}p_l(1-e^{-lu_{km}}),\ \text{a.s.}
\end{aligned}
\tag{19}
$$

In view of (17)-(19), it follows that

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{i=M(nt_{k-1})+1}^{M(nt_k)}\theta_{ni}\\
&=\sum_{m=1}^{s}\mu_1((t_{k-1},t_k]\times(x_{m-1},x_m])\sum_{l=1}^{\infty}p_l(1-e^{-lu_{km}}),\ \text{a.s.}
\end{aligned}
\tag{20}
$$

Finally, (8) follows from (10), (13) and (20). This completes the proof of this theorem. ∎

In the above theorem, by letting $t=1$ in (5), under weaker conditions, the one-dimensional point process Z_n, where $Z_n((x,x'])=\sum_{i=1}^{M(n)}N_{ni}((x,x'])$, $x<x'$, converges weakly to a one-dimensional compound Poisson process. The result is given below.

Corollary 2. *Let $\{k_n, n\geq 1\}$ be a sequence of integers with $k_n\to\infty$ as $n\to\infty$. Assume for any $x<x'$,*

(i) $\lim\limits_{n\to\infty}\max\limits_{1\leq i\leq k_n}P(A_{ni}((x,x'])\geq 1)=0,$

(ii) $\lim\limits_{n\to\infty}\sum_{i=1}^{k_n}P(A_{ni}((x,x'])\geq 2)=0,$ *and*

(iii) $\lim\limits_{n\to\infty}\sum_{i=1}^{k_n}p_{nil}P(A_{ni}((x,x'])=1)=p_l\Lambda((x,x']),\forall l\geq 1,$ *where* $p_l\geq 0$, $\sum_{l=1}^{\infty}p_l=1$, *and Λ is a non-atomic measure in R.*

If

$$M(n)/k_n\to 1,\ a.s.,\ as\ n\to\infty, \tag{21}$$

then Z_n converges weakly to a compound Poisson process with parameter measure Λ and compounding distribution $\{p_l, l \geq 1\}$.

On the other hand, in Theorem 1 assume the index set of i is replaced by $\{0, \pm 1, \cdots\}$, and the index set of M is replaced by $\mathbb{R}$. By letting $M(x) = M((0,x])$, if $x \geq 0$, $= -M((x,0])$, if $x < 0$, the point process N_n can be extended to be defined in the whole plane, namely,

$$N_n((t,t'] \times (x,x']) = \sum_{i=M(nt)+1}^{M(nt')} N_{ni}((x,x']), \quad t < t', x < x'. \tag{22}$$

Then after a suitable modification, the conclusion of Theorem 1 still holds. Furthermore, we have the following more general result which allows the point process N_n to be defined in any region T in $\mathbb{R}^2$. The proof is essentially the same as in Theorem 1, hence is omitted.

Theorem 1′. *For a given region T in $\mathbb{R}^2$, assume there exists some function $f(t) = b|t|^c + o(|t|^c)$ as $t \to \infty$, where $b, c > 0$, such that for any $(t,t'] \times (x,x'] \subset T$,*

(i) $\lim_{n\to\infty} \max_{[f(nt)] \leq i \leq [f(nt')]} P(A_{ni}((x,x']) \geq 1) = 0,$

(ii) $\lim_{n\to\infty} \sum_{i=[f(nt)]}^{[f(nt')]} P(A_{ni}((x,x']) \geq 2) = 0,$ *and*

(iii) $\lim_{n\to\infty} \sum_{i=[f(nt)]}^{[f(nt')]} p_{nil} P(A_{ni}((x,x']) = 1) = p_l \mu((t,t'] \times (x,x']), \forall l \geq 1,$ *where* $p_l \geq 0$, $\sum_{l=1}^{\infty} p_l = 1$ *and μ is a non-atomic measure defined in T.*

If

$$|M(t)|/|t|^c \to \lambda, \text{ a.s., as } |t| \to \infty, \tag{23}$$

then N_n (defined as in (22)) converges weakly to a compound Poisson process in the region T' with parameter measure μ_1 and compounding distribution $\{p_l, l \geq 1\}$, where $T' = \{((b/\lambda)^{\frac{1}{c}} w, v) | (w,v) \in T\}$ and μ_1 is a measure in T' satisfying

$$\mu_1((t,t'] \times (x,x']) = \mu(((\lambda/b)^{\frac{1}{c}} t, (\lambda/b)^{\frac{1}{c}} t'] \times (x,x']), \forall (t,t'] \times (x,x'] \subset T'.$$

3. Examples

Now, we give some examples of the applications of Theorems 1 and 1′.

Example 1. Suppose

(a) A_{ni} can only have a point at $x = 1$ with probability $\theta_{ni} > 0$, $\forall n, i \geq 1$;
(b) $P(X_{nij} = l) = p_{ni}(l)/\theta_{ni}$, $\forall n, i, l \geq 1$, $j = 0, \pm 1, \pm 2, \cdots$, where $p_{ni}(l) \geq 0$ and $\sum_{l=1}^{\infty} p_{ni}(l) = \theta_{ni}$.

Assume $f(t) = t$ and

(A) $\lim_{n\to\infty} \max_{1\leq i\leq [nt]} \theta_{ni} = 0$, $\forall t > 0$,
(B) $\lim_{n\to\infty} \sum_{i=1}^{[nt]} p_{ni}(l) = p_l a(t)$, $\forall l \geq 1$, where $p_l \geq 0$, $\sum_{l=1}^{\infty} p_l = 1$ and a is a continuous and nondecreasing function.

Then it can be seen easily that conditions (i)-(iii) of Theorem 1 hold and

$$\mu((0,t] \times (x, x']) = \begin{cases} a(t), & \text{if } 1 \in (x, x'], \\ 0, & \text{if } 1 \notin (x, x']. \end{cases}$$

If the point process M satisfies (6) with $c = 1$, then from Theorem 1, $N_n((0,t] \times (x, x']) = \sum_{i=1}^{M(nt)} N_{ni}((x, x'])$ converges weakly to a compound Poisson process with parameter measure μ_1 and compounding distribution $\{p_l, l \geq 1\}$, where $\mu_1((0,t] \times (x, x']) = \mu((0, \lambda t] \times (x, x'])$. Now let $(x, x']$ be a fixed interval which contains 1. Then $\tilde{N}_n(t) = N_n((0,t] \times (x, x']) = \sum_{i=1}^{M(nt)} N_{ni}((x, x']) = \sum_{i=1}^{M(nt)} Y_{ni}$ converges weakly to a compound Poisson process with parameter measure $\tilde{a}$ and compounding distribution $\{p_l, l \geq 1\}$, where $\tilde{a}(t) = a(\lambda t)$, $\forall t > 0$. Note that

$$Y_{ni} = N_{ni}((x, x']), \quad i \geq 1,$$

are independent random variables with $P(Y_{ni} = l) = p_{ni}(l)$, $l \geq 1$, and $P(Y_{ni} = 0) = 1 - \theta_{ni}$. Hence we obtain a generalization of Westcott (1976).

Example 2. For each $n \geq 1$, let $\{X_{ni}, i = 1, 2, \cdots\}$ be a sequence of independent random variables. Assume there exists a family of distribution functions $\{G_t, t \geq 0\}$, not all identical, such that as $n \to \infty$, $\max\{X_{n1}, \cdots, X_{n[nt]}\}$ converges weakly to G_t, $\forall t > 0$. Let $T = \{(t, x) | t > 0, x > x(t)\}$, where $x(t)$ is the left end of the support of G_t. Following from Lemma 1 of Weissman (1975), the random variables $\{X_{ni}\}$ are right-negligible as $n \to \infty$, i.e., for each $(t, x) \in T$,

$$\lim_{n\to\infty} \max_{1\leq i\leq [nt]} P(X_{ni} > x) = 0. \tag{24}$$

Now let $X_{nij} = 1$, $\forall n, i, j$, $f(t) = t$, $\forall t > 0$, and for each n, define A_{ni}, $i = 1, 2, \cdots$, to be

$$A_{ni}(t) = \begin{cases} 0, & \text{if } t < X_{ni}, \\ 1, & \text{if } t \geq X_{ni}. \end{cases}$$

Then it is easy to see that $N_{ni} = A_{ni}$ and the condition (ii) of Theorem $1'$ holds. Also for any $(t,t'] \times (x,x'] \subset T$, by (24) and

$$0 \le P(A_{ni}((x,x']) \ge 1) = P(x < X_{ni} \le x') \le P(X_{ni} > x), \tag{25}$$

the condition (i) of Theorem $1'$ follows. As $\max\{X_{n1}, \cdots, X_{n[nt]}\}$ converges weakly to G_t, $\forall t > 0$,

$$\prod_{i=1}^{[nt]}(1-(1-F_{ni}(x))) = \prod_{i=1}^{[nt]} F_{ni}(x) \to G_t(x), \tag{26}$$

where F_{ni} is the distribution function of X_{ni}. From (24) and (26), we have

$$\sum_{i=1}^{[nt]}(1-F_{ni}(x)) \to -\log G_t(x). \tag{27}$$

By (27), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=[nt]+1}^{[nt']} P(A_{ni}((x,x']) = 1) \\
&= \lim_{n\to\infty} \sum_{i=[nt]+1}^{[nt']} ((1-F_{ni}(x)) - (1-F_{ni}(x'))) \\
&= -\log \frac{G_{t'}(x)G_t(x')}{G_t(x)G_{t'}(x')},
\end{aligned}$$

so the condition (iii) of Theorem $1'$ holds with

$$\mu((t,t'] \times (x,x']) = -\log \frac{G_{t'}(x)G_t(x')}{G_t(x)G_{t'}(x')}.$$

Now if $M(t)/t \to \lambda$, a.s., as $t \to \infty$, then N_n converges weakly to a Poisson process in T' with parameter measure μ, where $T' = \{(w/\lambda, v) | (w,v) \in T\}$ and $N_n((t,t'] \times (x,x']) = \sum_{M(nt)+1}^{M(nt')} N_{ni}((x,x'])$, $\forall (t,t'] \times (x,x'] \subset T'$. Note that if we choose $M(t) = [t]$ and define $I_n(t,x) = \#\{X_{ni} > x, i = 1,2,\cdots,[nt]\}$, $\forall (t,x) \in T$, then $T' = T$,

$$I_n(t,x) = \sum_{i=1}^{[nt]} N_{ni}((x,\infty)) = N_n((0,t] \times (x,\infty)),$$

and Weissman's conclusion follows immediately.

Example 3. In Bai and Huang (1995), they considered the regression model:

$$Y = \theta(X) + \varepsilon,$$

where θ is a continuous real function defined on $[0,1]$, with a unique global maximum at $x_0 \in [0,1]$. The objective is to determine x_0 based on n observations $(X_1, Y_1), \cdots, (X_n, Y_n)$ with $Y_i = \theta(X_i) + \varepsilon_i$. Here $\{\varepsilon_i\}$ is a sequence of i.i.d. random variables, and $\{X_i\}$ are uniformly chosen from the interval $[0,1]$ (nonrandom case) or are i.i.d. and uniformly distributed over $[0,1]$. Let $Y_{(1)} \le \cdots \le Y_{(n)}$ be the order statistics of $\{Y_i\}$ and $Y_{(i)} = Y_{l_i}$, $i = 1, \cdots, n$. Now x_0 is estimated by $\hat{x}_0(r) = \sum_{i=0}^{r-1} X_{l_{n-i}}/r$, the average of those X_i's corresponding to the r largest order statistics $Y_{(i)}$'s. Under the assumption that ε belongs to the class of domain of attraction $D(g)$ with normalizing constants $\{A_n\}, \{B_n\}$, they proved that $\hat{x}_0(r)$ is consistent to x_0 if and only if $B_n \to 0$ as $n \to \infty$. Their proof is based on Weissman (1975), namely the limiting joint distributions of $B_n^{-1}(\varepsilon_{(n)} - A_n), \cdots, B_n^{-1}(\varepsilon_{(n-k)} - A_n))$ can be derived through Poisson process.

Now by Corollary 2, it is not difficult to extend Bai and Huang's result to the situation that $\{\varepsilon_i\}$ are nonidentically distributed, and the sample size n is replaced by $M(n)$, where $\{M(n), n \ge 1\}$ is a suitable point process. Under this setup, some other interesting problems such as the waiting time of obtaining enough sample sizes for the estimator to be more useful can be discussed.

4. Conclusion

In order to have a Poisson distribution as the limit of the sum of a sequence of random variables, the independence of those random variables are not necessary. For example, as stated in Section 1, Su and Huang (1995) considered the model where the random variables $\{\chi_{pi}, i \ge 1\}$ are assumed to be Markovian. It can be seen that our theorems hold under the conditions that both the process $\{A_{ni}\}$ and the random variables $\{X_{nij}\}$ are independent. So it is worth investigating how to relax the independence assumption so that some convergence results still hold.

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