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ON A STUDY OF RENEWAL PROCESS CONNECTED WITH CERTAIN CONDITIONAL MOMENTS

By WEN-JANG HUANG*
and
JYH-CHERNG SU
National Sun Yat-sen University, Kaohsiung

SUMMARY. We prove that the inter-arrival times of a renewal process $\{A(t), t \geq 0\}$, with S_k being the kth arrival time, have a gamma distribution if for some integers $n \geq 2$, $r \geq 2$, and $1 \leq k_1 < k_2 < \cdots < k_r \leq n$, $E(S_{k_i}^r | A(t) = n)$ is proportional to $E(S_{k_r}^r | A(t) = n)$, for every $t > 0$ and $i = 1, ..., r-1$. Under stronger conditions, characterizations of the Poisson process can be obtained. We also study the cases with negative order of conditional moments.

1. Introduction

Let $\{X_k, k \geq 1\}$ be a sequence of independent and identically distributed positive random variables with common continuous distribution function F. For every $n \geq 1$, define $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$, and let $A(t)$ be the integer k such that $S_k \leq t < S_{k+1}$. Defining in such way, $\{A(t), t \geq 0\}$ is known as a renewal process with X_k and S_k denoting the k-th inter-arrival time and k-th arrival time, respectively. Let $\delta_t = t - S_{A(t)}$ and $\gamma_t = S_{A(t)+1} - t$.

Chung (1972), Çinlar and Jagers (1973), Huang *et al.* (1993) and Li *et al.* (1994) have characterized Poisson process among the class of renewal processes through some conditional expectations about S_k, δ_t or γ_t. In particular, Çinlar and Jagers (1973) proved that if for every integer $n \geq 1$ and for some $1 \leq k \leq n$,

$$E(S_k | A(t) = n) = kt/(n+1), \quad \forall t > 0, \qquad \ldots (1)$$

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then $\{A(t), t \geq 0\}$ is a Poisson process. Huang *et al.* (1993) improved this result and proved that as long as (1) holds for a single integer $n \geq 1$, then $\{A(t), t \geq 0\}$ is a Poisson process. Wesolowski (1989), (1990) characterized that two independent non-degenerate positive random variables are gamma distributed by the conditions

$$E(X^r|X+Y) = a(X+Y)^r \quad \text{and} \quad E(X^s|X+Y) = b(X+Y)^s \qquad \ldots(2)$$

for (r, s) being either $(1, 2)$ or $(1, -1)$. Inspired by Wesolowski's works, Li et al. (1994) obtained a similar result when $(r, s) = (-1, -2)$. Also they characterized a renewal process $\{A(t), t \geq 0\}$ to be a Poisson process, by two identities such as

$$E(S_k^r|A(t) = n) = at^r \quad \text{and} \quad E(S_k^s|A(t) = n) = bt^s, \forall t > 0, \qquad \ldots(3)$$

for some fixed integers $1 \leq k \leq n$ and (r, s) being one of the elements in the set $B = \{(1, 2), (1, -1), (-1, -2)\}$, where a and b are constants, although they are not given in the first place yet can be determined afterwards. Note that except $(r, s) \in B$, Li *et al.* (1994) do not have other similar results. Characterizations of the Poisson process in the class of nonhomogeneous Poisson process using certain conditional expectations can also be found in Huang and Li (1993) and the references therein.

On the other hand, Hall and Simons (1969) characterized gamma distributions by using

$$E(X^2|X+Y) = a(X+Y)^2 \quad \text{and} \quad E(Y^2|X+Y) = b(X+Y)^2. \qquad \ldots(4)$$

In the same paper, stated in terms of reverse martingale, they also provided a characterization of the gamma distribution from the assumptions

$$b_{n_j} E(S_{n_j}^r|S_{n_{j+1}}) = b_{n_{j+1}} S_{n_{j+1}}^r, \quad j = 1, 2, \ldots, r-1, \qquad \ldots(5)$$

for some integers $r \geq 2$ and $1 \leq n_1 < n_2 < \cdots < n_r$.

Based on the idea of Hall and Simons (1969), in the present paper we will investigate some properties of the conditional moments of the arrival times of a renewal process. We prove that the inter-arrival times of a renewal process have a gamma distribution if for some integers $n \geq 2$, $r \geq 2$, and $1 \leq k_1 < k_2 < \cdots < k_r \leq n$, $E(S_{k_i}^r|A(t) = n)$ is proportional to $E(S_{k_r}^r|A(t) = n)$, for every $t > 0$ and $i = 1, \ldots, r-1$. Under stronger conditions, a characterization of the Poisson process will be obtained. We also study the cases with negative order of conditional moments, where only positive order was considered in Hall and Simons (1969). Finally we give some extensions of the results in Li *et al.* (1994).

2. A characterization of the gamma distribution

Before we study the case of renewal process started from the next section, we give a characterization of the gamma distribution, by using conditional moments with negative orders. The result can be compared with that of Hall and Simons (1969), where the condition (4) was used.

THEOREM 1. *Let X and Y be two independent non-degenerate random variables with $E(|X|^r) < \infty$ and $E(|Y|^r) < \infty$, for $r = 1, -1$. If*

$$E(X^{-1}|X+Y) = a(X+Y)^{-1} \text{ and } E(Y^{-1}|X+Y) = b(X+Y)^{-1} \qquad \ldots(6)$$

hold for some constants a and b, then (i) $a > 1, b > 1, ab - a - b > 0$; (ii) *$X$ and Y, or $-X$ and $-Y$ have gamma distributions with the same scale parameter.*

PROOF. From (6) we have

$$E(Y/X|X+Y) = a - 1, \qquad \ldots(7)$$

and

$$E(X/Y|X+Y) = b - 1. \qquad \ldots(8)$$

For every $\theta \in R$, let $f(\theta) = E(X^{-1}e^{i\theta X})$ and $g(\theta) = E(Y^{-1}e^{i\theta Y})$, where $i = \sqrt{-1}$. Then (7) and (8) imply

$$f(\theta)g''(\theta) = (a-1)f'(\theta)g'(\theta), \qquad \ldots(9)$$

and

$$f''(\theta)g(\theta) = (b-1)f'(\theta)g'(\theta), \qquad \ldots(10)$$

respectively. As both X and Y are non-degenerate, (9) and (10) imply $a \neq 1$ and $b \neq 1$. Furthermore, from (9) and (10), we obtain

$$f(\theta) = E(X^{-1})(i^{-1}g'(\theta))^{(a-1)^{-1}}, \qquad \ldots(11)$$

and

$$f'(\theta) = i(E(Y^{-1}))^{1-b}(g(\theta))^{b-1}. \qquad \ldots(12)$$

Substituting (11) and (12) into (9), yields

$$\frac{a-1}{a}(i)^{-(a-1)^{-1}}E(X^{-1})(g'(\theta))^{a/(a-1)} = \frac{a-1}{b}i(E(Y^{-1}))^{1-b}(g(\theta))^{b}+K_1, \quad \ldots(13)$$

where K_1 is a constant. Letting $\theta \to 0$ and noting that $E(X^{-1})/a = E(Y^{-1})/b = E(X+Y)^{-1}$, $\lim_{\theta\to 0} g'(\theta) = i$ and $\lim_{\theta\to 0} g(\theta) = E(Y^{-1})$, we have $K_1 = 0$. Thus

$$(g(\theta))^{-b(a-1)/a}g'(\theta) = i(E(Y^{-1}))^{-b(a-1)/a}. \qquad \ldots(14)$$

Now if $b(a-1)/a = 1$, then $g(\theta) = E(Y^{-1})\exp\{i(E(Y^{-1}))^{-1}\theta\}$, which in turn implies Y is degenerate. Therefore $b(a-1)/a \neq 1$, and

$$g(\theta) = \left(\left(E(Y^{-1})\right)^{(a+b-ab)/a} + i\frac{a+b-ab}{a}\left(E(Y^{-1})\right)^{-b(a-1)/a}\theta\right)^{a/(a+b-ab)}$$

Consequently,

$$E(e^{i\theta Y}) = \frac{1}{i}g'(\theta) = (1 - i\frac{ab-a-b}{aE(Y^{-1})}\theta)^{-b(a-1)/(ab-a-b)}, \qquad \dots(15)$$

and

$$E(e^{i\theta X}) = (1 - i\frac{ab-a-b}{bE(X^{-1})}\theta)^{-a(b-1)/(ab-a-b)}. \qquad \dots(16)$$

This completes the proof.

3. Main results

Let the renewal process $\{A(t), t \geq 0\}$ be defined as in Section 1. Also for a gamma distributed random variable with parameters α and β (denote this distribution by $\Gamma(\alpha,\beta)$), let $G_{\alpha,\beta}(t)$ denote its distribution function, that is

$$G_{\alpha,\beta}(t) = \int_0^t \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx. \qquad \dots(17)$$

We now present two preliminary results that will be needed in establishing our main results. First we give a lemma which can be proved by using standard technique of conditional expectations.

LEMMA 1. *Let the common distribution function F of the inter-arrival times of the renewal process $\{A(t), t \geq 0\}$ have a $\Gamma(\alpha,\beta)$ distribution. Then for every $t > 0$, integers $r > -k\alpha$ and $1 \leq k \leq n$,*

$$E(S_k^r | A(t) = n) = C_{r,k}\beta^r \frac{G_{n\alpha+r,\beta}(t) - G_{(n+1)\alpha+r,\beta}(t)}{G_{n\alpha,\beta}(t) - G_{(n+1)\alpha,\beta}(t)}, \qquad \dots(18)$$

where

$$C_{r,k} = \begin{cases} \prod_{j=0}^{r-1}(k\alpha+j) & ,r \geq 1, \\ 1 & ,r = 0, \\ \prod_{j=1}^{-r}(k\alpha-j)^{-1} & ,-1 \geq r > k\alpha. \end{cases} \qquad \dots(19)$$

Also $E(S_k^r | A(t) = n)$ does not exist when $r \leq -k\alpha$.

The next lemma provides a sufficient condition for a gamma renewal process to be Poisson.

LEMMA 2. *As in Lemma 1, let F have a $\Gamma(\alpha,\beta)$ distribution. Given the integers $s \neq 0$, r and $1 \leq k_1, k_2 \leq n$, if for some constant $a > 0$,*

$$t^s E(S_{k_1}^r | A(t) = n) = aE(S_{k_2}^{r+s} | A(t) = n), \quad \forall t > 0, \qquad \ldots(20)$$

then $\{A(t), t \geq 0\}$ is a Poisson process, and

$$a = \begin{cases} \frac{C_{r,k_1}}{C_{r+s,k_2}} \prod_{j=1}^{s}(n+r+j) & , s \geq 1, \\ \frac{C_{r,k_1}}{C_{r+s,k_2}} \prod_{j=1}^{-s}(n+r+s+j)^{-1} & , s \leq -1. \end{cases} \qquad \ldots(21)$$

PROOF. First assume $s \geq 1$. By Lemma 1, (20) implies

$$\begin{aligned} & t^s C_{r,k_1} \beta^r \frac{G_{n\alpha+r,\beta}(t) - G_{(n+1)\alpha+r,\beta}(t)}{G_{n\alpha,\beta}(t) - G_{(n+1)\alpha,\beta}(t)} \\ &= aC_{r+s,k_2} \beta^{r+s} \frac{G_{n\alpha+r+s,\beta}(t) - G_{(n+1)\alpha+r+s,\beta}(t)}{G_{n\alpha,\beta}(t) - G_{(n+1)\alpha,\beta}(t)}, \quad \forall t > 0. \end{aligned} \qquad \ldots(22)$$

Thus

$$\frac{aC_{r+s,k_2}}{C_{r,k_1}} = \frac{\beta^{-s} t^s (G_{n\alpha+r,\beta}(t) - G_{(n+1)\alpha+r,\beta}(t))}{G_{n\alpha+r+s,\beta}(t) - G_{(n+1)\alpha+r+s,\beta}(t)}, \quad \forall t > 0. \qquad \ldots(23)$$

Letting $t \to 0$ in the right side of (23) and using L'Hospital's rule repeatedly, yields

$$\begin{aligned} \frac{aC_{r+s,k_2}}{C_{r,k_1}} &= \beta^{-s} \sum_{l=0}^{s} \frac{s!}{(s-l)!} \lim_{t \to 0} \frac{t^{s-l}\left(G'_{n\alpha+r,\beta}(t) - G'_{(n+1)\alpha+r,\beta}(t)\right)}{G^{(l+1)}_{n\alpha+r+s,\beta}(t) - G^{(l+1)}_{(n+1)\alpha+r+s,\beta}(t)} \\ &= \beta^{-s} \sum_{l=0}^{s} \frac{s!}{(s-l)!} \beta^s \prod_{j=0}^{s-l-1} (n\alpha + r + j) \\ &= \prod_{j=1}^{s} (n\alpha + r + j), \end{aligned} \qquad \ldots(24)$$

where $\prod_{j=0}^{-1}$ is defined to be 1, the superscript $(l+1)$ denotes the $(l+1)$th derivative with respect to t, and we have used here that for $0 \leq l \leq s-1$,

$$\lim_{t \to 0} \frac{t^{s-l}\left(G'_{n\alpha+r,\beta}(t) - G'_{(n+1)\alpha+r,\beta}(t)\right)}{G^{(l+1)}_{n\alpha+r+s,\beta}(t) - G^{(l+1)}_{(n+1)\alpha+r+s,\beta}(t)} = \beta^s \prod_{j=0}^{s-l-1} (n\alpha + r + j). \qquad \ldots(25)$$

Similarly, by letting $t \to \infty$ in the right side of (23), it follows

$$\frac{aC_{r+s,k_2}}{C_{r,k_1}} = \prod_{j=0}^{s-1} (n\alpha + \alpha + r + j). \qquad \ldots(26)$$

Therefore, by comparing (24) and (26), we obtain $\alpha = 1$. This proves the assertion that $\{A(t), t \geq 0\}$ is a Poisson process. Substituting $\alpha = 1$ into (26) the constant a can be obtained immediately.

Finally, when $s \leq -1$, by letting $s' = -s \geq 1$ and $r' = r + s$, then the equation (20) is equivalent to that for the case $s \geq 1$. Hence we obtain the assertions immediately again.

We now characterize the common inter-arrival distribution function to be gamma distributed, under the assumption that certain conditional moments of the arrival times with the same order are assumed to be proportional to each other.

THEOREM 2. *Assume for some integers $n \geq 2$, $r \geq 2$, $1 \leq k_1 < k_2 < \cdots < k_r \leq n$, and positive constants $a_i, i = 1, ..., r-1$,*

$$a_i E(S_{k_i}^r | A(t) = n) = E(S_{k_r}^r | A(t) = n), \quad i = 1, ..., r-1, \qquad \ldots (27)$$

for every $t > 0$ whenever $P(A(t) = n) > 0$. Also assume $E(X_1^r) < \infty$. Then F has a $\Gamma(\alpha, \beta)$ distribution for some constants α and β. Moreover

$$a_i = \frac{\prod_{j=0}^{r-1}(k_r\alpha + j)}{\prod_{j=0}^{r-1}(k_i\alpha + j)}, \quad i = 1, ..., r-1. \qquad \ldots (28)$$

PROOF. From (27) we obtain (by letting $a_r = 1$)

$$\begin{aligned} &a_i \int_0^t x^r \left(F_{n-k_i}(t-x) - F_{n-k_i+1}(t-x)\right) dF_{k_i}(x) \\ &= a_{i+1} \int_0^t x^r \left(F_{n-k_{i+1}}(t-x) - F_{n-k_{i+1}+1}(t-x)\right) dF_{k_{i+1}}(x), \end{aligned} \qquad \ldots (29)$$

$i = 1, ..., r-1$, where F_j is the j-fold convolution of F with itself, $j \geq 1$. By taking the Laplace transforms, (29) can be converted into

$$\begin{aligned} &a_i(\phi^{k_i}(\theta))^{(r)} \frac{\phi^{n-k_i}(\theta) - \phi^{n-k_i+1}(\theta)}{\theta} \\ &= a_{i+1}(\phi^{k_{i+1}}(\theta))^{(r)} \frac{\phi^{n-k_{i+1}}(\theta) - \phi^{n-k_{i+1}+1}(\theta)}{\theta}, \end{aligned} \qquad \ldots (30)$$

where

$$\phi(\theta) = \int_0^\infty e^{-\theta x} dF(x), \theta > 0. \qquad \ldots (31)$$

After cancelling the common factors, (30) turns to

$$a_i \phi^{k_{i+1}-k_i}(\theta) \left(\phi^{k_i}(\theta)\right)^{(r)} = a_{i+1} \left(\phi^{k_{i+1}}(\theta)\right)^{(r)}, \qquad \ldots (32)$$

for every $\theta > 0$ and $i = 1, ..., r-1$. Now (32) has the form (5) of Hall and Simons (1969), and it is given there that the solution of (32) is

$$\phi(\theta) = (1 + \beta\theta)^{-\alpha}, \qquad \ldots (33)$$

for some $\alpha, \beta > 0$. This proves that F has a $\Gamma(\alpha, \beta)$ distribution. Using Lemma 1 the constants a_i's are also obtained.

In view of Theorem 2 and Lemma 2, the stronger conditions that $E(S^r_{k_i}|A(t) = n)$ is proportional to t^r, $\forall\, i = 1, ..., r$, will yield the process $\{A(t), t \geq 0\}$ is Poisson. We state the result in the following.

THEOREM 3. *In Theorem 2, if the conditions in (27) are replaced by*

$$c_i E(S^r_{k_i}|A(t) = n) = t^r, \quad i = 1, ..., r, \qquad \ldots (34)$$

for every $t > 0$ whenever $P(A(t) = n) > 0$, where $c_i, i = 1, ..., r$, are positive constants, then $\{A(t), t \geq 0\}$ is a Poisson process and

$$c_i = \frac{\prod_{j=1}^{r}(n + j)}{\prod_{j=0}^{r-1}(k_i + j)}, \quad i = 1, ..., r. \qquad \ldots (35)$$

Next we have a result which is slightly different from Theorem 3 and can be shown by following the steps of the previous theorem. A remark will be given after the theorem.

THEOREM 4. *Assume for some integers $n \geq 2$, $r \geq 2$, $1 \leq k_1 < k_2 < \cdots < k_{r-1} \leq n$, $n_1 \geq 1$, $1 \leq k_r \leq n_1$, and positive constants c_i, $i = 1, ..., r$,*

$$c_i E(S^r_{k_i}|A(t) = n) = t^r, i = 1, ..., r-1, \qquad \ldots (36)$$

and

$$c_r E(S_{k_r}|A(t) = n_1) = t, \qquad \ldots (37)$$

for every $t > 0$ whenever $P(A(t) = n) > 0$ and $P(A(t) = n_1) > 0$. Also assume $E(X_1^r) < \infty$. Then $\{A(t), t \geq 0\}$ is a Poisson process.

Note that (37) implies $c_r(k_r/k)E(S_k|A(t) = n_1) = t$, $\forall\ 1 \leq k \leq n_1$. So that when $r = 2$ and $n = n_1$, (36) and (37) can be reduced to (12) and (13) of Li *et al.* (1994). Therefore we have obtained a generalization of Theorem 3 of Li *et al.* (1994).

4. Results based on negative order of conditional moments

In using the reverse martingale assumption such as (5) to characterize the gamma distribution, Hall and Simons (1969) wondered whether there is a solution for F when r is positive and non-integral. Although we cannot answer the question for the case of non-integral r, in this section we will give some results related to negative order of conditional moments, which has the same flavor as Theorem 1 for the case of renewal process. Again let $\{A(t), t \geq 0\}$ be a renewal process as defined in Section 1.

THEOREM 5. *Assume there exist integers* $1 \leq k_1 < k_2 \leq n$ *and a constant* $a_1 > 0$, *such that*

$$a_1 E(S_{k_1}^{-1} | A(t) = n) = E(S_{k_2}^{-1} | A(t) = n), \qquad \ldots (38)$$

for every $t > 0$, *whenever* $P(A(t) = n) > 0$. *Also assume* $E(X_1) < \infty$ *and* $E(S_{k_1}^{-1}) < \infty$. *Then* $a_1 < k_1/k_2$ *and* F *has a* $\Gamma(\alpha, \beta)$ *distribution, where* $\alpha > 0$ *and* $\beta = (a_1 - 1)/(a_1 k_2 - k_1)$.

PROOF. From (38) we obtain

$$\begin{aligned} & a_1 \int_0^t x^{-1} \left(F_{n-k_1}(t-x) - F_{n-k_1+1}(t-x) \right) dF_{k_1}(x) \\ &= \int_0^t x^{-1} \left(F_{n-k_2}(t-x) - F_{n-k_2+1}(t-x) \right) dF_{k_2}(x), \end{aligned} \qquad \ldots (39)$$

which in turn implies

$$a_1 h_1(\theta) \frac{\phi^{n-k_1}(\theta) - \phi^{n-k_1+1}(\theta)}{\theta} = h_2(\theta) \frac{\phi^{n-k_2}(\theta) - \phi^{n-k_2+1}(\theta)}{\theta}, \qquad \ldots (40)$$

where

$$h_1(\theta) = \int_0^\infty x^{-1} e^{-\theta x} dF_{k_1}(x), \qquad \ldots (41)$$

and

$$h_2(\theta) = \int_0^\infty x^{-1} e^{-\theta x} dF_{k_2}(x). \qquad \ldots (42)$$

Since $h_1'(\theta) = -\phi^{k_1}(\theta)$ and $h_2'(\theta) = -\phi^{k_2}(\theta)$, (40) can be rewritten as

$$\frac{h_1'(\theta)}{h_1(\theta)} = a_1 \frac{h_2'(\theta)}{h_2(\theta)}. \qquad \ldots (43)$$

Thus

$$h_1(\theta) = c h_2^{a_1}(\theta), \qquad \ldots (44)$$

where c is a constant. Differentiating both sides of (44) twice, with respect to θ, yields

$$\phi^{(k_1-a_1k_2)/(a_1-1)-1}(\theta)\phi'(\theta) = \alpha_1, \qquad \ldots(45)$$

for some constant α_1.

Now if $k_1 - a_1k_2 = 0$, then $\phi(\theta) = e^{\alpha_1\theta}$, which contradicts the assumption that F is continuous. Hence $k_1 - a_1k_2 \neq 0$ and

$$\phi(\theta) = (1+\alpha\theta)^{-(a_1-1)/(a_1k_2-k_1)}, \qquad \ldots(46)$$

where $\alpha = \alpha_1(k_1 - a_1k_2)/(a_1 - 1)$. Finally, in order that $\phi(\theta)$ is a Laplace transform, $\beta = (a_1 - 1)/(a_1k_2 - k_1)$ must be positive. Also as obviously $a_1 < 1$, we have $a_1 < k_1/k_2$.

Again under stronger conditions the renewal process will become Poisson. Since it can be proved along the lines of Theorem 3, we only state the result.

THEOREM 6. *Assume there exist integers* $1 \le k_1 < k_2 < n$ *and constants* a *and* b, *such that*

$$E(S_{k_1}^{-1}|A(t) = n) = at^{-1}, \qquad \ldots(47)$$

and

$$E(S_{k_2}^{-1}|A(t) = n) = bt^{-1}, \qquad \ldots(48)$$

for every $t > 0$ *whenever* $P(A(t) = n) > 0$. *Also assume* $E(X_1) < \infty$ *and* $E(S_{k_1}^{-1}) < \infty$. *Then* $k_1 \ge 2$, $a = n/(k_1 - 1)$, $b = n/(k_2 - 1)$, *and* $\{A(t), t \ge 0\}$ *is a Poisson process.*

When $\{A(t), t \ge 0\}$ is a Poisson process, for integers $1 \le k \le n$ and $r > -k$,

$$a_1E(S_k^r|A(t) = n) = E((t - S_k)^r|A(t) = n), \quad \forall t > 0, \qquad \ldots(49)$$

where a_1 is a suitable constant. Yet when F is just gamma distributed, (49) may not be true. We now give a converse result concerned with the case that $r = -1$.

THEOREM 7. *Assume there exist two integers* $2 \le k \le n - 1$ *and constants* a *and* b, *such that*

$$E(S_k^{-1}|A(t) = n) = at^{-1}, \qquad \ldots(50)$$

and

$$E((t - S_k)^{-1}|A(t) = n) = bt^{-1}, \qquad \ldots(51)$$

for every $t > 0$ *whenever* $P(A(t) = n) > 0$. *Also assume* $E(X_1) < \infty$, $E(S_k^{-1}) < \infty$ *and*

$$\int_0^1 t^{-1}F_{n-k}(t)dt < \infty. \qquad \ldots(52)$$

Then $\{A(t), t \geq 0\}$ is a Poisson process and $a = n/(k-1)$, $b = n/(n-k)$.

PROOF. From (50) and (51), we obtain for every $t > 0$,

$$\int_0^t x^{-1}\left(F_{n-k}(t-x) - F_{n-k+1}(t-x)\right) dF_k(x) = at^{-1}\left(F_n(t) - F_{n+1}(t)\right), \quad \ldots (53)$$

and

$$\int_0^t (t-x)^{-1}\left(F_{n-k}(t-x) - F_{n-k+1}(t-x)\right) dF_k(x) = bt^{-1}\left(F_n(t) - F_{n+1}(t)\right). \quad \ldots (54)$$

Taking the Laplace transforms of both sides of (53) and (54), respectively, it follows

$$\xi'(\theta)\eta(\theta) = -a \int_0^\infty e^{-\theta t} t^{-1}\left(F_n(t) - F_{n+1}(t)\right) dt, \quad \ldots (55)$$

and

$$\xi(\theta)\eta'(\theta) = -b \int_0^\infty e^{-\theta t} t^{-1}\left(F_n(t) - F_{n+1}(t)\right) dt, \quad \ldots (56)$$

where for $\theta > 0$,

$$\xi(\theta) = \int_0^\infty t^{-1} e^{-\theta t}\left(F_{n-k}(t) - F_{n-k+1}(t)\right) dt, \quad \ldots (57)$$

and

$$\eta(\theta) = \int_0^\infty t^{-1} e^{-\theta t} dF_k(t). \quad \ldots (58)$$

Differentiating both sides of (55) and (56), respectively, we have

$$\xi''(\theta)\eta(\theta) = (a-1)\xi'(\theta)\eta'(\theta), \quad \ldots (59)$$

and

$$\xi(\theta)\eta''(\theta) = (b-1)\xi'(\theta)\eta'(\theta). \quad \ldots (60)$$

Also (55) and (56) imply

$$\xi'(\theta)\eta(\theta) = (a/b)\xi(\theta)\eta'(\theta). \quad \ldots (61)$$

Solving (59), (60) and (61) we obtain the assertions.

5. Some extensions of the results by Li et al. (1994)

In this section we give some simple extensions of the results in Li et al. (1994). First we extend Theorem 3 of the above paper.

THEOREM 8. *Assume for some fixed integers* $1 \le k_1 \le n_1$, *and* $1 \le k_2 \le n_2$,

$$E(S_{k_1}|A(t) = n_1) = at, \qquad \ldots (62)$$

and

$$E(S_{k_2}^2|A(t) = n_2) = bt^2 + ct, \qquad \ldots (63)$$

hold for some constants a, b *and* c, *for every* $t > 0$, *whenever* $P(A(t) = n_i) > 0$, $i = 1, 2$. *Also assume* $E(X_1^2) < \infty$. *Then*

(i) $a = k_1/(n_1+1)$, $b = k_2(k_2+1)/[(n_2+1)(n_2+2)]$ *and* $c = 0$;

(ii) $\{A(t), t \ge 0\}$ *is a Poisson process.*

PROOF. As in the proof of Theorem 3 of Li et al. (1994), (62) implies

$$\frac{1-\phi(\theta)}{\theta} = \mu_1 \phi^{k_1 a^{-1} - n_1}(\theta). \qquad \ldots (64)$$

Hence

$$\frac{1-\phi(\theta)}{\theta} = \mu_1 \phi^{k_2(k_1 a^{-1} - n_1 + n_2)/k_2 - n_2}(\theta). \qquad \ldots (65)$$

From this we have

$$E(S_{k_2}|A(t) = n_2) = a't, \qquad \ldots (66)$$

where $a' = k_2/(k_1 a^{-1} - n_1 + n_2)$. Now using Theorem 3 of Li *et al.* (1994), (66) and (63) together imply the assertions.

Thus although (62) and (63) look more general than (12) and (13) of Li *et al.* (1994), basically there are not much difference between these two pairs of conditions.

Similarly, we have the following parallel extension of Theorem 4 of Li et al. (1994).

THEOREM 9. *Assume for some fixed integers* $1 \le k_1 \le n_1$ *and* $2 \le k_2 \le n_2$,

$$E(S_{k_1}|A(t) = n_1) = at, \qquad \ldots (67)$$

and

$$E(S_{k_2}^{-1}|A(t) = n_2) = bt^{-1}, \qquad \ldots (68)$$

hold for some constants a *and* b, *for every* $t > 0$ *whenever* $P(A(t) = n_i) > 0$, $i = 1, 2$. *Also assume* $E(X_1) < \infty$ *and* $E(S_{k_2}^{-1}) < \infty$. *Then*

(i) $a = k_1/(n_1+1)$ *and* $b = n_2/(k_2+1)$;

(ii) $\{A(t), t \ge 0\}$ *is a Poisson process.*

Theorems 3, 4 and 5 of Li *et al.* (1994) are special cases of the following theorem (corresponding to $r = 0$, $r = -1$ and $r = -2$, respectively). This theorem can also be compared with Theorem 3 of the present paper, where there are r equations, and here only two equations are needed.

THEOREM 10. *Assume for some integers r and $1 \leq k \leq n$,*

$$tE(S_k^r|A(t)=n) = aE(S_k^{r+1}|A(t)=n), \qquad \ldots(69)$$

and

$$tE(S_k^{r+1}|A(t)=n) = bE(S_k^{r+2}|A(t)=n), \qquad \ldots(70)$$

hold for some constants a and b, for every $t > 0$ whenever $P(A(t)=n) > 0$. Also assume $E(X_1^{r+2}) < \infty$ if $r \geq 0$, or $E(X_1) < \infty$ and $E(S_k^r) < \infty$ if $r < 0$. Then

(i) $r > -k$, $a = (n+r+1)/(k+r)$, $b = (n+r+2)/(k+r+1)$;

(ii) $\{A(t), t \geq 0\}$ *is a Poisson process.*

PROOF. From (69) and (70), it follows

$$\frac{((\phi^{n-k}(\theta) - \phi^{n-k+1}(\theta))/\theta)'}{(\phi^{n-k}(\theta) - \phi^{n-k+1}(\theta))/\theta} = (a-1)\frac{q'(\theta)}{q(\theta)}, \qquad \ldots(71)$$

and

$$\frac{((\phi^{n-k}(\theta) - \phi^{n-k+1}(\theta))/\theta)'}{(\phi^{n-k}(\theta) - \phi^{n-k+1}(\theta))/\theta} = (b-1)\frac{q''(\theta)}{q'(\theta)}, \qquad \ldots(72)$$

where

$$q(\theta) = \int_0^\infty x^r e^{-\theta x} dF_k(x). \qquad \ldots(73)$$

Note that

$$q(\theta) = (\phi^k(\theta))^{(r)}, \quad r \geq 0, \qquad \ldots(74)$$

and

$$(q(\theta))^{(-r)} = \phi^k(\theta), \quad r < 0. \qquad \ldots(75)$$

Also it is easy to see that both a and $b \neq 1$. Hence

$$\frac{q''(\theta)}{q'(\theta)} = \frac{a-1}{b-1} \cdot \frac{q'(\theta)}{q(\theta)}, \qquad \ldots(76)$$

which has the solution $q(\theta) = (m_1\theta + m_2)^e$, where m_1, m_2 are constants and $e = (a-1)/(b-1)$. This together with (74) or (75) imply $\phi(\theta) = (1+\beta\theta)^{-\alpha}$ for some α, $\beta > 0$. Finally, the assertions (i) and (ii) are achieved by using Lemma 2.

6. Concluding remark

Inspired by Theorem 10, under certain conditions, we can use

$$(X+Y)E(X^r|X+Y) = aE(X^{r+1}|X+Y), \qquad \ldots(77)$$

and

$$(X+Y)E(X^{r+1}|X+Y) = bE(X^{r+2}|X+Y), \qquad \ldots(78)$$

where a and b are constants, to characterize X and Y to be gamma distributed. It is easy to see that this is a generalization of Wesolowski (1989), (1990), and Theorem 1 of Li *et al.* (1994), where they used (2), for a pair of $(r,s) \in B$, to characterize X and Y to be gamma distributed.

In Theorem 2, for each $i = 1, \ldots, r-1$, we can replace n by n_i in the ith equation of (27) and still obtain similar characterizations, if some suitable modifications about the conditions for the integers $\{n_i\}$ and $\{k_i\}$ are made. That is Theorem 2 can be further generalized. We state the theorem in the present form as it is simpler and easier to understand. On the other hand, the conditions

$$E(S_k^2|A(t)=n) = at^2 \text{ and } E((t-S_k)^2|A(t)=n) = bt^2 \qquad \ldots(79)$$

are equivalent to

$$E(S_k^2|A(t)=n) = at^2 \text{ and } E(S_k|A(t)=n) = ct, \qquad \ldots(80)$$

where $c = (1+a-b)/2$. Also it is already known that using the two equations in (80), F can be characterized. Yet for the cases such as given

$$E(S_{k_1}^{-2}|A(t)=n) = at^{-2} \text{ and } E(S_{k_2}^{-2}|A(t)=n) = bt^{-2}, \qquad \ldots(81)$$

or

$$E(S_k^{-2}|A(t)=n) = at^{-2} \text{ and } E((t-S_k)^{-2}|A(t)=n) = bt^{-2}, \qquad \ldots(82)$$

as the computations become very complicated, we are still unable to determine the distribution function F from either (81) or (82).

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DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL SUN YAT-SEN UNIVERSITY
KAOHSIUNG, TAIWAN 80424
R.O.C.