ON A STUDY OF RENEWAL PROCESS
CONNECTED WITH CERTAIN CONDITIONAL
MOMENTS

By WEN-JANG HUANG

and

JYI-CHEWNG SU

National Sun Yat-sen University, Kaohsiung

SUMMARY. We prove that the inter-arrival times of a renewal process \( \{A(t), t \geq 0\} \), with \( S_k \) being the \( k \)th arrival time, have a gamma distribution if for some integers \( n \geq 2, r \geq 2, \) and \( 1 \leq k_1 < k_2 < \cdots < k_r \leq n \), \( E(S_k | A(t) = n) \) is proportional to \( E(S_k | A(t) = n) \), for every \( t > 0 \) and \( i = 1, \ldots, r - 1 \). Under stronger conditions, characterizations of the Poisson process can be obtained. We also study the cases with negative order of conditional moments.

1. Introduction

Let \( \{X_k, k \geq 1\} \) be a sequence of independent and identically distributed positive random variables with common continuous distribution function \( F \). For every \( n \geq 1 \), define \( S_n = \sum_{k=1}^{n} X_k, S_0 = 0 \), and let \( A(t) \) be the integer \( k \) such that \( S_k \leq t < S_{k+1} \). Defining in such way, \( \{A(t), t \geq 0\} \) is known as a renewal process with \( X_k \) and \( S_k \) denoting the \( k \)th inter-arrival time and \( k \)th arrival time, respectively. Let \( \delta_k = t - A(t) \) and \( \gamma_k = A(t) - t \).

Chung (1972), Çinlar and Jagers (1973), Huang et al. (1993) and Li et al. (1994) have characterized Poisson process among the class of renewal processes through some conditional expectations about \( S_k, \delta_k \) or \( \gamma_k \). In particular, Çinlar and Jagers (1973) proved that if for every integer \( n \geq 1 \) and for some \( 1 \leq k \leq n \),

\[
E(S_k | A(t) = n) = kt/(n + 1), \quad \forall t > 0,
\]

then \( \delta_k \) and \( \gamma_k \) is a renewal process, that is, \( \{A(t), t \geq 0\} \) is a renewal process.

\[
\delta_k = t - A(t) \quad \text{and} \quad \gamma_k = A(t) - t.
\]

for every \( t > 0 \). Let \( \delta_k \) and \( \gamma_k \) be independent, not necessarily identically distributed gamma random variables of means \( (r, s) \). In order to characterize the renewal process, the following result holds.

In this paper, we consider the characterization of renewal process through conditional moments.

Let \( B \) be a renewal process with \( \delta_k \) and \( \gamma_k \) not necessarily identically distributed gamma random variables of means \( (r, t) \). In this case, the following result holds.

\[
E(S_k | A(t) = n) = kt/(n + 1), \quad \forall t > 0
\]

for every \( t > 0 \) and for some \( 1 \leq k \leq n \).
RENEWAL PROCESS

\[ g \{ A(t), t \geq 0 \}, \text{ with} \]
\[ g n \geq 2, r \geq 2, \text{ and} \]
\[ g n = n \}, \text{ for every } t > 0 \]
\[ g \text{ Poisson process can be} \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
\[ g \]
2. A characterization of the gamma distribution

Before we study the case of renewal process started from the next section, we give a characterization of the gamma distribution, by using conditional moments with negative orders. The result can be compared with that of Hall and Simons (1969), where the condition (4) was used.

**THEOREM 1.** Let $X$ and $Y$ be two independent non-degenerate random variables with $E(|X|^r) < \infty$ and $E(|Y|^r) < \infty$, for $r = 1, -1$. If

$$E(X^{-1}|X + Y) = a(X + Y)^{-1} \text{ and } E(Y^{-1}|X + Y) = b(X + Y)^{-1}$$  \hspace{1cm} (6)

hold for some constants $a$ and $b$, then (i) $a > 1, b > 1, ab - a - b > 0$; (ii) $X$ and $Y$, or $-X$ and $-Y$ have gamma distributions with the same scale parameter.

**PROOF.** From (6) we have

$$E(Y/X|X + Y) = a - 1, \hspace{1cm} (7)$$

and

$$E(X/Y|X + Y) = b - 1. \hspace{1cm} (8)$$

For every $\theta \in R$, let $f(\theta) = E(X^{-1}e^{\theta X})$ and $g(\theta) = E(Y^{-1}e^{\theta Y})$, where $i = \sqrt{-1}$. Then (7) and (8) imply

$$f(\theta)g'(\theta) = (a - 1)f'(\theta)g'(\theta), \hspace{1cm} (9)$$

and

$$f''(\theta)g(\theta) = (b - 1)f'(\theta)g'(\theta), \hspace{1cm} (10)$$

respectively. As both $X$ and $Y$ are non-degenerate, (9) and (10) imply $a \neq 1$ and $b \neq 1$. Furthermore, from (9) and (10), we obtain

$$f(\theta) = E(X^{-1})(i^{-1}g'(\theta))^{(a-1)-1}, \hspace{1cm} (11)$$

and

$$f'(\theta) = i(E(Y^{-1}))^{1-b}g(\theta)^{b-1}. \hspace{1cm} (12)$$

Substituting (11) and (12) into (9), yields

$$\frac{a-1}{a}(i)^{(a-1)-1}E(X^{-1})(g'(\theta))^{a/(a-1)} = \frac{a-1}{b}iE(Y^{-1})^{1-b}(g(\theta))^{b} + K_1, \hspace{1cm} (13)$$

where $K_1$ is a constant. Letting $\theta \to 0$ and noting that $E(X^{-1})/a = E(Y^{-1})/b = E(X + Y)^{-1}$, $\lim_{\theta \to 0} g'(\theta) = i$ and $\lim_{\theta \to 0} g(\theta) = E(Y^{-1})$, we have $K_1 = 0$. Thus

$$(g(\theta))^{-b(a-1)/a}g'(\theta) = iE(Y^{-1})^{-b(a-1)/a}. \hspace{1cm} (14)$$

Now if $b(\theta)$ implies $Y$

$$g(\theta) = \frac{1}{b(\theta)},$$

Consequently

and

This completes the proof.

Let $X$ and $Y$ be independent gamma distributed random variables with parameters $\alpha$ and $\beta$, respectively. We can write

$$f(\theta) = \frac{1}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-\theta x}dx = \frac{1}{\Gamma(\alpha)}\int_{\theta^{-1}}^{\infty} (\theta^{-1})^{\alpha-1}e^{-x}dx = \frac{1}{\Gamma(\alpha)}\theta^{\alpha-1}$$

for every $\theta > 0$.

**LEMMA 1.** Let $X$ and $Y$ be independent gamma distributed random variables with parameters $\alpha$ and $\beta$, respectively. Then, for every $\theta > 0$,

$$f(\theta) = \frac{1}{\Gamma(\alpha)}\theta^{\alpha-1},$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$$

Also $E(X^r) = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}$ for every $r > 0$.

Theorem 1. Let $X$ and $Y$ be independent gamma distributed random variables with parameters $\alpha$ and $\beta$, respectively. Then

$$E(X^r|X + Y) = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}.$$

The proof is similar to that of Theorem 1.
RENTER: PROCESS

Now if \( b(a - 1)/a = 1 \), then \( g(\theta) = E(Y^{-1}) \exp\{i(E(Y^{-1}))^{-1}\theta \} \), which in turn implies \( Y \) is degenerate. Therefore \( b(a - 1)/a \neq 1 \), and

\[
g(\theta) = \left( (E(Y^{-1}))^{(a+b-ab)/a} + i\frac{a + b - ab}{a} (E(Y^{-1}))^{-b(a-1)/a} \theta \right)^{a/(a+b-ab)}
\]

Consequently,

\[
E(e^{i\theta Y}) = \frac{1}{i} g'(\theta) = (1 - i \frac{ab - a - b}{aE(Y^{-1})} \theta)^{-b(a-1)/(ab-a-b)}
\]  \( \ldots (15) \)

and

\[
E(e^{i\theta X}) = (1 - i \frac{ab - a - b}{bE(X^{-1})} \theta)^{-a(b-1)/(ab-a-b)}
\]  \( \ldots (16) \)

This completes the proof.

3. Main results

Let the renewal process \( \{A(t), t \geq 0\} \) be defined as in Section 1. Also for a gamma distributed random variable with parameters \( \alpha \) and \( \beta \) (denote this distribution by \( \Gamma(\alpha, \beta) \)), let \( G_{\alpha,\beta}(t) \) denote its distribution function, that is

\[
G_{\alpha,\beta}(t) = \int_0^t x^{\alpha-1} e^{-x/\beta} \frac{dx}{\Gamma(\alpha)\beta^\alpha}
\]  \( \ldots (17) \)

We now present two preliminary results that will be needed in establishing our main results. First we give a lemma which can be proved by using standard technique of conditional expectations.

**Lemma 1.** Let the common distribution function \( F \) of the inter-arrival times of the renewal process \( \{A(t), t \geq 0\} \) have a \( \Gamma(\alpha, \beta) \) distribution. Then for every \( t > 0 \), integers \( r > -k\alpha \) and \( 1 \leq k \leq n \),

\[
E(S^*_k|A(t) = n) = C_{r,k} \beta \frac{G_{n \alpha + r, \beta}(t) - G_{n + 1 \alpha, \beta}(t)}{G_{n \alpha, \beta}(t) - G_{n + 1 \alpha, \beta}(t)},
\]  \( \ldots (18) \)

where

\[
C_{r,k} = \left\{
\begin{array}{ll}
\prod_{j=0}^{r-1}(k\alpha + j), & r \geq 1, \\
1, & r = 0,
\end{array}
\right.
\]  \( \ldots (19) \)

Also \( E(S^*_k|A(t) = n) \) does not exist when \( r \leq -k\alpha \).

The next lemma provides a sufficient condition for a gamma renewal process to be Poisson.
Lemma 2. As in Lemma 1, let $F$ have a $\Gamma(\alpha, \beta)$ distribution. Given the integers $s \neq 0$, $r$ and $1 \leq k_1, k_2 \leq n$, if for some constant $a > 0$,

$$t^r E(S_{k_1}^n | A(t) = n) = aE(S_{k_1}^{n+r} | A(t) = n), \quad \forall t > 0,$$

then $\{A(t), t \geq 0\}$ is a Poisson process, and

$$a = \begin{cases}
\frac{C_{r+k_1}}{C_{r+k_2}} \prod_{j=1}^{r} (n + r + j), & s \geq 1, \\
\frac{C_{r+k_1}}{C_{r+k_2}} \prod_{j=1}^{-s} (n + r + s + j)^{-1}, & s \leq -1.
\end{cases}$$

Proof. First assume $s \geq 1$. By Lemma 1, (20) implies

$$t^r C_{r,k_1} \beta^s \frac{G_{n \alpha + r + s, \beta}(t) - G_{(n+1) \alpha + r + s, \beta}(t)}{G_{n \alpha, \beta}(t) - G_{(n+1) \alpha, \beta}(t)}, \quad \forall t > 0.$$ \hspace{1cm} (22)

Thus

$$\frac{aC_{r+s,k_2}}{C_{r,k_1}} = \frac{\beta^{-r} t^r (G_{n \alpha + r + s, \beta}(t) - G_{(n+1) \alpha + r + s, \beta}(t))}{G_{n \alpha, \beta}(t) - G_{(n+1) \alpha, \beta}(t)}, \quad \forall t > 0.$$ \hspace{1cm} (23)

Letting $t \to 0$ in the right side of (23) and using L'Hospital's rule repeatedly, yields

$$\frac{aC_{r+s,k_2}}{C_{r,k_1}} = \beta^{-r} \sum_{i=0}^{s} \frac{s!}{(s-i)!} \lim_{t \to 0} \frac{t^{s-i} (G_{n \alpha + r + s, \beta}(t) - G_{(n+1) \alpha + r + s, \beta}(t))}{G_{n \alpha + r + s, \beta}(t) - G_{(n+1) \alpha + r + s, \beta}(t)}$$

$$= \beta^{-r} \sum_{i=0}^{s} \frac{s!}{(s-i)!} \beta^s \prod_{j=0}^{i-1} (n \alpha + r + j) \quad \text{for some}\ \frac{s+1}{s+2} \frac{C_{r+s,k_2}}{C_{r,k_1}} = \prod_{j=0}^{s-1} (n \alpha + r + j),$$

where $\prod_{j=0}^{-1}$ is defined to be 1, the superscript $(l + 1)$ denotes the $(l + 1)$th derivative with respect to $t$, and we have used here that for $0 \leq l \leq s - 1$,

$$\lim_{t \to 0} \frac{t^{l-1} (G_{n \alpha + r + s, \beta}(t) - G_{(n+1) \alpha + r + s, \beta}(t))}{G_{n \alpha + r + s, \beta}(t) - G_{(n+1) \alpha + r + s, \beta}(t)} = \beta^s \prod_{j=0}^{l-1} (n \alpha + r + j).$$ \hspace{1cm} (25)

Similarly, by letting $t \to \infty$ in the right side of (23), it follows

$$\frac{aC_{r+s,k_2}}{C_{r,k_1}} = \prod_{j=0}^{s-1} (n \alpha + r + j).$$ \hspace{1cm} (26)
Therefore, by comparing (24) and (26), we obtain \( \alpha = 1 \). This proves the
assertion that \( \{A(t), t \geq 0\} \) is a Poisson process. Substituting \( \alpha = 1 \) into (26)
the constant \( a \) can be obtained immediately.

Finally, when \( s \leq -1 \), by letting \( s' = -s \geq 1 \) and \( r' = r + s' \), then the
equation (20) is equivalent to that for the case \( s \geq 1 \). Hence we obtain the
assertions immediately again.

We now characterize the common inter-arrival distribution function to be
gamma distributed, under the assumption that certain conditional moments of
the arrival times with the same order are assumed to be proportional to each
other.

**Theorem 2.** Assume for some integers \( n \geq 2 \), \( r \geq 2 \), \( 1 \leq k_1 < k_2 < \ldots < k_r \leq n \), and positive constants \( a_i, i = 1, \ldots, r - 1 \),

\[
a_i E(S_i | A(t) = n) = E(S_i | A(t) = n), \quad i = 1, \ldots, r - 1,
\]

for every \( t > 0 \) whenever \( P(A(t) = n) > 0 \). Also assume \( E(X_1^r) < \infty \). Then \( F \) has a \( \Gamma(\alpha, \beta) \) distribution for some constants \( \alpha \) and \( \beta \). Moreover

\[
a_i = \frac{\Pi_{j=0}^{i-1}(k_i \alpha + j)}{\Pi_{j=0}^{i-1}(k_1 \alpha + j)}, \quad i = 1, \ldots, r - 1.
\]

**Proof.** From (27) we obtain (by letting \( a_r = 1 \))

\[
a_i \int_0^t x^r (F_{n-k_i}(t-x) - F_{n-k_{i+1}}(t-x)) \, dF_{k_i}(x)
\]

\[
= a_{i+1} \int_0^t x^r (F_{n-k_{i+1}}(t-x) - F_{n-k_{i+2}}(t-x)) \, dF_{k_{i+1}}(x),
\]

\( i = 1, \ldots, r - 1 \), where \( F_j \) is the \( j \)-fold convolution of \( F \) with itself, \( j \geq 1 \). By
taking the Laplace transforms, (29) can be converted into

\[
a_i (\phi^k(\theta))^{(r)} \frac{\phi^{n-k}(\theta)}{\theta} = a_{i+1} (\phi^{k+1}(\theta))^{(r)} \frac{\phi^{n-k-1}(\theta)}{\theta},
\]

where

\[
\phi(\theta) = \int_0^\infty e^{-\theta x} dF(x), \theta > 0.
\]

After cancelling the common factors, (30) turns to

\[
a_i (\phi^{k+1}(\theta))^{(r)} = a_{i+1} (\phi^{k}(\theta))^{(r)}
\]
for every $\theta > 0$ and $i = 1, \ldots, r - 1$. Now (32) has the form (5) of Hall and Simons (1969), and it is given there that the solution of (32) is

$$\phi(\theta) = (1 + \beta \theta)^{-\alpha},$$  \hspace{1cm} (33)

for some $\alpha, \beta > 0$. This proves that $F$ has a $\Gamma(\alpha, \beta)$ distribution. Using Lemma 1 the constants $a_i$'s are also obtained.

In view of Theorem 2 and Lemma 2, the stronger conditions that $E(S_k^i|A(t) = n)$ is proportional to $t^i$, $\forall i = 1, \ldots, r$, will yield the process $\{A(t), t \geq 0\}$ is Poisson. We state the result in the following.

**Theorem 3.** In Theorem 2, if the conditions in (27) are replaced by

$$c_i E(S_k^i|A(t) = n) = t^i, \quad i = 1, \ldots, r,$$  \hspace{1cm} (34)

for every $t > 0$ whenever $P(A(t) = n) > 0$, where $c_i, i = 1, \ldots, r$, are positive constants, then $\{A(t), t \geq 0\}$ is a Poisson process and

$$c_i = \frac{\prod_{j=1}^{i}(n + j)}{\prod_{j=0}^{i-1}(k_j + j)}, \quad i = 1, \ldots, r.$$  \hspace{1cm} (35)

Next we have a result which is slightly different from Theorem 3 and can be shown by following the steps of the previous theorem. A remark will be given after the theorem.

**Theorem 4.** Assume for some integers $n \geq 2$, $r \geq 2$, $1 \leq k_1 < k_2 < \cdots < k_{r-1} \leq n$, $n_1 \geq 1$, $1 \leq k \leq n_1$, and positive constants $c_i, i = 1, \ldots, r,$

$$c_i E(S_k^i|A(t) = n) = t^i, i = 1, \ldots, r - 1,$$  \hspace{1cm} (36)

and

$$c_r E(S_k|A(t) = n_1) = t,$$  \hspace{1cm} (37)

for every $t > 0$ whenever $P(A(t) = n) > 0$ and $P(A(t) = n_1) > 0$. Also assume $E(X^i) < \infty$. Then $\{A(t), t \geq 0\}$ is a Poisson process.

Note that (37) implies $c_r (k_r / k) E(S_k|A(t) = n_1) = t$, $\forall 1 \leq k \leq n_1$. So that when $r = 2$ and $n = n_1$, (36) and (37) can be reduced to (12) and (13) of Li et al. (1994). Therefore we have obtained a generalization of Theorem 3 of Li et al. (1994).
4. Results based on negative order of conditional moments

In using the reverse martingale assumption such as (5) to characterize the gamma distribution, Hall and Simons (1969) wondered whether there is a solution for \( r \) when \( r \) is positive and non-integral. Although we cannot answer the question for the case of non-integral \( r \), in this section we will give some results related to negative order of conditional moments, which has the same flavor as Theorem 1 for the case of renewal process. Again let \( \{A(t), t \geq 0\} \) be a renewal process as defined in Section 1.

**Theorem 5.** Assume there exist integers \( 1 \leq k_1 < k_2 \leq n \) and a constant \( a_1 > 0 \), such that

\[
a_1 E(S_{k_1}^{-1} | A(t) = n) = E(S_{k_2}^{-1} | A(t) = n),
\]

for every \( t > 0 \), whenever \( P(A(t) = n) > 0 \). Also assume \( E(X_1) < \infty \) and \( E(S_{k_1}^{-1}) < \infty \). Then \( a_1 < k_1/k_2 \) and \( F \) has a \( \Gamma(\alpha, \beta) \) distribution, where \( \alpha > 0 \) and \( \beta = (a_1 - 1)/(a_1 k_2 - k_1) \).

**Proof.** From (38) we obtain

\[
a_1 \int_0^t x^{-1} \left( F_{n-k_1}(t-x) - F_{n-k_1-1}(t-x) \right) dF_{k_1}(x)
= \int_0^t x^{-1} \left( F_{n-k_2}(t-x) - F_{n-k_2-1}(t-x) \right) dF_{k_2}(x),
\]

which in turn implies

\[
a_1 h_1(\theta) \frac{\phi^{n-k_1}(\theta) - \phi^{n-k_1+1}(\theta)}{\theta} = h_2(\theta) \frac{\phi^{n-k_2}(\theta) - \phi^{n-k_2+1}(\theta)}{\theta},
\]

where

\[
h_1(\theta) = \int_0^\infty x^{-1} e^{-\theta x} dF_{k_1}(x),
\]

and

\[
h_2(\theta) = \int_0^\infty x^{-1} e^{-\theta x} dF_{k_2}(x).
\]

Since \( h_1'(\theta) = -\phi^{k_1}(\theta) \) and \( h_2'(\theta) = -\phi^{k_2}(\theta) \), (40) can be rewritten as

\[
\frac{h_1'(\theta)}{h_1(\theta)} = \frac{h_2'(\theta)}{h_2(\theta)} = a_1.
\]

Thus

\[
h_1(\theta) = ch_2(\theta),
\]
where \( c \) is a constant. Differentiating both sides of (44) twice, with respect to \( \theta \), yields

\[
\phi^{(k_1-a_1k_2)/(a_1-1)-1}(\theta)\phi'(\theta) = a_1,
\]

for some constant \( a_1 \).

Now if \( k_1 - a_1 k_2 = 0 \), then \( \phi(\theta) = e^{a_1 \theta} \), which contradicts the assumption that \( F \) is continuous. Hence \( k_1 - a_1 k_2 \neq 0 \) and

\[
\phi(\theta) = (1 + \alpha \theta)^{-1/(a_1 k_2 - k_1)},
\]

where \( \alpha = a_1 (k_1 - a_1 k_2)/(a_1 - 1) \). Finally, in order that \( \phi(\theta) \) is a Laplace transform, \( \beta = (a_1 - 1)/(a_1 k_2 - k_1) \) must be positive. Also as obviously \( a_1 < 1 \), we have \( a_1 < k_1/k_2 \).

Again under stronger conditions the renewal process will become Poisson. Since it can be proved along the lines of Theorem 3, we only state the result.

Theorem 6. Assume there exist integers \( 1 \leq k_1 < k_2 < n \) and constants \( a \) and \( b \), such that

\[
E(S_{k_1}^{-1}|A(t) = n) = at^{-1},
\]

and

\[
E(S_{k_2}^{-1}|A(t) = n) = bt^{-1},
\]

for every \( t > 0 \) whenever \( P(A(t) = n) > 0 \). Also assume \( E(X_1) < \infty \) and \( E(S_{k_2}^{-1}) < \infty \). Then \( k_1 \geq 2 \), \( a = n/(k_1 - 1) \), \( b = n/(k_2 - 1) \), and \( \{A(t), t \geq 0\} \) is a Poisson process.

When \( \{A(t), t \geq 0\} \) is a Poisson process, for integers \( 1 \leq k \leq n \) and \( \tau > -k \),

\[
a_1 E(S_{k}^{\tau}|A(t) = n) = E((t - S_k)^{\tau}|A(t) = n), \quad \forall t > 0,
\]

where \( a_1 \) is a suitable constant. Yet when \( F \) is just gamma distributed, (49) may not be true. We now give a converse result concerned with the case that \( \tau = -1 \).

Theorem 7. Assume there exist two integers \( 2 \leq k \leq n-1 \) and constants \( a \) and \( b \), such that

\[
E(S_{k}^{-1}|A(t) = n) = at^{-1},
\]

and

\[
E((t - S_k)^{-1}|A(t) = n) = bt^{-1},
\]

for every \( t > 0 \) whenever \( P(A(t) = n) > 0 \). Also assume \( E(X_1) < \infty \), \( E(S_{k}^{-1}) < \infty \) and

\[
\int_0^1 t^{-1} F_{n-k}(t)dt < \infty.
\]
Then \( \{A(t), t \geq 0\} \) is a Poisson process and \( a = n/(k - 1), \ b = n/(n - k) \).

PROOF. From (50) and (51), we obtain for every \( t > 0 \),

\[
\int_0^t x^{-1} \left( F_{n-k}(t-x) - F_{n-k+1}(t-x) \right) dF_k(x) = a t^{-1} \left( F_n(t) - F_{n+1}(t) \right), \quad \ldots (53)
\]

and

\[
\int_0^t (t-x)^{-1} \left( F_{n-k}(t-x) - F_{n-k+1}(t-x) \right) dF_k(x) = b t^{-1} \left( F_n(t) - F_{n+1}(t) \right). \quad \ldots (54)
\]

Taking the Laplace transforms of both sides of (53) and (54), respectively, it follows

\[
\xi'(\theta)\eta(\theta) = -a \int_0^\infty e^{-\theta t} t^{-1} \left( F_n(t) - F_{n+1}(t) \right) dt, \quad \ldots (55)
\]

and

\[
\xi(\theta)\eta'(\theta) = -b \int_0^\infty e^{-\theta t} t^{-1} \left( F_n(t) - F_{n+1}(t) \right) dt, \quad \ldots (56)
\]

where for \( \theta > 0 \),

\[
\xi(\theta) = \int_0^\infty t^{-1} e^{-\theta t} \left( F_{n-k}(t) - F_{n-k+1}(t) \right) dt, \quad \ldots (57)
\]

and

\[
\eta(\theta) = \int_0^\infty t^{-1} e^{-\theta t} dF_k(t). \quad \ldots (58)
\]

Differentiating both sides of (55) and (56), respectively, we have

\[
\xi''(\theta)\eta(\theta) = (a - 1)\xi'(\theta)\eta'(\theta), \quad \ldots (59)
\]

and

\[
\xi(\theta)\eta''(\theta) = (b - 1)\xi'(\theta)\eta'(\theta). \quad \ldots (60)
\]

Also (55) and (56) imply

\[
\xi'(\theta)\eta(\theta) = (a/b)\xi(\theta)\eta'(\theta). \quad \ldots (61)
\]

Solving (59), (60) and (61) we obtain the assertions.

5. Some extensions of the results by Li et al. (1994)

In this section we give some simple extensions of the results in Li et al. (1994). First we extend Theorem 3 of the above paper.
THEOREM 8. Assume for some fixed integers $1 \leq k_1 \leq n_1$, and $1 \leq k_2 \leq n_2$,
\[ E(S_{k_1} \mid A(t) = n_1) = at, \quad \ldots (62) \]
and
\[ E(S_{k_2}^2 \mid A(t) = n_2) = bt^2 + ct, \quad \ldots (63) \]
hold for some constants $a$, $b$, and $c$, for every $t > 0$, whenever $P(A(t) = n_i) > 0$, $i = 1, 2$. Also assume $E(X_i^2) < \infty$. Then
(i) $a = k_1/(n_1 + 1)$, $b = k_2(k_2 + 1)/((n_2 + 1)(n_2 + 2))$ and $c = 0$; 
(ii) $\{A(t), t \geq 0\}$ is a Poisson process.

PROOF. As in the proof of Theorem 3 of Li et al. (1994), (62) implies
\[ \frac{1 - \phi(\theta)}{\theta} = \mu_1 \phi^{k_1a^{-1} - n_1} \phi(\theta), \quad \ldots (64) \]
Hence
\[ \frac{1 - \phi(\theta)}{\theta} = \mu_1 \phi^{k_1(k_1a^{-1} - n_1 + n_2)/k_2 - n_2} \phi(\theta), \quad \ldots (65) \]
From this we have
\[ E(S_{k_1} \mid A(t) = n_2) = a't, \quad \ldots (66) \]
where $a' = k_2/(k_1a^{-1} - n_1 + n_2)$. Now using Theorem 3 of Li et al. (1994), (66) and (63) together imply the assertions.
Thus although (62) and (63) look more general than (12) and (13) of Li et al. (1994), basically there are not much difference between these two pairs of conditions.
Similarly, we have the following parallel extension of Theorem 4 of Li et al. (1994).

THEOREM 9. Assume for some fixed integers $1 \leq k_1 \leq n_1$ and $2 \leq k_2 \leq n_2$,
\[ E(S_{k_1} \mid A(t) = n_1) = at, \quad \ldots (67) \]
and
\[ E(S_{k_2}^{-1} \mid A(t) = n_2) = bt^{-1}, \quad \ldots (68) \]
hold for some constants $a$ and $b$, for every $t > 0$ whenever $P(A(t) = n_i) > 0$, $i = 1, 2$. Also assume $E(X_i) < \infty$ and $E(S_{k_i}^{-1}) < \infty$. Then
(i) $a = k_1/(n_1 + 1)$ and $b = n_2/(k_2 + 1)$; 
(ii) $\{A(t), t \geq 0\}$ is a Poisson process.

Theorems 3, 4 and 5 of Li et al. (1994) are special cases of the following theorem (corresponding to $r = 0$, $r = -1$ and $r = -2$, respectively). This theorem can also be compared with Theorem 3 of the present paper, where there
are \( r \) equations, and here only two equations are needed.

**Theorem 10.** Assume for some integers \( r \) and \( 1 \leq k \leq n \),

\[
t E(S_k^r | A(t) = n) = a E(S_{k+1}^r | A(t) = n), \tag{69}
\]

and

\[
t E(S_k^{r+1} | A(t) = n) = b E(S_{k+2}^{r+1} | A(t) = n), \tag{70}
\]

hold for some constants \( a \) and \( b \), for every \( t > 0 \) whenever \( P(A(t) = n) > 0 \).

Also assume \( E(X_1^{r+2}) < \infty \) if \( r \geq 0 \), or \( E(X_1) < \infty \) and \( E(S_k^r) < \infty \) if \( r < 0 \).

Then

(i) \( r > -k, a = (n + r + 1)/(k + r), b = (n + r + 2)/(k + r + 1) \);

(ii) \( \{A(t), t \geq 0\} \) is a Poisson process.

**Proof.** From (69) and (70), it follows

\[
\frac{\left(\phi^{n-k}(\theta) - \phi^{n-k+1}(\theta)\right)/\theta}{\left(\phi^{n-1}(\theta) - \phi^{n-k+1}(\theta)\right)/\theta} = (a - 1) \frac{q'(\theta)}{q(\theta)}, \tag{71}
\]

and

\[
\frac{\left(\phi^{n-k}(\theta) - \phi^{n-k+1}(\theta)\right)/\theta}{\left(\phi^{n-k}(\theta) - \phi^{n-k+1}(\theta)\right)/\theta} = (b - 1) \frac{q''(\theta)}{q'(\theta)}, \tag{72}
\]

where

\[
q(\theta) = \int_0^\infty x^r e^{-\beta x} d F_k(x). \tag{73}
\]

Note that

\[
q(\theta) = (\phi^k(\theta))^{(r)}, \quad r \geq 0, \tag{74}
\]

and

\[
(q(\theta))^{(-r)} = \phi^k(\theta), \quad r < 0. \tag{75}
\]

Also it is easy to see that both \( a \) and \( b \neq 1 \). Hence

\[
\frac{q''(\theta)}{q'(\theta)} = \frac{a - 1}{b - 1} \frac{q'(\theta)}{q(\theta)} \tag{76}
\]

which has the solution \( q(\theta) = (m_1 \theta + m_2)^e \), where \( m_1, m_2 \) are constants and \( e = (a - 1)/(b - 1) \). This together with (74) or (75) imply \( \phi(\theta) = (1 + \beta \theta)^{-\alpha} \) for some \( \alpha, \beta > 0 \). Finally, the assertions (i) and (ii) are achieved by using Lemma 2.
6. Concluding remark

Inspired by Theorem 10, under certain conditions, we can use

\[ (X + Y)E(X^{r} | X + Y) = aE(X^{r+1} | X + Y), \]  \hspace{1cm} \ldots (77)

and

\[ (X + Y)E(X^{r+1} | X + Y) = bE(X^{r+2} | X + Y), \]  \hspace{1cm} \ldots (78)

where \( a \) and \( b \) are constants, to characterize \( X \) and \( Y \) to be gamma distributed.

It is easy to see that this is a generalization of Wesolowski (1989), (1990), and Theorem 1 of Li et al. (1994), where they used (2), for a pair of \((r, s) \in B\), to characterize \( X \) and \( Y \) to be gamma distributed.

In Theorem 2, for each \( i = 1, \ldots, r - 1 \), we can replace \( n \) by \( n_i \) in the \( i \)th equation of (27) and still obtain similar characterizations, if some suitable modifications about the conditions for the integers \( \{ n_i \} \) and \( \{ k_i \} \) are made. That is Theorem 2 can be further generalized. We state the theorem in the present form as it is simpler and easier to understand. On the other hand, the conditions

\[ E(S_{k}^2 | A(t) = n) = at^2 \text{ and } E((t - S_k)^2 | A(t) = n) = bt^2 \]  \hspace{1cm} \ldots (79)

are equivalent to

\[ E(S_{k}^2 | A(t) = n) = at^2 \text{ and } E(S_k | A(t) = n) = ct, \]  \hspace{1cm} \ldots (80)

where \( c = (1 + a - b)/2 \). Also it is already known that using the two equations in (80), \( F \) can be characterized. Yet for the cases such as given

\[ E(S_{k}^{-2} | A(t) = n) = at^{-2} \text{ and } E(S_{k}^{-2} | A(t) = n) = bt^{-2}, \]  \hspace{1cm} \ldots (81)

or

\[ E(S_{k}^{-2} | A(t) = n) = at^{-2} \text{ and } E((t - S_k)^{-2} | A(t) = n) = bt^{-2}, \]  \hspace{1cm} \ldots (82)

as the computations become very complicated, we are still unable to determine the distribution function \( F \) from either (81) or (82).

Acknowledgements. The authors are grateful to the referee for his helpful comments and the Managing Editor for his assistance.
RENWEAL PROCESS

References


DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL SUN YAT-SEN UNIVERSITY
KAOSHING, TAIWAN 80424
R.O.C.