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# ON A STUDY OF SOME PROPERTIES OF POINT PROCESSES

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*SUMMARY*. In this article we study some interesting properties of point processes. First we provide an elementary proof of the characterization of point processes with the order statistics property. Then we give certain characterization results for point processes with both the Markov and exchangeable properties. Among others, we also give a discrete time version of the order statistics property and obtain a characterization of the mixed geometric process.

## 1. Introduction

Let $\{N(t),\ t \geqslant 0\}$ be a point process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $N(0) = 0$, $N(t) < \infty$, $\forall\ t \geqslant 0$, a.s., and right continuous sample paths having successive unit steps at times $S_1 = \xi_1$, $S_2 = \xi_1 + \xi_2$, ..., where $\xi_i$ is the $i$th inter-arrival time. The case with $N(t) = 0$, for every $t > 0$, with positive probability is being excluded at the outset from all the following considerations. First, we give the following definitions.

*Definition* 1. $\{N(t),\ t \geqslant 0\}$ is said to have the exchangeable property E if for every positive integer $k$, whenever $P(N(t) = k) > 0$, $P(\xi_i \leqslant x_i,\ i = 1, \ldots, k \mid N(t) = k)$ is symmetric in $x_1, \ldots, x_k$.

*Definition* 2. $\{N(t),\ t \geqslant 0\}$ is said to be a $\mu$-mixed renewal process if for every positive integer $n$,

$$P(\zeta_i \leqslant x_i,\ i = 1, \ldots, n) = \int_{\Lambda} \prod_{i=1}^{n} P_{\lambda}(\xi_i \leqslant x_i) d\mu,$$

where the family $\{P_{\lambda},\ \lambda \epsilon \Lambda\}$ is such that $P_{\lambda}(\xi_i < 0) = 0$ for every $\lambda \epsilon \Lambda$ and $\mu$ is a probability measure on $B(\Lambda)$, the smallest $\sigma$-algebra of subsets of $\Lambda$ over which all the $\lambda$-functions $P_{\lambda}(A)$ are measurable, for all $A \epsilon \mathcal{F}$.

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In Definition 2, in particular, if for almost all $\lambda \epsilon \Lambda$, $P_\lambda(\xi_t \leqslant x) = 1 - \exp\{-b(\lambda)x\}$, for some function $b(\lambda) > 0$, then $\{N(t), t \geqslant 0\}$ is said to be a mixed Poisson process. For simplicity and without loss of generality, in the following when we mention a mixed Poisson process, we always let $\Lambda$ be a subset of $(0, \infty)$ and $b(\lambda) = \lambda$. For a point process $\{N(t), t \geqslant 0\}$, it will be said to have the property P, if conditional on $N(t) = k$, the successive jump times $\{S_1, S_2, \ldots, S_k\}$ are distributed as the order statistics of $k$ i.i.d. r.v.'s with the common $\mathcal{U}(0, t]$ distribution. Feigin (1979) proves that a point process $\{N(t), t \geqslant 0\}$ has the property P if and only if it is a mixed Poisson process. By using a result from Freedman (1962), we will give an elementary proof of this fact in Section 2.

In Huang (1990) it is shown that if $\lim_{t \to \infty} N(t) = \infty$, a.s., then $\{N(t), t \geqslant 0\}$ has the property E if and only if it is a mixed renewal process. Also it has been proved that, under some suitable conditions, a mixed renewal process is Markovian if and only if it is a mixed Poisson process. In Section 3, we will give a shorter proof of this result. On the other hand we will also characterize a point process to posses both the exchangeable and Markov properties when $\lim_{t \to \infty} N(t) = K$, a finite constant.

Inspired by the property P, we may ask whether there is a corresponding property for the discrete time processes. In order to distinguish easily, let $\{A(t), t = 0, 1, 2, \ldots\}$ denote a discrete time process throughout this paper. As we assume unit step jumps, obviously, $\{A(t), t = 0, 1, 2, \ldots\}$ cannot have the property P. We give the following definition instead.

*Definition* 3. $\{A(t), t = 0, 1, 2, \ldots\}$ is said to have the property T if for every positive integers $k$ and $t$, $0 < k \leqslant t$, whenever $P(A(t) = k) > 0$

$$P(S_i = s_i,\ i = 1, 2, \ldots, k \mid A(t)=k)=C_{k,t},\ \forall 0 < s_1 < \ldots < s_k \leqslant t, \quad \ldots \quad (1)$$

where $C_{k,t}$ is a constant depending only on $k$ and $t$.

When $\{A(t), t = 0, 1, 2, \ldots\}$ has the property T,

$$\sum_{1 \leqslant s_1 < s_2 < \ldots < s_k \leqslant t} P(S_i=s_i, i=1,\ldots,k \mid A(t)=k) = \sum_{1 \leqslant s_1 < s_2 < \ldots < s_k \leqslant t} C_{k,t} = 1, \quad \ldots \quad (2)$$

thus $C_{k,t} = \binom{t}{k}^{-1}$. That is

$$P(S_i = s_i,\ i = 1, \ldots, k \mid A(t) = k) = \binom{t}{k}^{-1}. \quad \ldots \quad (3)$$

or equivalently, conditional by on $A(t) = k$, $S_1, \ldots, S_k$ are distributed as the order statistics of a sample drawn from the set $\{1, 2, \ldots, t\}$, without replacement. Therefore, the property T can be viewed as a discrete time version of the property P.

In Section 4, we prove that the property T characterizes the mixed geometric process. In Section 5, we will show that in the class of discrete time processes, a mixed renewal process is a mixed geometric process if and only if it is a Markov process, which is a discrete time version of the result in Section 3.

## 2. Order statistics property

In this section we will characterize point processes with the property P. We need the following theorem which was due to Freedman (1962).

Theorem 1. *The process $\{X_n, n \geqslant 1\}$ is a sequence of mixture of Poisson variables, namely, $\forall n \geqslant 1$*

$$P(X_1 = a_1, \ldots, X_n = a_n) = \int_0^\infty \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{a_i}}{a_i!} d\mu \qquad \ldots \quad (4)$$

*if and only if for some function $\varphi(.,.)$*

$$P(X_i = a_i, i = 1, \ldots, n) = \left( \prod_{i=1}^n a_i! \right)^{-1} \varphi \left(n, \sum_{i=1}^n a_i \right). \qquad \ldots \quad (5)$$

Here the meaning of mixture is defined similarly as in Definition 2. That is $\{X_n, n \geqslant 1\}$ satisfies (4) if and only if there exist a set $\Lambda \subset (0, \infty)$, and a probability measure $\mu$ on $B(\Lambda)$, such that given $\lambda \in \Lambda$, $\{X_n, n \geqslant 1\}$ are i.i.d. Poisson r.v.'s with parameter $\lambda$. On the other hand, Diaconis and Freedman (1980) have pointed out that (5) is equivalent to :

For every $n \geqslant 1$, the joint distribution of $X_1, \ldots, X_n$, given $W_n = \sum_{i=1}^n X_i$, is multinomial with $p = 1/n$, $\forall i = 1, \ldots, n$.

Using the above result, we are going to give an elementary proof that the property P characterizes the class of mixed Poisson processes.

Theorem 2. *The point process $\{N(t), t \geqslant 0\}$ has the property* P *if and only if it is a mixed Poisson process.*

*Proof.* The "if" part is omitted. We now prove the "only if" part. Let $Y_i = N(i) - N(i-1)$, $i = 1, 2, \ldots$. Then by using the property P we can easily obtain that for every $n \geqslant 1$, the joint distribution of $Y_1, Y_2, \ldots, Y_n$, given $W_n = \sum_{i=1}^n Y_i$, is multinomial with $p_i = 1/n$, $\forall i = 1, \ldots, n$. Hence by Theorem 1, we obtain $\{Y_i, i \geqslant 1\}$ is a sequence of mixture of Poisson variables, i.e., there exist a set $\Lambda \subset (0, \infty)$, a probability measure $\mu$ on $B(\Lambda)$, and a random variable $\Gamma$ defined on $\Lambda$ with $\Gamma(\lambda) = \lambda$, such that given $Y_i$, $i \geqslant 1$, are i.i.d. Poisson variables with parameter $\lambda$.

Next, let $m \geqslant 2$ be any integer and let $Z_i = N(i/m) - N((i-1)/m)$, $i = 1, 2, \ldots$. Again we obtain that $\{Z_i, i \geqslant 1\}$ is also a sequence of mixtures of Poisson variables associated with a probability measure $\mu'$ and a random variable $\Gamma'$ with $\Gamma'(\lambda) = \lambda$, such that given $\Gamma' = \lambda'$, $Z_i$, $i \geqslant 1$, are i.i.d. Poisson variables with parameter $\lambda'$.

Since $\{Y_i, i \geqslant 1\}$ are both exchangeable, by the (conditional) strong law of large numbers, we have as $n \to \infty$

$$\frac{\sum_{i=1}^{n} Y_i}{n} \to E(Y_1 \mid \Gamma) = \Gamma, \text{ a.s., and } \frac{\sum_{n=1}^{n} Z_i}{n} \to E(Z_1 \mid \Gamma') = \Gamma', \text{ a.s.} \qquad \ldots (6)$$

Furthermore, since

$$\frac{\sum_{i=1}^{n} Y_i}{n} = \frac{N(n)}{n} \text{ and } \frac{\sum_{i=1}^{n} Z_i}{n} = \frac{N(n/m)}{n},$$

(6) implies that $\Gamma' = \Gamma/m$, a.s.

From the above discussion, we conclude that for any rational numbers $0 = t_0 < t_1 < \ldots < t_n$, given $\Gamma = \lambda$, $N(t_1)$, $N(t_2) - N(t_1)$, $\ldots$, $N(t_n) - N(t_{n-1})$ are independent Poisson r.v.'s with $E(N(t_i) - N(t_{i-1})) = \lambda(t_i - t_{i-1})$, $i = 1, \ldots, n$.

On the other hand, given $0 < t_1 < t_2 < \ldots < t_n$, for any $k \geqslant 1$, we can find rational numbers $0 < x_{1k} < x_{2k} < \ldots < x_{nk}$, such that $0 < x_{ik} - t_i < 1/k$, $\forall i = 1, \ldots, n$. Also the property P in turn implies

$$P(N(x_{ik}) - N(t_i) > 0) \to 0, \text{ as } k \to \infty, \qquad \ldots (7)$$

$\forall i = 1, \ldots, n$. Therefore, by a standard argument it can be shown that $(N(x_{1k}), \ldots, N(x_{nk}))$ converges in distribution to $(N(t_1), \ldots, N(t_n))$ as $k \to \infty$. Hence, given $\lambda \in \Lambda$, $N(t_1)$, $N(t_2) - N(t_1)$, $\ldots$, $N(t_n) - N(t_{n-1})$ are independent Poisson r.v.'s with $E(N(t_i) - N(t_{i-1}) = \lambda(t_i - t_{i-1})$, $i = 1, \ldots, n$, respectively. This proves $\{N(t), t \geqslant 0\}$ is a mixed Poisson process.

## 3. Mixed poisson process

In this section, we will prove that a mixed renewal process is a mixed Poisson process if and only if it is a Markov process. Also we will characterize a terminating process to possess both the Markov and exchangeable properties. Throughout this section for a Markov process $\{N(t),\ t \geqslant 0)\}$ we assume it admits instantaneous rates defined by

$$\lambda_n(t) = \lim_{h \downarrow 0} \frac{1}{h} P(N(t+h) = n+1 \mid N(t) = n),\ n = 0, 1, 2, \ldots, t \geqslant 0. \quad \ldots \quad (8)$$

Also assume $F'_n$ is absolutely continuous with $F'_n(t) = f_n(t)$ exists, where $1 - F_n(t) = \exp(-\int_0^t \lambda_n(u)du)$.

First, we give the following lemma which was due to Pfeifer and Heller (1987).

Lemma 1. *Let $\{N(t),\ t \geqslant 0\}$ be an elementary pure birth process with finite birth rates $\{\lambda_n(t),\ t \geqslant 0\}$, $\forall n = 0, 1, 2, \ldots$. Then $\{N(t),\ t \geqslant 0\}$ is a mixed Poisson process if and only if there exist constants $c_n > 0$, $n = 0, 1, 2, \ldots$, such that $1 - F_{n+1}(t) = c_n f_n(t)$, $\forall t \geqslant 0$.*

The following theorem was due to Huang (1990), using Lemma 1 here we give a shorter proof.

Theorem 3. *Let $\{N(t),\ t \geqslant 0\}$ be a $\mu$-mixed renewal process with the given family $\{P_\lambda,\ \lambda \in \Lambda\}$ of probability measures and the mixing probability measure $\mu$ on $B(\Lambda)$. Then $\{N(t),\ t \geqslant 0\}$ is a Markov process if and only if $\{N(t),\ t \geqslant 0\}$ is a mixed Poisson process.*

*Proof.* The "if" part is obvious, it remains to prove the "only if" part.

Since $\{N(t), t \geqslant 0\}$ is a mixed renewal process, it is also exchangeable. Hence for every $n = 1, 2, \ldots$, and $t_n, t_{n+1}, s \geqslant 0$,

$$P(t_n < \xi_n \leqslant t_n + dt_n,\ t_{n+1} < \xi_{n+1} \leqslant t_{n+1} + dt_{n+1} \mid \sum_{i=1}^{n-1} \xi_i = s)$$

$$= P(t_{n+1} < \xi_n \leqslant t_{n+1} + dt_{n+1},\ t_n < \xi_{n+1} \leqslant t_n + dt_{n+} \mid \sum_{i=1}^{n-1} \xi_i = s), \quad \ldots \quad (9)$$

where as before $\xi_n$ is the $n$-th inter-arrival time. In terms of the instantaneous rate, (9) implies

$$\exp\left(-\int_{s+t_n}^{s+t_{n+1}} \lambda_n(\tau)d\tau\right) = \frac{\lambda_{n-1}(s+t_{n+1})}{\lambda_{n-1}(s+t_n)} \exp\left(-\int_{s+t_n}^{s+t_{n+1}} \lambda_{n-1}(\tau)d\tau\right), \ldots \quad (10)$$

or

$$\exp\left(-\int_{w_1}^{w_2} \lambda_n(\tau)d\tau\right) = \frac{\lambda_{n-1}(w_2)}{\lambda_{n-1}(w_1)} \exp\left(-\int_{w_1}^{w_2} \lambda_{n-1}(\tau)d\tau\right), \quad \ldots \quad (11)$$

$\forall n = 1, 2, \ldots$, and $w_1, w_2 \geqslant 0$. From (11), we obtain for every $n = 0, 1, 2, \ldots$, and $t \geqslant 0$,

$$1 - F_{n+1}(t) = [\lambda_n(0)]^{-1} f_n(t). \quad \ldots \quad (12)$$

The result now follows from Lemma 1.

Next we give a result which is concerned with the case that $\lim_{t\to\infty} N(t)$ is finite.

Theorem 4. *Let* $N(t) \uparrow K$, *a.s., at* $t \to \infty$, *where* $K$ *is a positive integer. Assume that* $\{N(t), t \geqslant 0\}$ *has the exchangeable and Markov properties. Then* $\{N(t), t \geqslant 0\}$ *is a process whose inter-arrival times form a sequence of mixture of* $K$ *exponential r.v.'s.*

*Proof.* By a similar approach as in Theorem 3 yields (11) hold for $n = 1, 2, \ldots, K-1$. Therefore, if we continue to generate this process after the $K$-th arrival occurs, letting the successive instantaneous rates $\lambda_n(t)$, $n \geqslant K$, satisfying (11), then we obtain a pure birth process, say $\{J(t), t \geqslant 0\}$, with $F_{n-1}$ and $f_n$ satisfying (12) for every $n = 0, 1, 2, \ldots$, and $t \geqslant 0$. By Lemma 1, this in turn implies $\{J(t), t \geqslant 0\}$ is a mixed Poisson process. The theorem is now established by the fact that the arrival times of the process $\{N(t), t \geqslant 0\}$ are just the first $K$ arrival times of the process $\{J(t), t \geqslant 0\}$

## 4. Mixed geometric process

In this section we will characterize the processes with the property T. We prove that a process has the property T if and only if it is a mixed geometric process. First we give the following definition.

*Definition* 4. A process $\{A(t), t = 0, 1, \ldots\}$ is called a mixed geometric process if and only if there exist a set $\Theta \subset (0, 1)$ and a probability measure $\mu$ on $B(\Theta)$, such that given $\theta \in \Theta$, the inter-arrival times are i.i.d. with the common mass function $P_\theta(X = i) = \theta(1-\theta)^{i-1}$, $i = 1, 2, \ldots$ .

The following lemma can be found in Freedman (1962).

Lemma 2. *The process* $\{X_n, n \geqslant 1\}$ *is a sequence of mixture of binomial variables, namely, for every* $n \geqslant 1$ *and* $0 \leqslant j_i \lessgtr N$, $i = 1, \ldots, n$,

$$P(X_i = j_i, i = 1, \ldots, n\} = \int_0^1 \left[ \prod_{i=1}^n \binom{N}{j_i} \theta^{j_i}(1-\theta)^{(N-j_i)} \right] du \quad \ldots \quad (13)$$

*if and only if for some function* $\phi$ (., .)

$$P(X_i = j_i, i = 1, \ldots, n) = \left[ \prod_{i=1}^n \binom{N}{j_i} \right] \phi\left(n, \sum_{i=1}^n j_i\right). \quad \ldots \quad (14)$$

In order to prove our main result of this section we need the following lemma.

Lemma 3. *A process* $\{A(t), t = 0, 1, \ldots\}$ *is a mixed geometric process if and only if there exist a set* $\Theta \subset (0, 1)$ *and a probability measure* $\mu$ *on* $B(\Theta)$ *such that, given* $\theta \in \Theta$, *for any* $0 = N_0 < N_1 < N_2 < \ldots < N_k$, $A(N_1)$, $A(N_2) - A(N_1), \ldots, A(N_k) - A(N_{k-1})$ *are independent* $\mathcal{B}(N_i - N_{i-1}, \theta)$ *r.v.'s*, $i = 1, \ldots, k$.

*Proof.* The "only if" part can be proved very easily, and the "if" part can be obtained immediately by the following observation. The assumption implies that given $\theta \in \Theta$, for any positive integer $n$

$$P_\theta(A(n+1) - A(n) = 1 \mid A(t), t = 1, \ldots, n) = P_\theta(A(n+1) - A(n) = 1) = \theta. \quad \ldots \quad (15)$$

Now, we are ready to characterize the property T.

Theorem 5. *A process* $\{A(t), t = 0, 1, \ldots\}$ *is a mixed geometric process if and only if it has the property* T.

*Proof.* Again we only prove the "if" part. The property T implies, for every integers $k$ and $t$, $0 < k \leqslant t$, whenever $P(A(t) = k) > 0$,

$$P(S_i = s_i, i = 1, \ldots, k \mid A(t) = k) = \binom{t}{k}^{-1}. \quad \ldots \quad (16)$$

For any positive integer $m$, let $Y_i = A(im) - A((i-1)m), i = 1, 2, \ldots$. Then for every $k \geqslant 1$ and $j_i \leqslant m, i = 1, \ldots, k$, from (16) it follows that

$$P\left(Y_1 = j_1, Y_2 = j_2, \ldots, Y_k = j_k \mid \sum_{i=1}^{k} Y_i = \sum_{i=1}^{k} j_i\right) = \prod_{i=1}^{k} \binom{m}{j_i} \binom{km}{\sum_{i=1}^{k} j_i}^{-1} \quad \ldots \quad (17)$$

or

$$P(Y_1 = j_1, Y_2 = j_2, \ldots, Y_k = j_k) = P\left(\sum_{i-1}^{k} Y_i = \sum_{i-1}^{k} j_i\right) \binom{km}{\sum_{i=1}^{k} j_i}^{-1} \left[\prod_{i=1}^{k} \binom{m}{j_i}\right] \quad \ldots \quad (18)$$

Hence, in view of Lemma 2, we obtain $\{Y_i, i \geqslant 1\}$ is a sequence of mixture of $\mathcal{B}(m, \theta)$ r.v.'s, i.e., there exist a set $\Theta \subset (0, 1)$, a probability measure $\mu$, and a random vairable $\Upsilon$ defined on $\Theta$ with $\Upsilon(\theta) = \theta$, such that given $\Upsilon = \theta$, $Y_1$, $Y_2, \ldots$ are i.i.d. $\mathcal{B}(m, \theta)$ r.v.'s.

Similarly, for any positive integers $m' \neq m$, let $Z_i = A(im') - A((i-1)m')$, $i = 1, 2, \ldots$, we obtain $\{Z_i, i \geqslant 1\}$ is a sequence of mixture of $\mathcal{B}(m', \theta')$ r.v.'s, i.e., there exist a set $\Theta' \subset (0, 1)$, a probability measure $\mu'$ and a random variable $\Upsilon'$ with $\Upsilon'(\theta) = \theta$, such that given $\Upsilon' = \theta'$, $Z_1, Z_2, \ldots$ are i.i.d. r.v.'s with the common $\mathcal{B}(m', \theta')$ distribution.

Now as in Theorem 2, by the strong law of large numbers, we obtain

$$\lim_{n\to\infty} \frac{A(n)}{n} = \Upsilon = \Upsilon', \text{ a.s.} \qquad \dots \quad (19)$$

It follws that for any integers $0 = N_0 < N_1 < N_2 < \dots < N_k$, given $\theta \in \Theta$, $A(N_1)$, $A(N_2)-A(N_1)$, ..., $A(N_k)-A(N_{k-1})$ are independent with $A(N_i)-A(N_i)$ having a $\mathcal{B}(N_i-N_{i-1}, \theta)$ distribution, $i = 1, \dots, k$. Thus by Lemma 3, this proves that $\{A(t), t = 0,1, \dots\}$ is a mixed geometric process as required.

## 5. A FURTHER DISCUSSION OF THE MIXED GEOMETRIC PROCESS

In this section, for a mixed renewal process with discrete inter-arrival times, we will prove that if this process possesses the Markov porperty then it will be a mixed geometric process.

Theorem 6. *A $\mu$-mixed renewal process $\{A(t), t = 0, 1, \dots\}$ is a mixed geometric process if and only if it is a Markov process.*

*Proof.* As before the "only if" part is obvious. To prove the "if" part, for any $0 \leqslant u < t, t \geqslant 1$ and $v \geqslant 0$, by the Markov property we have

$$\begin{aligned} &P(A(t+v) = A(t) = A(t-u) = 1) \\ &= P(A(t) = A(t-u) = 1)P(A(t+v) = 1 \mid A(t) = 1). \end{aligned} \qquad \dots \quad (20)$$

Since $\{A(t), t = 0, 1, \dots\}$ is a mixed renewal pocess, there exist a set $\Lambda \subset (0, \infty)$ and a prbability measure $\mu$ on $B(\Lambda)$, such that given $\lambda \in \Lambda$, $\{A(t), t = 0, 1, \dots\}$ is renewal process. From this, (20) can be rewritten as

$$\frac{E\left[\sum_{i=1}^{t-u} P_\lambda(X = i)P_\lambda(X > t+v-i)\right]}{E\left[\sum_{i=1}^{t-u} P_\lambda(X = i)P_\lambda(X > t-i)\right]} = \frac{E\left[\sum_{i=1}^{t} P_\lambda(X = i)P_\lambda(X > t+v-i)\right]}{E\left[\sum_{i=1}^{t} P_\lambda(X = i)P_\lambda(X > t-i)\right]}. \qquad \dots \quad (21)$$

Let $f(u) = E\left[\sum_{i=1}^{t-u} P_\lambda(X = i)P_\lambda(X > t+v-i)\right]$ and $g(u) = E\left[\sum_{i=1}^{t-u} P_\lambda(X = i)\right.$ $\left.P_\lambda(X > t-i)\right]$. As the right side of (21) is independent of $u$, the left side, which is equal to $f(u)/g(u)$, is also independent of $u$. Hence

$$\frac{f(u)}{g(u)} = \frac{f(u)-f(u+1)}{g(u)-g(u+1)}. \qquad \dots \quad (22)$$

Now the right side of (22) is equal to

$$\frac{E[P_\lambda(X = t-u)P_\lambda(X > v+u)]}{E[P_\lambda(X = t-u)P_\lambda(X > u)]}, \qquad \dots \quad (23)$$

which is also independent of $u$. Thus by letting $u = 0$ and $u = t-1$ in (23) respectively, we obtain

$$\frac{E[P_\lambda(X = 1)P_\lambda(X > t-1)]}{E[P_\lambda(X = t)]} = \frac{E[P_\lambda(X = 1)P_\lambda(X > v+t-1)]}{E[P_\lambda(X = t)P_\lambda(X > v)]} . \quad \dots (24)$$

Similarly, the right side of (24), since it is still independent of $v$, can be rewritten as

$$\frac{E[P_\lambda(X = 1)P_\lambda(X = t+v)]}{E[P_\lambda(X = t)P_\lambda(X = v+1)]}, \quad \dots (25)$$

which is equal to 1, as can be seen by letting $v = 0$. Consequently

$$\frac{E[P_\lambda(X = 1)P_\lambda(X > v+t-1)]}{E[P_\lambda(X = t)P_\lambda(X > v)]} = \frac{E[P_\lambda(X = 1)P_\lambda(X > t-1)]}{E[P_\lambda(X = t)]} = 1. \quad \dots (26)$$

On the other hand, for any $0 \leqslant u < t, v \geqslant 0$ and $w \geqslant 1$, by considering the following two equations

$$\begin{aligned} &P(A(t-u) = A(t) = A(t+v) = 1, A(t+v+w) = 2) \\ &= P(A(t-u) = A(t) = 1)P(A(t+v) = 1 \,|\, A(t) = 1) \\ &.P(A(t+v+w) = 2 \,|\, A(t+w) = 1) \end{aligned} \quad \dots (27)$$

and

$$\begin{aligned} &P(A(t-u) = A(t) = 1, A(t+w) = A(t+w+v) = 2) \\ &= P(A(t-u) = A(t) = 1)P(A(t+w) = 2 \,|\, A(t) = 1) \\ &P(A(t+w+v) = 2 \,|\, (A(t+w) = 2), \end{aligned} \quad \dots (28)$$

after some manipulations, we obtain

$$E[P_\lambda^2(X = 1)P_\lambda(X > t+v-1)] = E[P_\lambda(X = t)P_\lambda(X = 1)(P_\lambda(X > v)] \quad \dots (29)$$

and

$$E[P_\lambda^2(X = 1)P_\lambda(X > v)(P_\lambda(X > t-1)] = E(P_\lambda(X = 1)P_\lambda(X = t)P_\lambda(X > v)]. \quad \dots (30)$$

Using (26), (29) and (30) we obtain

$$E[P_\lambda(X = t)-P_\lambda(X = 1)P_\lambda(X > t-1)]^2 = 0, \forall t \geqslant 1. \quad \dots (31)$$

This in turn implies

$$P_\lambda(X = t) = P_\lambda(X = 1)P_\lambda(X > t-1), \quad \text{a.s.} \quad \dots (32)$$

If follows that $P_\lambda(X = t) = P_\lambda(X = 1)\,[P_\lambda(X > 1)]^{t-1}, \forall t = 1, 2, \dots,$ for almost all $\lambda \varepsilon \Lambda$, which completes the proof.

## 5. Conclusion

In this paper we investigate the order statistics property and obtain some characterizations of the mixed Poisson precesses. In Pfeifer and Heller (1987) it is shown that an elementary pure birth process is a mixed Poisson precess if and only if the sequence of the post-jump intensities forms a martingale with respect to the $\sigma$-algebras generated by the jump times of the process. As mixed sample processes also have the order statistics property (see Puri (1982)), it is natural to study a similar characterization in the class of mixed sample processes. This, and actually a more general result, has been given by Pfeifer (1987).

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