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CHARACTERIZATION RESULTS BASED ON RECORD VALUES

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Abstract: The constancy of the conditional expectation of some suitable functions of record values on some others, is used to characterize the exponential or geometric distribution among the class of continuous or discrete distributions, respectively. Some characterizations of the homogeneous Poisson process in the class of nonhomogeneous Poisson processes through properties of arrival times, current life and residual life are also given.

Key words and phrases: Characterization, exponential distribution, geometric distribution, Poisson process, record values.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables having the common distribution function F. Throughout this paper, unless otherwise stated, we assume $F(0) = 0$ and $F(x) < 1$, $\forall x > 0$. The sequence $\{L(n), n \geq 0\}$ defined by $L(0) = 1$ and $L(n) = \min\{j | X_j > X_{L(n-1)}\}$, $n = 1, 2, \ldots$, is called the sequence of (upper) record times, while the corresponding sequence $\{R_n, n \geq 0\}, R_n = X_{L(n)}$, $n \geq 0$, is called the sequence of (upper) record values. Shorrock (1972a,b) studies the behavior of the bivariate sequence $\{(R_n, L(n+1)-L(n)), n \geq 0\}$. He also proves that when F is continuous the point process $\{N(t),\, t \geq 0\}$, defined by $N(t) = \#\{n | R_n \leq t\}$, $t \geq 0$, is a nonhomogeneous Poisson process with $E(N(t)) = -\ln(1 - F(t))$.

In recent years much has been written concerning record values. Especially, various authors including Ahsanullah (1978, 1979), Dallas (1981), Gupta (1984), Kirmani and Gupta (1989), and Nagaraja et al. (1989) give results on characterization of the exponential and geometric distributions by properties about $\{R_n, n \geq 0\}$ or $\{N(t),\, t \geq 0\}$.

The purpose of this paper is to investigate some extensions of the above results. In Section 2, for the nonhomogeneous Poisson process $\{N(t),\, t \geq 0\}$, we give some conditions, related to the current life $\delta_t^* = t - R_{N(t)-1}$, and residual life $\gamma_t^* = R_{N(t)} - t$, to characterize $\{N(t), t \geq 0\}$ as a homogeneous Poisson process.

In Section 3, we determine F from properties such as $E(G(R_j - R_{j-1})|R_j = x) = E(G(R_0)|R_j = x)$ or $E(G(R_i - R_{i-1})|R_j = x) = E((R_{i-1} - R_{i-2})|R_j = x)$, for every $x > 0$, for some fixed integers $j \geq i \geq 1$, and G is a monotone function, where R_{-1} is defined to be 0. Finally, Section 4 characterizes the distribution function F from the property that $E(G(R_{j+k+1} - R_{j+k})|R_j = x)$ or $E(R_{j+k+2} - R_{j+k}|R_j = x)$ equals a constant for every $x \geq 0$, where j and k are fixed non-negative integers.

Before starting our discussions, we give some densities related to the random variables $R_0, R_1, \ldots$. First, we introduce the notation $R(x) = -\ln(\bar{F}(x))$, where $\bar{F}(x) = 1 - F(x)$. Also, if F is an absolutely continuous distribution function with density function $f(x) = F'(x)$, let $r(x) = dR(x)/dx = f(x)/\bar{F}(x)$. Shorrock (1972a) has shown that the process $\{R_n, n \geq 0\}$ is a Markov chain with $P(R_0 \leq a) = F(a)$, and $P(R_{n+1} > a|R_n = b) = \bar{F}(a)/\bar{F}(b)$. Furthermore, when $f(x) = F'(x)$ exists, Resnick (1973) gives the joint density of $R_0, R_1, \ldots, R_n$ in the form

$$\prod_{i=0}^{n} f(x_i) \Big/ \prod_{i=0}^{n-1} \bar{F}(x_i) = f(x_n) \prod_{i=0}^{n-1} r(x_i), \quad 0 < x_0 < \cdots < x_n. \tag{1.1}$$

Using this and the Markov property of $\{R_n, n \geq 0\}$, the following densities, which will be used later, can be obtained easily (see also Ahsanullah (1979)). First of all, the density of R_n is

$$f_{R_n}(y) = \frac{1}{\Gamma(n+1)} R^n(y) f(y), \quad y > 0. \tag{1.2}$$

For $n > m$, the joint density of R_m and R_n is

$$\begin{aligned} &f_{R_m,R_n}(x,y) \\ &= \frac{1}{\Gamma(n-m)\Gamma(m+1)} R^m(x)(R(y) - R(x))^{n-m-1} r(x) f(y), \quad 0 < x < y, \end{aligned} \tag{1.3}$$

and the conditional density of R_n given $R_m = x$ is

$$f_{R_n|R_m=x}(y) = \frac{(R(y) - R(x))^{n-m-1} f(y)}{\Gamma(n-m)\bar{F}(x)}, \quad 0 < x < y. \tag{1.4}$$

Also for $j \geq 0$, $1 \leq k < l$, the conditional density of (R_{j+l}, R_{j+k}) given $R_j = x$ is

$$\begin{aligned} &f_{R_{j+l},R_{j+k}|R_j=x}(u,v) \\ &= \frac{(R(u) - R(v))^{l-k-1}(R(v) - R(x))^{k-1} f(u) r(v)}{\Gamma(l-k)\Gamma(k)\bar{F}(x)}, \quad 0 < x < v < u. \end{aligned} \tag{1.5}$$

Again, most of the results in this paper hold true in the "if and only if" form; yet, for simplicity, we only state the results in the "if" form.

2. Characterizations of the Homogeneous Poisson Process

As mentioned in Section 1, when the common distribution F of $\{X_n, n \geq 1\}$ is continuous (hence $R(t)$ is also continuous, $\forall t \geq 0$, with $R(0) = 0$), then the process $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process. For the process $\{N(t), t \geq 0\}$, R_n can be viewed as its $(n+1)$th arrival time. Kirmani and Gupta (1989) give some characterizations of the homogeneous Poisson process among the class of nonhomogeneous Poisson processes. In this section we will give some extensions and related results. Part (i) of the following theorem is a generalization of Theorem 10 of Kirmani and Gupta (1989). As $E(N(t)) = -\ln(\bar{F}(t)) = R(t)$, a characterization of the homogeneous Poisson process of $\{N(t), t \geq 0\}$ is equivalent to a characterization of the exponential distribution F.

Theorem 1. *Let G be a non-decreasing function such that for any $x > 0$, G has a point of increase in $(0, x)$.*
(i) *Assume $r(0+)$ exists and for some positive integer n,*

$$E(G(R_0)|N(t) = n) = E(G(\delta_t^*)|N(t) = n), \tag{2.1}$$

for every $t > 0$ whenever $P(N(t) = n) > 0$. Then $\bar{F}(t) = e^{-\lambda t}$, $t > 0$, where $\lambda = r(0+)$.
(ii) *Assume $R(t) > 0$ has a continuous derivative for every $t > 0$, and for some integers n and i, such that $1 \leq i \leq n-1$,*

$$E(G(R_i - R_{i-1})|N(t) = n) = E(G(R_{i-1} - R_{i-2})|N(t) = n), \tag{2.2}$$

for every $t > 0$ whenever $P(N(t) = n) > 0$. Then $\bar{F}(t) = e^{-\lambda t}$, $t > 0$, where $\lambda = r(0+)$.

Proof. (i) The independent increments of the Poisson process imply

$$\begin{aligned} &\int_0^t P(R_0 > x, N(t) = n)dG(x) \\ &= \int_0^t P(N(x) = 0)P(N(t) - N(x) = n)dG(x) \\ &= e^{-R(t)} \int_0^t (R(t) - R(x))^n dG(x)/n!. \end{aligned} \tag{2.3}$$

Similarly,

$$\int_0^t P(\delta_t^* > x, N(t) = n)dG(x) = e^{-R(t)} \int_0^t R^n(t-x)dG(x)/n!. \tag{2.4}$$

Therefore, (2.1) is equivalent to

$$\int_0^t (R^n(t-x) - (R(t) - R(x))^n)dG(x) = 0, \quad \forall t > 0. \tag{2.5}$$

As $R(x)$ is a continuous function, then, by Proposition 2.2 of Lau and Rao, and from (2.5) it follows that for every $t > 0$, there exists a $0 < x_0 < t$ such that

$$R(t - x_0) = R(t) - R(x_0). \tag{2.6}$$

Hence by Proposition 2.1 of Lau and Rao (1990), (2.6) leads to $R(t) = \lambda t$, $\forall t > 0$, where $\lambda = r(0+)$. This proves that $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process and $F(t) = 1 - e^{-\lambda t}$, $t > 0$.

(ii) First, we have

$$F_{R_i - R_{i-1} | R_{i-1} = x}(y) = 1 - e^{-R(x+y)+R(x)}, \tag{2.7}$$

and

$$dF_{R_{i-1}}(x) = \frac{e^{-R(x)}}{i!} dR^i(x). \tag{2.8}$$

Using (2.7)and (2.8), it follows that

$$\begin{aligned}
&\int_0^t P(R_i - R_{i-1} > y, N(t) = n) dG(y) \\
&= \int_0^t \int_0^{t-y} (1 - F_{R_i - R_{i-1} | R_{i-1} = x}(y)) P(N(t) - N(x+y) = n - i) dF_{R_{i-1}}(x) dG(y) \\
&= \frac{e^{-R(t)}}{(n-i)!i!} \int_0^t \int_0^{t-y} (R(t) - R(x+y))^{n-i} dR^i(x) dG(y). \qquad (2.9)
\end{aligned}$$

Similarly, we can obtain an expression for $\int_0^t P(R_{i-1} - R_{i-2} > y, N(t) = n) dG(y)$. Hence (2.2) implies

$$\begin{aligned}
&\frac{e^{-R(t)}}{(n-i)!i!} \int_0^t \int_0^{t-y} (R(t) - R(x+y))^{n-i} dR^i(x) dG(y) \\
&= \frac{e^{-R(t)}}{(n-i+1)!(i-1)!} \int_0^t \int_0^{t-y} (R(t) - R(x+y))^{n-i+1} dR^{i-1}(x) dG(y). \quad (2.10)
\end{aligned}$$

Differentiating both sides of (2.10) $(n - i + 1)$ times with respect to t yields

$$\int_0^t R^{i-1}(t-y)(r(t-y) - r(t)) dG(y) = 0, \quad \forall t > 0. \tag{2.11}$$

Again, for every $t > 0$, there exists a $0 < t_0 < t$ such that $R^{i-1}(t - t_0)(r(t - t_0) - r(t)) = 0$. Hence, using the assumption $R(t) > 0$, $\forall t > 0$, it follows that for every $t > 0$, there exists a $0 < t_0 < t$ such that $r(t) = r(t - t_0)$. This implies $r(t)$ equals to a constant, which proves the assertion again.

The next characterization is based on the residual life.

Theorem 2. *Let G be a non-decreasing function having non-lattice support on $x \geq 0$ with $G(0) = 0$ and $E(G(X_1)) < \infty$.*
(i) *If*

$$E(G(\gamma_t^*)) = c, \quad \forall t > 0, \tag{2.12}$$

and if

$$c < \int_0^\infty e^{-\xi x} dG(x) < \infty \tag{2.13}$$

for some $\xi > 0$, then $c = E(G(X_1))$ and $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.
(ii) *If for some fixed integer $n \geq 0$*

$$E(G(\gamma_t^*)|N(t) = n) = c, \tag{2.14}$$

for every $t > 0$ whenever $P(N(t) = n) > 0$, and if (2.13) holds for some $\xi > 0$, then $c = E(G(X_1))$ and $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.

Proof. (i) First, $c = E(G(X_1))$ is obvious. Next, since

$$E(G(\gamma_t^*)) = \int_0^\infty P(\gamma_t^* > x) dG(x) = \int_0^\infty e^{-R(t+x)+R(t)} dG(x), \tag{2.15}$$

(2.12) implies

$$ce^{-R(t)} = \int_0^\infty e^{-R(t+x)} dG(x), \tag{2.16}$$

or (since $R_0 = X_1$)

$$cP(X_1 > t) = \int_0^\infty P(X_1 > t + x) dG(x). \tag{2.17}$$

This, together with (2.13) implies, by Shimizu (1978) or (1979), X_1 is exponentially distributed. This completes the proof of part (i).
(ii) Again, by the independent increments of the Poisson process,

$$\begin{aligned} E(G(\gamma_t^*)|N(t) = n) &= \int_0^\infty P(\gamma_t^* > x|N(t) = n) dG(x) \\ &= \int_0^\infty P(N(t+x) - N(t) = 0) dG(x) = \int_0^\infty e^{-R(t+x)+R(t)} dG(x). \end{aligned} \tag{2.18}$$

Now (2.16) can be obtained by using (2.18), (2.13) and (2.14). The result then follows.

The following theorem shows if the variance of γ_t^* is constant, then the process $\{N(t), t \geq 0\}$ will also be a homogeneous Poisson process.

Theorem 3. *Assume $R(t)$ is differentiable with $r(t) \neq 0$, $\forall t > 0$, and $E(X_1^2) < \infty$. If*

$$\mathrm{Var}(\gamma_t^*) = c, \quad \forall t > 0, \tag{2.19}$$

then X_1 is exponentially distributed and $c = \mathrm{Var}(X_1)$.

Proof. First, we have $E(\gamma_t^{*2}) = \int_0^\infty P(\gamma_t^* > x)dx^2 = \int_0^\infty e^{-R(t+x)+R(t)}dx^2$ and $E(\gamma_t^*) = \int_0^\infty e^{-R(t+x)+R(t)}dx$. Thus, (2.19) implies

$$\int_0^\infty e^{-R(t+x)+R(t)}dx^2 - \left(\int_0^\infty e^{-R(t+x)+R(t)}dx\right)^2 = c. \tag{2.20}$$

It follows that

$$2e^{-R(t)}\int_t^\infty (x-t)e^{-R(x)}dx - \left(\int_t^\infty e^{-R(x)}dx\right)^2 = ce^{-2R(t)}. \tag{2.21}$$

Taking derivatives of both sides of (2.21) with respect to t leads to

$$2r(t)e^{-R(t)}\int_t^\infty (x-t)e^{-R(x)}dx = 2cr(t)e^{-2R(t)}. \tag{2.22}$$

Consequently, $\int_t^\infty (x-t)e^{-R(x)}dx = ce^{-R(t)}$. This in turn implies $\int_0^\infty e^{-R(t+x)}dx^2 = 2ce^{-R(t)}$. Therefore, by Shimizu (1978), X_1 has an exponential distribution.

Along the lines of the proof of the previous theorem, we have the following.

Corollary 1. *Let $R(t)$ and X_1 satisfy the conditions of Theorem 3. If for some fixed integer $n \geq 0$, $\mathrm{Var}(\gamma_t^*|N(t) = n) = c$, for every $t > 0$ whenever $P(N(t) = n) > 0$, then X_1 is exponentially distributed with $c = \mathrm{Var}(X_1)$.*

Theorem 4. *Let $G(t)$ be a non-constant non-decreasing differentiable function with $G(t) > 0$, $\forall t > 0$. Also, assume $R(t) > 0$, $\forall t > 0$. If for some fixed integer $n \geq 1$,*

$$E(G(R_{n-1})|N(t) = n) = cG(t), \tag{2.23}$$

for every $t > 0$ whenever $P(N(t) = n) > 0$, where $c > 0$ is a constant, then (i) *$0 < c < 1$,* (ii) *$\lim_{t\to\infty} G(t) = \infty$, and* (iii) *$R(t) = \lambda(G(t))^{c/(n(1-c))}$, for some constant $\lambda > 0$. In particular if $c = n/(n+1)$ and $G(t) = t$, then $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.*

Proof. Since

$$\begin{aligned} &E(G(R_{n-1})|N(t) = n) \\ = &\int_0^t P(R_{n-1} > x, N(t) = n)dG(x)/P(N(t) = n) \\ = &\int_0^t P(N(x) \leq n-1, N(t) = n)dG(x)/P(N(t) = n), \end{aligned} \tag{2.24}$$

we obtain from (2.23)

$$\int_0^t \sum_{i=0}^{n-1} P(N(x)=i, N(t)-N(x)=n-i)dG(x) = cG(t)P(N(t)=n). \quad (2.25)$$

This yields

$$\int_0^t \sum_{i=0}^{n-1} \frac{e^{-R(x)}R^i(x)}{i!} \frac{e^{-R(t)+R(x)}(R(t)-R(x))^{n-i}}{(n-i)!} dG(x) = cG(t)\frac{e^{-R(t)}R^n(t)}{n!}. \quad (2.26)$$

After simplification

$$\int_0^t (R^n(t) - R^n(x))dG(x) = cG(t)R^n(t). \quad (2.27)$$

As $G(x)$ is differentiable and $R(x)$ is continuous, the left side, and hence the right side, is differentiable (see, e.g., Apostol (1974) Theorem 7.32). Thus $R(t)$ is differentiable. Also since $R(t) > 0$, $\forall t > 0$, it is easy to see from (2.27) that $c < 1$. Taking the derivatives of both sides of (2.27) with respect to t yields

$$\frac{r(t)}{R(t)} = \frac{c}{n(1-c)} \frac{G'(t)}{G(t) - nG(0)/(1-c)}. \quad (2.28)$$

Thus $R(t) = \lambda(G(t) - nG(0)/(1-c))^{c/(n(1-c))}$, where $\lambda > 0$ is a constant. Now using $R(0) = 0$ it is easy to see $G(0) = 0$. Thus $R(t) = \lambda(G(t))^{c/(n(1-c))}$. This proves (iii). On the other hand, as $R(t) = -\ln(\bar{F}(t))$, $\lim_{t\to\infty} R(t) = \infty$, hence we obtain $\lim_{t\to\infty} G(t) = \infty$.

Finally, if $c = n/(n+1)$ and $G(t) = t$, then $R(t) = \lambda t$, and therefore $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.

Theorem 5. *Assume $G(t)$ and $R(t)$ satisfy the conditions of Theorem 4. Let $n \geq 2$ be a fixed integer. For $k = n-1, n$, if*

$$E(G(R_0)|N(t) = k) = c_k G(t), \quad (2.29)$$

for every $t > 0$ whenever $P(N(t) = k) > 0$, where the c_k are positive constants, then $c_{n-1} > c_n$, $\lim_{t\to\infty} G(t) = \infty$, and

$$R(t) = \lambda(G(t))^{c_n/(n(c_{n-1}-c_n))}, \quad (2.30)$$

for some constant $\lambda > 0$. In particular, if $G(t) = t$ and $c_k = (k+1)^{-1}$, $k = n-1, n$, then $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process.

Proof. As in the proof of Theorem 4, we have from (2.29)

$$\int_0^t (R(t)-R(x))^n dG(x) = c_n G(t) R^n(t) \tag{2.31}$$

and

$$\int_0^t (R(t)-R(x))^{n-1} dG(x) = c_{n-1} G(t) R^{n-1}(t). \tag{2.32}$$

Differentiating both sides of (2.31) with respect to t yields

$$nr(t)\int_0^t (R(t)-R(x))^{n-1} dG(x) = c_n(G'(t)R^n(t) + nG(t)R^{n-1}(t)r(t)). \tag{2.33}$$

Now by (2.32), the left side of (2.33) is equal to $nc_{n-1}G(t)R^{n-1}(t)r(t)$. If $c_{n-1} = c_n$, then $c_n G'(t)R^n(t) = 0$. Since $R(t) > 0$ and $G'(t) \neq 0$, this implies $c_n = 0$, which contradicts the assumption $c_n > 0$. Consequently, after simplification, it follows

$$r(t)/R(t) = (c_n/(n(c_{n-1}-c_n)))G'(t)/G(t). \tag{2.34}$$

Since $R(t)$ and $G(t)$ are both non-decreasing functions, we obtain from (2.34) that $c_{n-1} > c_n$. The solution of (2.34) is given exactly by (2.30).

Finally, we have a result for the residual life δ_t^*. In order to have a nice characterization assume $G(t) = t$. Since the proof of this theorem is similar to Theorems 4 and 5, it is omitted.

Theorem 6. *Let $R(t) > 0$, for every $t > 0$. For some fixed integer $n \geq 1$, if $E(\delta_t^*|N(t) = n) = bt$, for every $t > 0$ whenever $P(N(t) = n) > 0$, where $b > 0$ is a constant, then $0 < b < 1$ and $R(t) = \lambda t^{(1-b)/nb}$, for some $\lambda > 0$. In particular if $b = (n+1)^{-1}$, then $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.*

Note that in the above theorems, the condition G is non-decreasing can actually be replaced by G is monotone. For if G is non-increasing, just consider the function $-G$, then it is easy to modify the other conditions and the results still hold. On the other hand in Theorem 2, if $G(0) \neq 0$, then, by letting $G_1(x) = G(x) - G(0)$, $x \geq 0$, and $G_1(x) = 0$, $x < 0$, everything can be fixed similarly. Yet for simplicity, here and in the next two sections, we still assume G is non-decreasing and sometimes $G(0) = 0$.

3. Characterizations Related to the Backward Record Values

Çinlar and Jagers (1973) and Huang et al. (1993) characterize the distribution of the inter-arrival times among the renewal processes through properties of backward recurrence times. Motivated by these results, in this section, we will characterize the common distribution F, by some properties of the conditional distribution of $R_i - R_{i-1}$ given R_j, $i \leq j$. First we have

Theorem 7. *Assume $F(x)$ has density $f(x)$. Let G be a non-decreasing function such that for every $x > 0$, G has a point of increase in $(0, x)$. Assume for some fixed integer $j \geq 1$,*

$$E(G(R_j - R_{j-1})|R_j = x) = E(G(R_0)|R_j = x), \quad \forall x > 0; \tag{3.1}$$

then X_1 has an exponential distribution.

Proof. First, (3.1) implies

$$\int_0^x G(x-y) f_{R_{j-1}|R_j=x}(y)dy = \int_0^x G(y) f_{R_0|R_j=x}(y)dy, \tag{3.2}$$

or, by (1.2) and (1.3),

$$\int_0^x G(x-y) R^{j-1}(y) r(y) dy = \int_0^x G(y)(R(x) - R(y))^{j-1} r(y) dy. \tag{3.3}$$

Integration by parts implies $\int_0^x (R^j(x-y) - (R(x) - R(y))^j) dG(y) = 0$, $\forall x > 0$. Hence, by Propositions 2.1 and 2.2 of Lau and Rao (1990), $R(x) = \alpha x$, where $\alpha = r(0+)$. This in turn implies X_1 is exponentially distributed.

Another characterization is given below, where for convenience R_{-1} is defined to be zero.

Theorem 8. *Let F and G be as described in Theorem 7. Also, assume $f(x)$ is continuous, $F(x) > 0$ for $x > 0$ and $G(0) = 0$. If for some fixed integers $1 \leq i \leq j$,*

$$E(G(R_i - R_{i-1})|R_j = x) = E(G(R_{i-1} - R_{i-2})|R_j = x), \quad \forall x > 0, \tag{3.4}$$

then X_1 has an exponential distribution.

Proof. Again, if $1 \leq i \leq j-1$, then, using (1.2), (1.3) and (1.4), we have

$$\begin{aligned}
& E(G(R_i - R_{i-1})|R_j = x) \\
= & \int_0^x P(R_i - R_{i-1} > y | R_j = x) dG(y) \\
= & \int_0^x \int_y^x \int_z^x f_{R_i - R_{i-1}, R_i | R_j = x}(z, w) dw \, dz \, dG(y) \\
= & \frac{\Gamma(j+1)}{\Gamma(j-i)\Gamma(i) R^j(x)} \int_0^x (R(x) - R(w))^{j-i-1} r(w)
\end{aligned}$$

$$\cdot \int_0^w G(z)R^{i-1}(w-z)r(w-z)dz\,dw$$
$$= \frac{\Gamma(j+1)}{\Gamma(j-i)\Gamma(i+1)R^j(x)} \int_0^x (R(x)-R(w))^{j-i-1}r(w)$$
$$\cdot \int_0^w R^i(w-z)dG(z)dw, \tag{3.5}$$

where the last step is by an integration by parts. Similarly,

$$E(G(R_{i-1}-R_{i-2})|R_j=x)$$
$$= \frac{\Gamma(j+1)}{\Gamma(j-i+1)\Gamma(i)R^j(x)} \int_0^x (R(x)-R(w))^{j-i}r(w)\int_0^w R^{i-1}(w-z)dG(z)dw. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) yields

$$\frac{j-i}{i}\int_0^x (R(x)-R(w))^{j-i-1}r(w)\int_0^w R^i(w-z)dG(z)dw$$
$$= \int_0^x (R(x)-R(w))^{j-i}r(w)\int_0^w R^{i-1}(w-z)dG(z)dw. \tag{3.7}$$

Differentiating both sides of (3.7) with respect to x $(j-i+1)$ times yields

$$\int_0^x R^{i-1}(x-z)(r(x)-r(x-z))dG(z)=0. \tag{3.8}$$

Next, if $i=j$, then, from the proof of Theorem 7 we have

$$E(G(R_j-R_{j-1})|R_j=x) = \frac{1}{R^j(x)}\int_0^x R^j(x-y)dG(y); \tag{3.9}$$

and, by letting $i=j-1$ in (3.5), we have

$$E(G(R_{j-1}-R_{j-2})|R_j=x) = \frac{j}{R^j(x)}\int_0^x\int_y^x R^{j-1}(w-y)r(w)dw\,dG(y). \tag{3.10}$$

Hence, we obtain (3.8) again.

Now, by Proposition 2.2 of Lau and Rao (1990), noting that $G(x)\not\equiv 0$ for $x>0$, there exists a $y\in(0,x)$ such that $R^{i-1}(x-y)(r(x)-r(x-y))=0$. Since $F(x)>0$ for every $x>0$, we have, for every $y<x$, $R(x-y)>0$. Consequently, for every $x>0$, $r(x)=r(x-y)$, for some $y\in(0,x)$. Now, since $r(x)$ is continuous (because $f(x)$ is continuous), we obtain $r(x)\equiv\alpha$, where α is a positive constant. This proves that X_1 is exponential.

It is desirable to generalize the above results, namely, characterize F from

$$E(G(R_i-R_{i-1})|R_j=x)=E(G(R_k-R_{k-1})|R_j=x), \quad \forall x>0, \tag{3.11}$$

for some fixed integers $0 \leq k < i \leq j$; however, the computations are very cumbersome for solving the above general equation.

4. Characterizations Related to the Forward Record Values

Gupta (1984) proves F is exponential if and only if $E((R_{j+1}-R_j)^r|R_j=x)=c$ (independent of x) for fixed $j \geq 0$ and $r \geq 1$. This result has been extended by Rao and Shanbhag (1986), where they obtain the same characterization if, in Gupta's condition, the expression $(R_{j+1}-R_j)^r$ is replaced by $G(R_{j+1}-R_j)$, where G is a monotone function which satisfies certain conditions. On the other hand, when F is discrete, Nagaraja et al. (1989) give a characterization of the geometric tail distribution from $E(R_{j+2}-R_{j+1}|R_j=x)=c$. Here, we say X has a geometric tail distribution, and write X is GT(k,θ), if for some fixed integer $k \geq 1$, $p(x)=\theta S(x)$, $\forall x \geq k$, where $p(x)=P(X=x)$, and $S(x)=P(X \geq x)$.

The following two theorems are generalizations of the above results, where, instead of considering the difference of R_{j+1} and R_j, we consider the difference of any two abjacent record values after R_j.

Theorem 9. *Assume that $F(x)$ has density function $f(x)$ and $F(x) > 0$ for $x > 0$. Let G be a non-decreasing function having non-lattice support on $x > 0$ with $G(0)=0$ and $E(G(X_1)) < \infty$. If, for some fixed non-negative integers j and k,*

$$E(G(R_{j+k+1}-R_{j+k})|R_j=x)=c \tag{4.1}$$

for every $x > 0$, where $c > 0$ is a constant, and if for some $\xi > 0$ (2.13) holds, then $c = E(G(X_1))$ and X_1 is exponentially distributed.

Proof. First, by using (1.5), we have

$$\begin{aligned}
&E(G(R_{j+k+1}-R_{j+k})|R_j=x)\\
&= \int_0^\infty P(R_{j+k+1}-R_{j+k} > y|R_j=x)dG(y)\\
&= \int_0^\infty \int_y^\infty \int_x^\infty f_{R_{j+k+1},R_{j+k}|R_j=x}(z+w,w)dwdzdG(y)\\
&= \int_0^\infty \int_x^\infty G(z)\frac{(R(w)-R(x))^{k-1}r(w)f(w+z)}{\Gamma(k)\bar{F}(x)}dwdz\\
&= \frac{1}{\Gamma(k)\bar{F}(x)}\int_x^\infty (R(w)-R(x))^{k-1}r(w)\int_0^\infty G(z)f(w+z)dzdw\\
&= \frac{1}{\Gamma(k)\bar{F}(x)}\int_x^\infty (R(w)-R(x))^{k-1}r(w)\int_w^\infty \bar{F}(z)dG(z-w)dw. \qquad (4.2)
\end{aligned}$$

Here, since $E(G(X_1)) < \infty$, we have

$$0 \leq \lim_{z\to\infty} G(z-w)\bar{F}(z) \leq \lim_{z\to\infty} G(z)\bar{F}(z) \leq \lim_{z\to\infty}\int_z^\infty G(x)dF(x) = 0.$$

Using this and integrating by parts yields

$$\int_0^\infty G(z)f(w+z)dz = -\int_w^\infty G(z-w)d\bar{F}(z) = \int_w^\infty \bar{F}(z)dG(z-w). \tag{4.3}$$

Thus, the last equation of (4.2) holds. Now (4.1) implies

$$\int_x^\infty (R(w)-R(x))^{k-1}r(w)\int_w^\infty \bar{F}(z)dG(z-w)dw = c\Gamma(k)\bar{F}(x). \tag{4.4}$$

Taking the derivatives of both sides k times, with respect to x, it follows that

$$\int_0^\infty \bar{F}(z+x)dG(z) = c\bar{F}(x). \tag{4.5}$$

Now the solution of (4.5) is $F(x) = 1-e^{-\lambda x}$, $x > 0$, where λ is the positive number defined by $\int_0^\infty e^{-\lambda x}dG(x) = c$ (see Shimizu (1979) or Huang et al. (1993)). Finally, by letting $x \to 0$ in (4.5) we obtain $c = E(G(X_1))$. This completes the proof.

Next, we have a discrete version of the previous theorem, where for simplicity we assume the span of F equals 1.

Theorem 10. *Assume F is arithmetic with span 1, and G is a non-decreasing function with $G(0) = 0$. For some fixed non-negative integers j, k and l, if*

$$E(G(R_{j+k+1}-R_{j+k})|R_j = x) = c, \tag{4.6}$$

for $x = j+l+1, j+l+2, \ldots$, where $c > G(1)$ is a finite constant and $c < \sum_{n=0}^\infty e^{-\xi n}(G(n+1)-G(n)) < \infty$ for some $\xi > 0$, then X_1 has a $\mathrm{GT}(j+l+k+2, p)$ *distribution for some $0 < p < 1$.*

Proof. The assumption (4.6) implies, for $x = j+l+1, j+l+2, \ldots$, that

$$\sum_{y=1}^\infty G(y)P(R_{j+k+1}-R_{j+k} = y|R_j = x) = c. \tag{4.7}$$

In view of the Markov property of $\{R_n, n \geq 0\}$, and using the conditional probability $P(R_{i+1} = u|R_i = v) = p(u)/\bar{F}(v)$, (4.7) implies

$$\begin{aligned}
&\sum_{w_1=x+1}^\infty \sum_{w_2=w_1+1}^\infty \cdots \sum_{w_k=w_{k-1}+1}^\infty \sum_{y=1}^\infty \\
&G(y)P\Big(R_{j+k+1}-R_{j+k} = y, R_{j+k} = w_k, \ldots, R_{j+1} = w_1|R_j = x\Big) \\
=& \sum_{w_1=x+1}^\infty \sum_{w_2=w_1+1}^\infty \cdots \sum_{w_k=w_{k-1}+1}^\infty \sum_{y=1}^\infty G(y)\frac{p(y+w_k)}{\bar{F}(w_k)}\frac{p(w_k)}{\bar{F}(w_{k-1})}\cdots\frac{p(w_2)}{\bar{F}(w_1)}\frac{p(w_1)}{\bar{F}(x)} \\
=& c.
\end{aligned} \tag{4.8}$$

From this we obtain

$$\sum_{w_1=x+1}^{\infty} \cdots \sum_{w_k=w_{k-1}+1}^{\infty} \sum_{y=1}^{\infty} G(y)p(y+w_k) \prod_{i=1}^{k} \frac{p(w_i)}{\bar{F}(w_i)} = c\bar{F}(x). \tag{4.9}$$

Replacing x by $x+1$ to form a new equation and subtracting these two equations, we have

$$\sum_{w_2=x+2}^{\infty} \sum_{w_3=w_2+1}^{\infty} \cdots \sum_{w_k=w_{k-1}+1}^{\infty} \sum_{y=1}^{\infty} G(y)p(y+w_k) \prod_{i=2}^{k} \frac{p(w_i)}{\bar{F}(w_i)} = c\bar{F}(x+1). \tag{4.10}$$

Repeating the same procedure $(k-1)$ more times, we obtain

$$\sum_{y=1}^{\infty} G(y)p(y+x+k) = c\bar{F}(x+k), \quad x = j+l+1,\ j+l+2, \ldots. \tag{4.11}$$

Next, by letting $Y = X-k-j-l-1$, we obtain

$$\sum_{y=1}^{\infty} G(y)P(Y=y+x) = c(1-U(x)), \quad x = 0,1,2,\ldots, \tag{4.12}$$

where $U(x) = P(Y \le x)$. Finally, the solution of (4.12) is (see Huang et al. (1993)) $P(Y=i) = p^{i-1}(1-p)$, $i \ge 1$, where $0<p<1$ is defined by $\sum_{i=0}^{\infty}(G(i+1)-G(i))(1-p)^i = c$. This proves $X = Y+k+j+l+1$ is $\mathrm{GT}(k+j+l+2, p)$.

In the previous two theorems, we characterize the distribution of X_1 by the spacings of two adjacent record values. The case of non-adjacent record values has seldom been considered by other authors. Dallas (1987) has some partial results for non-adjacent order statistics under centain very restricted conditions.

In the following, first we give a useful lemma and then use this lemma to characterize X_1 to be exponential based on two non-adjacent record values.

Lemma 1. *Assume that $F(x)$ has density $f(x)$ and $F(x)>0$ for $x>0$. If for some fixed integers $j \ge 0$ and $k, l \ge 1$,*

$$E(R_{j+k+l} - R_{j+k} | R_j = x) = c, \tag{4.13}$$

for every $x>0$, where $c>0$ is a constant, then $E(R_{i+l} - R_i | R_i = x) = c$, for every $x>0$ and $i \ge 0$.

Proof. From (4.13), by using (1.5), we obtain

$$\begin{aligned} & E\Big(R_{j+k+l} - R_{j+k} | R_j = x\Big) \\ = & \int_x^{\infty}\int_0^{\infty}\int_0^{z} \frac{(R(z+w)-R(w))^{l-1}(R(w)-R(x))^{k-1} f(z+w) r(w)}{\Gamma(l)\Gamma(k)\bar{F}(x)} dy dz dw \\ = & \ c. \end{aligned} \tag{4.14}$$

Hence

$$\begin{aligned}&\int_x^\infty (R(w)-R(x))^{k-1}r(w)\int_0^\infty z(R(z+w)-R(w))^{l-1}f(z+w)dz\,dw\\ &= c\Gamma(l)\Gamma(k)\bar{F}(x).\end{aligned} \tag{4.15}$$

Differentiating both sides k times, with respect to x, it follows that for every $x>0$,

$$\int_x^\infty \frac{(z-x)(R(z)-R(x))^{l-1}f(z)dz}{\Gamma(l)\bar{F}(x)}=c,\quad \forall x>0. \tag{4.16}$$

The lemma then follows immediately, as the right side of (4.16) is exactly $E(R_{l+i}-R_i|R_i=x)$.

Along the lines of the proof of Lemma 1, it is easy to obtain a corresponding result when F is arithmetic.

Now we give an extension of Theorem 9 under the condition that $G(x)=x$.

Theorem 11. *Assume that $F(x)$ has a differentiable density $f(x)$ with $f(0+)>0$ and $F(x)>0$ for $x>0$. Also assume $E(X_1)<\infty$. If for some fixed integers $j,k\geq 0$,*

$$E(R_{j+k+2}-R_{j+k}|R_j=x)=c, \tag{4.17}$$

for every $x>0$, where $c>0$ is a constant, then X_1 is exponentially distributed.

Proof. By Lemma 1 and (1.4), (4.17) implies

$$\int_x^\infty z(R(z)-R(x))f(z)dz=(c+x)\bar{F}(x). \tag{4.18}$$

Differentiating with respect to x, the above becomes

$$\begin{aligned}\bar{F}(x)-(c+x)f(x)&=-r(x)\int_x^\infty zf(z)dz\\ &= r(x)\int_x^\infty zd\bar{F}(z)=-xr(x)\bar{F}(x)-r(x)\int_x^\infty \bar{F}(z)dz.\end{aligned} \tag{4.19}$$

On the other hand, since $0<F(x)<1$ for every $x>0$, and noting that $f(x)=r(x)\bar{F}(x)$, we have from (4.19) $r(x)>0$ for every $x>0$. Thus (4.19) implies, on replacing $f(x)$ by $r(x)\bar{F}(x)$,

$$\int_x^\infty \bar{F}(z)dz=c\bar{F}(x)-\bar{F}(x)/r(x). \tag{4.20}$$

Differentiating with respect to x, gives $2+r'(x)/r^2(x)=cf(x)/\bar{F}(x)$. From this we obtain

$$R(x)=-\ln(\bar{F}(x))=(2/c)x-1/cr(x)+1/cr(0+), \tag{4.21}$$

or

$$(cR(x) - 2x - 1/r(0+))r(x) = -1. \tag{4.22}$$

Substituting

$$v(x) = cR(x) - 2x + c/2 - 1/r(0+) \tag{4.23}$$

into (4.22) yields

$$v(x)v'(x) - (c/2)v'(x) + 2v(x) = 0. \tag{4.24}$$

If $R(x) = 2x/c$, $\forall x > 0$, then X_1 is exponential with $F(x) = 1 - e^{-2x/c}$. Suppose that $R(x) \neq 2x/c$ for some $x_0 > 0$. Since $R(x)$ is a continuous function we have $R(x) \neq 2x/c$ for x belonging to some interval contains x_0. Let $a_1 = \inf\{x|R(x) \neq 2x/c\}$, $a_2 = \inf\{x|x > a_1$ and $R(x) = 2x/c\}$. Then $0 \leq a_1 < a_2 \leq \infty$ (a_2 is defined to be ∞ if $R(x) \neq 2x/c$ for every $x > a_1$) and $R(x) \neq 2x/c$, $\forall x \in (a_1, a_2)$. Assume $v(x) \neq 0$, $\forall x \in (a_1, a_2)$. Then (4.24) can be rewritten as

$$v'(x) - (c/2)v'(x)/v(x) + 2 = 0, \quad \forall x \in (a_1, a_2). \tag{4.25}$$

Thus $v(x)$ satisfies

$$v(x) = \exp\{(2/c)(v(x) + 2x - b)\}, \quad \forall x \in (a_1, a_2), \tag{4.26}$$

for some constant b. If $v(x_1) = 0$ for some $x_1 \in (a_1, a_2)$, then, from the continuity of $v(x)$ it is easy to obtain a contradiction. Hence $v(x) \neq 0$, $\forall x \in (a_1, a_2)$, and (4.26) holds. From (4.23) and (4.26), we obtain, for $\forall x \in (a_1, a_2)$,

$$cR(x) - 2x + c/2 - 1/r(0+) = \exp\{(2/c)(cR(x) + c/2 - 1/r(0+) - b)\}. \tag{4.27}$$

Since $R(x)$ is continuous, $\forall x > 0$, and $R(0) = 0$, it follows that $R(a_1) = 2a_1/c$. Also, if $a_2 < \infty$, then $R(a_2) = 2a_2/c$. By letting $x = a_1$ and a_2 in (4.27) respectively, we have

$$c/2 - 1/r(0+) = \exp\{(2/c)(2a_1 + c/2 - 1/r(0+) - b)\}, \tag{4.28}$$

and

$$c/2 - 1/r(0+) = \exp\{(2/c)(2a_2 + c/2 - 1/r(0+) - b)\}. \tag{4.29}$$

This in turn implies $a_1 = a_2$, which is a contradiction. Hence $a_2 = \infty$. Moreover from (4.28) we find that $c/2 - 1/r(0+) > 0$. Comparing the right side of (4.27) with the right side of (4.28) yields

$$cR(x) - 2x + c/2 - 1/r(0+) = e^{2R(x) - 4a_1/c}(c/2 - 1/r(0+)). \tag{4.30}$$

Differentiating (4.30), we obtain

$$r(x)\Big(c - 2(c/2 - 1/r(0+))e^{-4a_1/c}e^{2R(x)}\Big) = 2, \quad \forall x \in (a_1, \infty). \tag{4.31}$$

Since, as mentioned before, $r(x) > 0$, $\forall x > 0, c/2 - 1/r(0+) > 0$ and $\lim_{x\to\infty} R(x) = \infty$, (4.31) cannot hold for every $x > a_1$, which yields a contradiction again. Therefore, we conclude that $R(x) = 2x/c$, $\forall x > 0$. This completes the proof.

Remark. In this section, when F is continuous, we have characterized the distribution of X_1 by using the constant regression of $R_{j+l} - R_{j+k}$ on R_j, for $l = k+1$ or $k+2$, respectively. Yet when $l - k \geq 3$, it is still unknown what class of distributions will be characterized. Also if F is arithmetic with span 1, then, from $E(R_{j+k+2} - R_{j+k}|R_j = x) = c$, $\forall x = j+1, j+2, \ldots$, corresponding to (4.20) we obtain,

$$\sum_{y=2}^{\infty} yp(x+y) = c\bar{F}(x+1) - \bar{F}(x+1)/r(x+1), \quad \forall x = j+1,\, j+2, \ldots, \tag{4.32}$$

and we fail to solve it.

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