1. Let $X_1, \cdots, X_n$ be a sample with pdf $f(x|\theta)$, and let $W(\mathbf{X}) = W(X_1, \cdots, X_n)$ be any estimator satisfying

$$\frac{d}{d\theta} E_\theta W(\mathbf{X}) = \int_\chi \frac{\partial}{\partial\theta}[W(\mathbf{x})f(\mathbf{x}|\theta)]d\mathbf{x},$$

and $Var_\theta W(\mathbf{X}) < \infty$. Then prove

$$Var_\theta W(\mathbf{X}) \geq \frac{(\frac{d}{d\theta} E_\theta W(\mathbf{X}))^2}{E_\theta((\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta))^2)}.$$

2. Let $X_1, \cdots, X_n$ be a sample with pdf $f(x|\lambda) = \lambda\exp(-\lambda x), x > 0$. Here assume that we only observe $Y_1, \cdots, Y_n$ with $Y_i = I_{X_i<M}$, where M is a fixed constant. Based on Y_i, find the MLE of λ.

3. Suppose that $Y_1, \cdots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i, i = 1, \cdots, n$$

where $x_1, \cdots, x_n$ are fixed constants, and $\varepsilon_1, \cdots, \varepsilon_n$ are i.i.d. $N(0, \sigma^2)$ with unknown σ^2. (1) Find the MLE of β, $\hat{\beta}$, and show it is an unbiased estimator of β. (2) Let $\tilde{\beta} = \sum_i Y_i / \sum_i x_i$ be another estimator of β. Show $\tilde{\beta}$ is also unbiased estimator of β. (3) Compare the variances of $\hat{\beta}$ and $\tilde{\beta}$.

4. Let $X_1, \cdots, X_n$ be a sample with pdf $f(x|\theta) = \exp(-(x-\theta)), x \geq \theta$. We are interested in testing $H_0 : \theta \leq \theta_0$ v.s. $H_1 : \theta > \theta_0$. Derive the LRT for this hypothesis.

5. Suppose that $X_1, \cdots, X_n$ are i.i.d. with a $Beta(\mu, 1)$ pdf and $Y_1, \cdots, Y_m$ are i.i.d. with a $Beta(\theta, 1)$ pdf. Also assume that X_i are independent of Y_i. Find a LRT of $H_0 : \theta = \mu$ v.s. $H_1 : \theta \neq \mu$, and show this LRT is based on the following statistic,

$$T = \frac{\sum_i \log X_i}{\sum_i \log X_i + \sum_j \log Y_j}.$$

6. Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution with unknown parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$. We are interesting in testing

$$H_0 : \mu_X = \mu_Y \text{ v.s. } H_1 : \mu_X \neq \mu_Y.$$

(1) Show that $W_i = X_i - Y_i$ are i.i.d. $N(\mu_W, \sigma_W^2)$. (2) Derive the LRT for this hypothesis, and show this LRT is based on the statistic

$$T_W = \frac{\bar{W}}{\sqrt{\frac{1}{n}S_W^2}},$$

where $\bar{W} = \frac{1}{n}\sum_i W_i$ and $S_W^2 = \frac{1}{(n-1)}\sum_i (W_i - \bar{W})^2$. (3) Show that under H_0, $T_W \sim$ Student's t with $n-1$ degree of freedom.

7. Let $X_1, \cdots, X_n$ be a random sample with pdf $f(x|p,\theta) = \frac{1}{\Gamma(p)\theta^p} x^{p-1} \exp(-x/\theta)$. Here p is assumed to be known and θ is an unknown parameter. We are interesting in testing

$$H_0 : \theta \leq \theta_0 \text{ v.s. } H_1 : \theta > \theta_0.$$

   Find the UMP level α test.

8. Let X follow the binomial distribution, i.e. $B(n, \theta)$, with known n and unknown θ. Now we are interesting in testing

$$H_0 : \theta \leq \theta_0 \text{ v.s. } H_1 : \theta > \theta_0.$$

   Find the UMP level α test.

9. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$ with unknown μ and σ^2. Construct a $1-\alpha$ lower confidence bound for μ.