1. (10 points) Let $M_X(t)$ be the moment generating function of X, and define $S(t) = \log(M_X(t))$. Show that

$$\frac{d}{dt}S(t) \mid_{t=0}= E(X) \text{ and } \frac{d^2}{dt^2}S(t)|_{t=0} = Var(X).$$

2. (10 points) Let $(X_1, \ldots, X_n)$ have a multivariate distribution with m trials and cell probabilities, $p_1, \ldots, p_n$. Show that, for each i and j,

$$\begin{aligned} X_i|X_j = x_j &\sim \text{Binomial}(m - x_i, \frac{p_i}{1-p_j}) \\ X_j &\sim \text{Binomial}(m, p_j) \end{aligned}$$

3. (10 points) Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f_X(x) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent.

4. (10 points) Let $X_1, \ldots, X_n$ be iid $\text{N}(\mu, \sigma^2)$. Find a function of S^2, the sample variance, say $g(S^2)$ that satisfies $Eg(S^2) = \sigma$. (Hint: Try $g(S^2) \propto \sqrt{S^2}$.)

5. (20 points) Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0.$$

   (a) Is $\sum_i X_i$ sufficient for θ?

   (b) Find a complete sufficient statistic for θ.

6. (20 points) Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x|\mu) = e^{(-(x-\mu))}$, where $-\infty < \mu < x < \infty$.

   (a) Show that $X_{(1)}$ is a complete sufficient statistic.

   (b) Use Basu's Theorem to show that $X_{(1)}$ and S^2 are independent.

7. (10 points) **(a)** Let $X_1, \ldots, X_n$ be a random sample from a Gamma(α, β) population. Find a sufficient statistic for (α, β).
   (10 points) **(b)** Let $X_1, \ldots, X_n$ be a random sample from a Uniform$(\theta, \theta+1)$ population. Find a minimal sufficient statistic for θ.

8. (10 points) Suppose X_1 and X_2 are iid observation from the pdf

$$f(x|\alpha) = \alpha x^{\alpha-1}\exp(-x^{\alpha}), x > 0, \alpha > 0.$$

Show that $(\log X_1)/(\log X_2)$ is an ancillary statistic.

9. (20 points) Let $X_1, \ldots, X_n$ be iid with geometric distribution, i.e.

$$P_\theta(X = x) = \theta(1-\theta)^x, x = 1, 2, \ldots, 0 < \theta < 1.$$

Show that $\sum_i X_i$ is sufficient for θ, and find the family of distribution of $\sum_i X_i$. Is this family is complete?