1. Let $X_i, i = 1, 2, 3$, be independent with $\mathrm{N}(i, i^2)$ distributions. For each of the following situations, use the X_is to construct a statistic with the indicated distribution.

   (1) Chi-square distribution with 3 degrees of freedom (5%)

   (2) t distribution with 2 degrees of freedom (5%)

   (3) F distribution with 1 and 2 degrees of freedom (5%)

2. Suppose $X_1, \ldots, X_n$ are iid uniform observations on the interval $(\theta, \theta+1)$, $-\infty < \theta < \infty$. Show that $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ, but it is not complete. (10%)

3. Prove the following statements.

   (1) The statistic $\sum_i X_i^2$ is minimal sufficient in the $\mathrm{N}(\mu, \mu)$ family. (5%)

   (2) The statistic $(\sum X_i, \sum X_i^2)$ is minimal sufficient in the $\mathrm{N}(\mu, \mu^2)$ family. (5%)

4. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \frac{1}{\theta}, 0 \leq x \leq \theta, \theta > 0.$$

   Estimate θ using both the method of moments and maximum likelihood. Calculate the means and variances of the two estimators. (10%)

5. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \theta x^{\theta-1}, 0 \leq x \leq 1, 0 < \theta.$$

   Find the method of moment estimator and the MLE of θ. (10%)

6. Let $X_1, \ldots, X_n$ be iid $\mathrm{N}(\theta, 1)$. Show that the best unbiased estimator of θ^2 is $\bar{X}_n - 1/n$, and show that its variance is greater than the Cramer-Rao Lower Bound. (10%)

7. Suppose $X_i, i = 1, \ldots, n$ are iid Bernoulli(p).

   (1) Show that the variance of the MLE of p attains the Cramer-Rao Lower Bound. (10%)

   (2) For $n \geq 4$, show that the product $X_1X_2X_3X_4$ is an unbiased estimator of p^4, and use this fact to find the best unbiased estimator of p^4. (10%)

8. Let $f(x|\theta)$ be the logistic location pdf,

$$f(x|\theta) = \frac{\exp(x-\theta)}{(1+\exp(x-\theta))^2}, -\infty < x < \infty, -\infty < \theta < \infty.$$

   Show that this family has an MLR. (10%)

9. Let $X_1, \dots, X_n$ be a random sample from a $\mathrm{N}(\theta, \sigma^2)$ population. Consider testing

$$H_0 : \theta \leq \theta_0 \text{ versus } H_1 : \theta > \theta_0.$$

(1) If σ^2 is known, show that the test that rejects H_0 when

$$\bar{X}_n > \theta_0 + z_\alpha \sqrt{\sigma^2/n}$$

is a test of size α, and also prove that this test is a UMP test. (15%)

(2) If σ^2 is unknown, show that the test that rejects H_0 when

$$\bar{X}_n > \theta_0 + t_{n-1,\alpha} \sqrt{S^2/n}$$

is a test of size α. Show that this test can be derived as an LRT. (10%)