1. Assume the model is

$$y = X\beta + \varepsilon,$$

where $y = (y_1, \cdots, y_n)^T$ is the vector of the observations, $X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{pmatrix}$ is the design matrix, $\beta = (\beta_0, \beta_1, \cdots, \beta_p)^T$ is the vector of the unknown parameters, and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)^T \sim MN(\mathbf{0}, I_n)$. (a) Find the least-squares estimator of β, $\hat{\beta}$. Let the vector of fitted values be $\hat{y} = X\hat{\beta} = Hy$, where $H = X(X^TX)^{-1}X^T$ is the hat matrix. (b) Show $I_{p+1} - H$ is symmetric and idempotent, and (c) $H\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^TH = \mathbf{1}^T$, where $\mathbf{1} = (1, \cdots, 1)^T$ is a $(p+1)\times 1$ vector of ones.

2. Consider the $p\times p$ matrix X^TX and let x be the ith row of X. Find $(X^TX - xx^T)^{-1}$.

3. Show that if $A = A^T$, then A is a tripotent if and only if $A = A^-$.

4. Let $B = \begin{pmatrix} -3 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. (a) Find a real number t such that $B + tI_3$ is positive definite. Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 4 \end{pmatrix}$. (b) Find a positive number t_1 such that $A + t_1B$ is positive definite.

5. Let an orthogonal matrix $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$. Show that $P_1^TP_1$ is idempotent.

6. IF A is a positive definite $k \times k$ matrix and B is a nonnegative $k \times k$ matrix, then (a) show

$$\lambda_1 \le \frac{x'Bx}{x'Ax} \le \lambda_k$$

for each and every vector $x \neq \mathbf{0}$, where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$ are the roots of $|B - \lambda A| = 0$. (b) Find vectors x_1 and x_2 such that $\lambda_1 = \frac{x_1'Bx_1}{x_1'Ax_1}$ and $\frac{x_2'Bx_2}{x_2'Ax_2} = \lambda_k$.