1. Let A be an $m \times m$ symmetric matrix, $\lambda_1 \geq \ldots \geq \lambda_m$ be its eigenvalues, $P_1, \ldots, P_m$ be the corresponding eigenvectors. Thus we have the following equations,

$$\begin{aligned} A &= \lambda_1 P_1 P_1^T + \ldots + \lambda_m P_m P_m^T, \\ I_m &= P_1 P_1^T + \ldots + P_m P_m^T. \end{aligned}$$

Show the following propositions:

(a) Let X be an $m \times 1$ vector.

$$\sup_X \frac{X^T A X}{X^T X} = \lambda_1 \text{ and } \inf_X \frac{X^T A X}{X^T X} = \lambda_m.$$

(b)

$$\sup_{P_i^T X = 0, i=1,\ldots,k} \frac{X^T A X}{X^T X} = \lambda_{k+1} \text{ and } \inf_{P_i^T X = 0, i=1,\ldots,k} \frac{X^T A X}{X^T X} = \lambda_m.$$

(c) Let B be an $m \times k$ matrix. Then

$$\inf_B \sup_{B^T X = 0} \frac{X^T A X}{X^T X} = \lambda_{k+1}.$$

2. Find the inverse of A, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 3 & 2 & 1 & 2 & 4 \end{pmatrix}.$$

3. Let

$$B = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix}.$$

Show that $B = I_3 + bb^T$, where $b^T = (1, 2, 3)$, and then find the determinant of B.

4. Let the $n \times n$ matrix A be defined by

$$A = \begin{pmatrix} I_{n_1} & \mathbf{0} \\ B & I_{n_2} \end{pmatrix},$$

where B is an $n_1 \times n_2$ matrix and the size of the other submatrices are thus determined. Show that A^{-1} exists and find it.

5. Show that the matrix A below is positive definite.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

6. Let A be a non-negative matrix. Show that B is positive definite for each and every a such that $0 \leq a \leq 1$, where

$$B = aI + (1 - a)A.$$

7. Let Σ be the covariance matrix of an $m \times 1$ random vector, X. Show that Σ is non-negative matrix.

8. If $A^T = A$ and $A^n = A^{n+2m-1}$, where m and n are any positive number, show that A is symmetric idempotent.